CONDUCTORS WITH RESPECT TO HEREDITARY ORDERS

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1. INTRODUCTION

Let $\mathfrak{o}$ be a Dedekind ring with quotient field k, let A be a separable, finite-dimensional k-algebra, and let Λ be an $\mathfrak{o}$-order in A. Let $\mathfrak{o}_p$, A_p, k_p, Λ_p and so on denote the completions at p, where p is a prime ideal in $\mathfrak{o}$. If Γ is an $\mathfrak{o}$-order in A, we denote by $(\Lambda\colon \Gamma)_r$ the maximal right Γ-ideal in Λ, or equivalently,

$$(\Lambda\colon \Gamma)_r = \{x \in A\colon x\Gamma \subseteq \Lambda\}.$$

This ideal is called the right conductor of Γ in Λ. In a similar way we define the left conductor $(\Lambda\colon \Gamma)_l$ of Γ in Λ. These conductors are related to properties of $\mathrm{Ext}^1_\Lambda(M, N)$, for arbitrary Λ-lattices M and N. Let cen Λ be the center of Λ, and let $J(\Lambda)$ denote the set of elements x in cent Λ that satisfy the condition

$$x\,\mathrm{Ext}^1_\Lambda(M, N) = 0$$

for every pair of left Λ-lattices M and N. D. G. Higman [4] has proved that $J(\Lambda) \neq 0$. H. Jacobinski [5] has shown that $(\Lambda\colon \Gamma)_l \cap \mathrm{cen}\,\Lambda \subseteq J(\Lambda)$ for all hereditary $\mathfrak{o}$-orders Γ containing Λ. K. W. Roggenkamp [6] has proved that

$$J(\Lambda) \subseteq ((\Lambda\colon \Gamma)_l\, \Gamma) \cap \mathrm{cen}\,\Lambda,$$

for all hereditary $\mathfrak{o}$-orders Γ containing Λ. Therefore, if there exists a hereditary $\mathfrak{o}$-order $\Gamma \supseteq \Lambda$ such that the left conductor $(\Lambda\colon \Gamma)_l$ is a two-sided Γ-ideal, then

$$J(\Lambda) = (\Lambda\colon \Gamma)_l \cap \mathrm{cen}\,\Lambda.$$

In a special case, the existence of such an order is known. Let G be a finite group of order n, such that char $k \nmid n$, $A = kG$, and $\Lambda = \mathfrak{o}G$. Jacobinski [5] has proved that the left conductor $(\Lambda\colon \Gamma)_l$ is a two-sided Γ-ideal for all $\mathfrak{o}$-orders Γ containing Λ.

Let Λ be an $\mathfrak{o}$-order contained in only finitely many maximal orders. In this paper, we shall prove that there exists a hereditary $\mathfrak{o}$-order Γ containing Λ such that the left conductor $(\Lambda\colon \Gamma)_l$ of Γ in Λ is a two-sided Γ-ideal. In fact, we shall prove a slightly more general result. Let I be a full ideal in A, that is, a finitely generated $\mathfrak{o}$-module such that $kI = A$. Let Λ be the left order of I. If Γ is an order, let $(I\colon \Gamma)_r$ be the maximal right Γ-ideal in I.

THEOREM 1. *Let* I *be a full ideal in* A *with left order* Λ *such that* Λ *is contained in only finitely many maximal orders. Then there exists a hereditary* $\mathfrak{o}$-*order* Γ, *containing* Λ, *such that the right conductor* $(I\colon \Gamma)_r$ *is a two-sided* Γ-*ideal.*

Remark 1. By symmetry, a similar result holds for the left conductor.

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Remark 2. If the field k is an algebraic number field or an algebraic function field in one variable over a finite constant field, then the condition that Λ is contained in only finitely many maximal-orders is automatically satisfied.

2. REDUCTION TO THE CASE WHERE A IS A SIMPLE ALGEBRA OVER A p-ADICALLY COMPLETE FIELD

An order Γ is hereditary if and only if Γ_p is hereditary for all prime ideals in $\mathfrak{o}$. Conductors also localize well; that is, $(I_p : \Gamma_p)_r = ((I: \Gamma)_r)_p$ for every prime ideal p in $\mathfrak{o}$. An order Λ is contained in only finitely many maximal orders if and only if Λ_p is contained in only finitely many maximal orders, for every prime ideal in $\mathfrak{o}$. Therefore it is sufficient to prove Theorem 1 in the case where A is an algebra over a p-adically complete field. From now on we deal only with this case, and we can omit the subscript p and assume that $\mathfrak{o}$ is a p-adically complete ring. Let e_i $(i = 1, 2, \cdots, n)$ be the primitive central idempotents of A. If Γ is hereditary, then $e_i \in \Gamma$ and each Γe_i is a hereditary order in Ae_i. The converse holds if $\Gamma = \bigoplus_1^n \Gamma e_i$, where each Γe_i is hereditary. Now let Λ be the left order of I. Then clearly $I \cap Ae_i$ admits Λe_i. But $\Delta_i = I \cap Ae_i$ is a full ideal in Ae_i. If Theorem 1 is proved for simple algebras, we can find a hereditary order Γ_i in Ae_i, containing Λe_i, such that the conductor $(\Delta_i : \Gamma_i)_r$ is a two-sided Γ_i-ideal. But then $\Gamma = \bigoplus \Gamma_i$ is hereditary and contains $\bigoplus \Lambda e_i \supseteq \Lambda$, and $(I: \Gamma)_r = \bigoplus (\Delta_i : \Gamma_i)_r$ is a two-sided Γ-ideal. Now we see that it is enough to prove Theorem 1 in the case where A is a simple algebra.

3. HEREDITARY ORDERS

We now mention some known facts about hereditary orders in a simple algebra over a p-adically complete field. Let $A = (D)_t$, where D is a skew field with the unique maximal $\mathfrak{o}$-order Ω, and let $P = \text{rad}\, \Omega$. From now on we fix an irreducible left A-module W. This A-module is a right D-module. We denote by W_Ω the set of all right Ω-lattices V in W such that $kV = W$. For every $V \in W_\Omega$, $\text{End}_\Omega(V)$ is a maximal order in A. Conversely, the set $\{\text{End}_\Omega(V) \mid V \in W_\Omega\}$ contains all maximal orders in A. If O is a maximal order in A and $V \in W_\Omega$ is a left O-lattice, then every full right O-ideal I is equal to $\text{Hom}_\Omega(V, U)$, where $U \in W_\Omega$. In fact, $U = IV$, and U is uniquely determined by V. Clearly, $\text{End}_\Omega(U)$ is the left order of I.

LEMMA 2. *If* $V_1, V_2, U \in W_\Omega$, *then*

(i) $\text{Hom}_\Omega(V_1, U) + \text{Hom}_\Omega(V_2, U) = \text{Hom}_\Omega(V_1 \cap V_2, U)$,

(ii) $\text{Hom}_\Omega(U, V_1) + \text{Hom}_\Omega(U, V_2) = \text{Hom}_\Omega(U, V_1 + V_2)$.

Proof. There exists a right D-basis $\{e_1, e_2, \cdots, e_t\}$ of W such that

$$V_1 = \bigoplus_1^t e_i p^{\alpha_i} \quad \text{and} \quad V_2 = \bigoplus_1^t e_i p^{\beta_i}.$$

This implies that there are two right Ω-lattices V_1' and V_2' such that $V_1 \cap V_2 = V_1' \oplus V_2'$ and V_i' is an Ω-direct summand of V_i $(i = 1, 2)$. Thus

$$\mathrm{Hom}_\Omega(V_1 \cap V_2, U) = \mathrm{Hom}_\Omega(V_1', U) \oplus \mathrm{Hom}_\Omega(V_2', U)$$
$$\subseteq \mathrm{Hom}_\Omega(V_1, U) + \mathrm{Hom}_\Omega(V_2, U).$$

The reverse inclusion is trivial. The proof of (ii) is similar to that of (i).

Now, if M is a finite subset of W_Ω, then

$$\Lambda_M = \{x \in A \mid xV \subseteq V \text{ for all } V \in M\} = \bigcap_{V \in M} \mathrm{End}_\Omega(V).$$

Λ_M is an order. We see that if

$$\overline{M} = \{VP^\alpha \mid V \in M \text{ and } \alpha \in Z\},$$

then $\Lambda_M = \Lambda_{\overline{M}}$. Therefore we can always assume that $V \in M$ implies that $VP^\alpha \in M$ for all integers α. If $V_1, V_2 \in M$, then V_1 and V_2 are isomorphic as Λ_M-lattices if and only if $V_1 = V_2 P^\alpha$ for some integer α. By $\overline{V}$ we denote the set $\{VP^\alpha \mid \alpha$ an integer$\}$, that is, the isomorphism class of V.

THEOREM 3 (A. Brumer [1] and [2], M. Harada [3]; for a proof, see Jacobinski [6]). (i) *An order* Γ *in* A *is hereditary if and only if* $\Gamma = \Lambda_M$ *for some* $M \subseteq W_\Omega$, *where* M *is totally ordered by inclusion and closed under isomorphism.*

(ii) *Let* $\Gamma = \Lambda_M$ *be hereditary. Then every indecomposable left* Γ*-lattice is isomorphic to some* $V \in M$.

LEMMA 4. *Let* $M \subseteq W_\Omega$ *be closed under isomorphism. Then* Λ_M *is a hereditary order if and only if* M *satisfies the following condition. For each pair* $U, V \in M$, *there exists* U' *isomorphic to* U, *as left* Λ_M*-lattice, such that* $VP \subseteq U' \subseteq V$.

Proof. It is enough to prove that the condition means that M is totally ordered by inclusion. If M is totally ordered and $U, V \in M$, then the maximal element $U' \in \overline{U}$ such that $U' \subseteq V$ satisfies the condition $VP \subseteq U'$. On the other hand, if V and $U \in M$ and $U \not\subseteq V$, the least integer α such that $UP^\alpha \subseteq V$ is positive. But then $VP \subseteq UP^\alpha \subseteq V$ and thus $V \subseteq UP^{\alpha-1} \subseteq U$. Therefore M is totally ordered by inclusion.

4. CONDUCTORS

Let I be a full ideal in the simple algebra A.

LEMMA 5. *If* Γ *is a hereditary order and* O_i $(i = 1, 2, \cdots, n)$ *are the maximal orders containing it, then*

$$(I: \Gamma)_r = \sum_i (I: O_i)_r.$$

Proof. By Theorem 3, $(I: \Gamma)_r$ is a sum of right O_i-ideals. Thus $(I: \Gamma) \subseteq \sum_i (I: O_i)_r$. Since $\Gamma = \bigcap_i O_i$, we see that $\sum_i (I: O_i)_r$ is a right Γ-ideal, and therefore the opposite inclusion also holds.

Let Λ be the left order of I, and let M be the set of all left Λ-lattices $V \in W_\Omega$. If $V \in M$, then, by (ii) in Lemma 2, there exists a maximal element U of W_Ω such that $\mathrm{Hom}_\Omega(V, U) \subseteq I$. If O is the left order of V, we see that $\mathrm{Hom}_\Omega(V, U)$ is the maximal right O-ideal in I. If $\lambda \in \Lambda$, then $\lambda \,\mathrm{Hom}(V, U) \subseteq I$, and thus $\lambda \,\mathrm{Hom}_\Omega(V, U) \subseteq \mathrm{Hom}_\Omega(V, U)$. Therefore U is a left Λ-lattice. We see that the map $B: V \to U$ takes M into itself. Obviously, B has the property that

$$B(VP^\alpha) = (BV)P^\alpha \qquad (\alpha \text{ an integer}).$$

LEMMA 6. *If* $V_1, V_2 \in M$, *then* $B(V_1 \cap V_2) = BV_1 \cap BV_2$.

Proof. Because $\mathrm{Hom}_\Omega(V_i, BV_1 \cap BV_2) \subseteq \mathrm{Hom}_\Omega(V_i, BV_i)$ $(i = 1, 2)$,

$$\mathrm{Hom}_\Omega(V_1 \cap V_2, BV_1 \cap BV_2) = \mathrm{Hom}_\Omega(V_1, BV_1 \cap BV_2) + \mathrm{Hom}_\Omega(V_2, BV_1 \cap BV_2) \subseteq I.$$

We conclude that $BV_1 \cap BV_2 \subseteq B(V_1 \cap V_2)$. The opposite inclusion is an immediate consequence of the definition of B.

5. PROOF OF THEOREM 1

As before, I is a full ideal in the simple algebra A with left order Λ. If $U, V \in W_\Omega$ are Λ-lattices, then U and V are isomorphic as Λ-lattices if and only if $\mathrm{End}_\Omega(U) = \mathrm{End}_\Omega(V)$. The assumption that the number of maximal orders containing Λ is finite implies that the number of nonisomorphic Λ-lattices in W_Ω is finite. In fact, these numbers are equal. Now take a Λ-lattice T in W_Ω. The sequence $\overline{T}, \overline{BT}, \cdots$ must be periodic. We can assume that $T, BT, \cdots, B^{m-1}T$ are nonisomorphic but T and $B^m T$ are isomorphic. Therefore $B^m T = TP^\alpha$ for some integer α. Now we define a Λ-lattice V in W_Ω by the formula

$$V = \bigcap_{i=0}^{m-1} B^i T P^{\{-i\alpha/m\}},$$

where $B^0 T = T$ and $\{-i\alpha/m\}$ is the least integer that is not less than $-i\alpha/m$. From Lemma 5 we get the relation

$$BV = BT \cap B^2 T P^{\{-\alpha/m\}} \cap \cdots \cap B^m T P^{\{-(m-1)\alpha/m\}}.$$

But

$$B^m T P^{\{-i(m-1)\alpha/m\}} = T P^{\alpha + \{-(m-1)\alpha/m\}} = T P^{\{\alpha/m\}},$$

and therefore $BV = \bigcap_{i=0}^{m-1} B^i T P^{\{(1-i)\alpha/m\}}$. By induction, we find that

$$B^\mu V = \bigcap_{i=0}^{m-1} B^i T P^{\{(\mu-i)\alpha/m\}}.$$

If x and y are real numbers, then $\{x+y\} \le \{x\} + \{y\} \le \{x+y\} + 1$, and this implies that

$$\{(\nu - i)\alpha/m\} \le \{(\nu - \mu)\alpha/m\} + \{(\mu - i)\alpha/m\} \le \{(\nu - i)\alpha/m\} + 1,$$

where ν and μ are nonnegative integers. Therefore

$$\bigcap_{i=0}^{m-1} B^i T P^{\{(\nu-i)\alpha/m\}} \supseteq \bigcap_{i=0}^{m-1} B^i T P^{\{(\nu-\mu)\alpha/m\}+\{(\mu-i)\alpha/m\}}$$

$$\supseteq \bigcap_{i=0}^{m-1} B^i T P^{\{(\nu-i)\alpha/m\}+1},$$

and this is equivalent to

$$B^\nu V \supseteq B^\mu V P^{\{(\nu-\mu)\alpha/m\}} \supseteq B^\nu V P. \tag{1}$$

The sequence $\overline{V}$, $\overline{BV}$, $\cdots$ must be periodic, and we can assume that V, $\overline{BV}$, $\cdots$, $\overline{B^n V}$ are nonisomorphic and V and $B^{n+1}V$ are isomorphic. Put $M = \bigcup_{i=0}^n \overline{B^i V}$. By Lemma 4 and (1), we conclude that $\Gamma = \Lambda_M$ is a hereditary order. Let O_i be the left order of $B^i V$ $(i = 0, 1, \cdots, n)$. Then

$$\Gamma = \bigcap_{i=0}^{n} O_i \quad \text{and} \quad (I\colon \Gamma)_r = \sum_{i=0}^{n} (I\colon O_i)_r,$$

by Lemma 5. But the left order of $(I\colon O_i)_r$ is O_{i+1} for $i = 0, 1, \cdots, n-1$, and the left order of $(I\colon O_n)_r$ is O_0. Therefore $\Gamma = \bigcap_{i=0}^n O_i$ is contained in the left order of $(I\colon \Gamma)_r = \sum_{i=0}^n (I\colon O_i)_r$. Finally, Γ contains Λ, because all the $B^i V$ are Λ-lattices. Thus Theorem 1 is proved.

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