SUMMING SEQUENCES FOR AMENABLE SEMIGROUPS

Steven A. Douglass

In a wide variety of settings, the special set $\sigma_n = \{0, 1, 2, \cdots, n-1\}$ enters into computations involving the additive semigroup Z^+ of nonnegative integers. In this paper, we identify the significant mathematical properties of the sequence $\{\sigma_n: n = 1, 2, \cdots\}$, and we show that if G is a countable, cancellative, amenable semigroup, then there exists a sequence $\{S_n\}$ of finite subsets of G possessing exactly these properties. In Section 3, we examine some examples and obtain miscellaneous properties.

1. PRELIMINARIES

Let G be a semigroup, and let m(G) denote the Banach space of bounded, real-valued functions on G endowed with the supremum norm

$$\|f\|_\infty = \sup\{|f(g)|: g \in G\}.$$

We shall also be interested in the subspace $\ell_1(G)$ consisting of the functions f in m(G) with finite ℓ_1-norm $\|f\|_1 = \sum \{|f(g)|: g \in G\} < \infty$. Endowed with the convolution

$$(f_1 * f_2)(g) = \sum \{f_1(h')f_2(h''): h'h'' = g\},$$

$\ell_1(G)$ is a real Banach algebra.

A *weight* on G is a nonnegative function ϕ in $\ell_1(G)$ having finite support and such that $\|\phi\|_1 = 1$. A *simple weight* on G is a weight ϕ that is constant on its support; that is, ϕ is a simple weight provided $\phi = |A|^{-1}\chi_A$, where A, $|A|$, and χ_A denote the support of ϕ, the number of elements in the support, and its characteristic function. We denote the collection of all weights by Φ, the collection of all simple weights by Φ_s. For simplicity, given a g in G, we denote by g the simple weight with support $\{g\}$.

A *mean* on G is a real linear functional Λ on m(G) such that for each f in m(G),

$$\inf\{f(g): g \in G\} \leq \Lambda(f) \leq \sup\{f(g): g \in G\}.$$

Clearly, a mean Λ is a positive linear functional such that $\Lambda(1) = 1$, where 1 denotes the function $1(g) = 1$ for all g in G. If g is in G and f is in m(G), then ${}^g f$ and f^g are functions in m(G) defined by the equations

$${}^g f(h) = f(gh) \quad \text{and} \quad f^g(h) = f(hg),$$

respectively. A mean Λ on G is said to be *left* [*right*] *invariant* if $\Lambda({}^g f) = \Lambda(f)$ [if $\Lambda(f^g) = \Lambda(f)$] for all g in G and all f in m(G). Finally, G is said to be *amenable* if a left invariant mean and a right invariant mean exist on G.

Received May 31, 1972.

Michigan Math. J. 20 (1973).

2. SUMMING SEQUENCES

M. M. Day [1] has proved that G is amenable if and only if there exists a net $\{\phi_\alpha\}$ in Φ such that, for every g in G,

$$\lim \phi_\alpha * g - \phi_\alpha = \lim g * \phi_\alpha - \phi_\alpha = 0,$$

where the limit is taken in either the weak or the uniform topology on $\ell_1(G)$. We shall use this result, together with the following two lemmas of I. Namioka [2], to prove the existence in each countable, cancellative, amenable semigroup G of a sequence $\{S_n\}$ of finite subsets of G such that

(i) $S_n \subset S_{n+1}$ for $n = 1, 2, \cdots$,

(ii) $G = \bigcup_{n=1}^{\infty} S_n$,

(iii) for each g in G, $\lim_{n\to\infty} |S_n g \cap S_n|/|S_n| = \lim_{n\to\infty} |g S_n \cap S_n|/|S_n| = 1$.

A sequence $\{S_n\}$ of finite subsets of G having these properties will be called a *summing sequence* for G. Elsewhere, we shall examine several applications of summing sequences that give rise to a general concept of Cesàro summability. In all that follows, G is an amenable semigroup.

LEMMA 2.1 (Namioka). *Φ is the convex hull of Φ_s; in particular, for each ϕ in Φ, there exist finite subsets $A_1, A_2, \cdots, A_n$ of G and positive numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$, such that*

$$A_1 \supset A_2 \supset \cdots \supset A_n, \quad \sum_{j=1}^{n} \lambda_j = 1, \quad \textit{and } \phi = \sum_{j=1}^{n} \lambda_j \phi_j,$$

where ϕ_j is the simple weight having support A_j.

LEMMA 2.2 (Namioka). *If $\phi = \sum_{j=1}^{n} \lambda_j \phi_j$ is the decomposition of a weight described in Lemma 2.1, then for each g in G,*

$$\|\phi * g - \phi\|_1 \geq \sum_{j=1}^{n} \lambda_j |A_j g \setminus A_j|/|A_j| \tag{2.1}$$

and

$$\|g * \phi - \phi\|_1 \geq \sum_{j=1}^{n} \lambda_j |g A_j \setminus A_j|/|A_j| . \tag{2.2}$$

Because in [2] Namioka dealt only with right amenability, and because we want the full strength of two-sided amenability, we include a modified version of Namioka's elegant proof of the following theorem.

THEOREM 2.3 (E. Følner and A. Frey, Jr.). *Let G be an amenable semigroup. For each finite subset B of G and each $\varepsilon > 0$, there exists a finite subset S of G such that*

$$|Sg \setminus S| < \varepsilon |S| \quad \textit{and} \quad |gS \setminus S| < \varepsilon |S|$$

for all g *in* B.

Proof (Namioka [2]). Let $B = \{g_1, g_2, \cdots, g_k\}$. Day's theorem enables us to select a ϕ in Φ such that

$$\|\phi * g_i - \phi\|_1 < \varepsilon/2k \quad \text{and} \quad \|g_i * \phi - \phi\|_1 < \varepsilon/2k \tag{2.3}$$

for $i = 1, 2, \cdots, k$. By Lemma 2.1, the weight ϕ has a convex linear decomposition $\phi = \sum_{j=1}^n \lambda_j \phi_j$, where the ϕ_j are simple weights whose supports A_j satisfy the condition $A_1 \supset A_2 \supset \cdots \supset A_n$. The inequalities (2.1) and (2.2) are then valid for all g in G.

Define a measure μ on the collection of subsets I of $N = \{1, 2, \cdots, n\}$ by defining $\mu(I) = \left\{\sum \lambda_j : j \in I\right\}$. By convention, $\mu(\phi) = 0$; also, $\mu(N) = \sum_{j=1}^n \lambda_j = 1$, since the λ_j are the coefficients in a convex linear combination. Now, for $i = 1, 2, \cdots, k$, define

$$I_i = \{j \in N: [|A_j g_i \setminus A_j| + |g_i A_j \setminus A_j|]/|A_j| < \varepsilon\}.$$

Then, applying (2.1), (2.2), and (2.3), we see that for $i = 1, 2, \cdots, k$,

$$\begin{aligned} \varepsilon/k &> \|\phi * g_i - \phi\|_1 + \|g_i * \phi - \phi\|_1 \\ &\geq \sum_{j=1}^{n} \lambda_j [|A_j g_i \setminus A_j| + |g_i A_j \setminus A_j|]/|A_j| \geq \varepsilon \sum_{j \notin I_i} \lambda_j = \varepsilon \mu(N \setminus I_i). \end{aligned}$$

Therefore, $\mu(N \setminus I_i) < 1/k$ for $i = 1, 2, \cdots, k$. Consequently,

$$\mu\left(N \setminus \bigcap_{i=1}^k I_i\right) = \mu\left(\bigcup_{i=1}^k (N \setminus I_i)\right) \leq \sum_{i=1}^k \mu(N \setminus I_i) < k/k = 1,$$

so that $\mu\left(\bigcap_{i=1}^k I_i\right) > 0$ and $\bigcap_{i=1}^k I_i \neq \phi$. Pick any j_0 in $\bigcap_{i=1}^k I_i$. Then the subset $S = A_{j_0}$ satisfies the conditions of the theorem.

The second property of a summing sequence $\{S_n\}$ —to wit, that each finite subset C of G is contained in S_n for sufficiently large n —will be a direct consequence of the following improvement of Theorem 2.3. The improvement results from the hypothesis that G has both right and left cancellation.

THEOREM 2.4. *Let* G *be a cancellative, amenable semigroup. Let* B *and* C *be any two finite subsets of* G, *and let* r *be any number* $(0 < r < 1)$. *Then there exists a finite subset* S *of* G *such that* $S \supseteq C$ *and*

$$|Sg \cap S| > r|S| \quad \textit{and} \quad |gS \cap S| > r|S|$$

for all g *in* B.

Proof. Without loss of generality, we may assume that G is infinite. Since $0 < r < 1$, we can choose an n such that $n > r|C|/(1 - r)$, that is, $r(1 + |C|/n) < 1$.

Select a finite subset B_1 of G such that $B_1 \supseteq B$ and $|B_1| \geq n^2$. Let

$$\varepsilon = 1 - r(1 + |C|/n) > 0 .$$

By Theorem 2.3, there exists a finite subset S_1 of G such that

$$|S_1 g \setminus S_1| < \varepsilon |S_1| \quad \text{and} \quad |g S_1 \setminus S_1| < \varepsilon |S_1|$$

for all g in B_1.

Now $S_1 g$ is the disjoint union of $S_1 g \setminus S_1$ and $S_1 g \cap S_1$, so that

$$|S_1 g| = |S_1 g \setminus S_1| + |S_1 g \cap S_1| .$$

On the other hand, right cancellation in G implies that $|S_1 g| = |S_1|$. Therefore

$$|S_1| = |S_1 g \setminus S_1| + |S_1 g \cap S_1| < \varepsilon |S_1| + |S_1 g \cap S_1|$$

for all g in B_1. That is,

$$|S_1 g \cap S_1| > (1 - \varepsilon)|S_1| = r(1 + |C|/n)|S_1| \quad (g \in B_1).$$

Similarly,

$$|g S_1 \cap S_1| > r(1 + |C|/n)|S_1| \quad (g \in B_1).$$

For each g in B_1, the set $S_1 g \cap S_1$ is not empty, and therefore there is at least one ordered pair (h_1, h_2) in $S_1 \times S_1$ such that $h_1 g = h_2$. An ordered pair (h_1, h_2) cannot correspond to different elements g and g' of B_1, for if $h_1 g = h_2 = h_1 g'$, then the left-cancellation law for G implies that $g = g'$. Consequently, $|B_1| \leq |S_1 \times S_1| = |S_1|^2$. Since B_1 was chosen so that $|B_1| \geq n^2$, it follows that $|S_1| \geq n$.

Finally, put $S = S_1 \cup C$ to obtain the set promised by the theorem. For each g in $B \subseteq B_1$,

$$\begin{aligned} |Sg \cap S| &\geq |S_1 g \cap S_1| > r(1 + |C|/n)|S_1| \geq r(1 + |C|/|S_1|)\,|S_1| \\ &= r(|S_1| + |C|) \geq r\,|S| . \end{aligned}$$

Analogously, $|gS \cap S| > r\,|S|$ for each g in B.

THEOREM 2.5. *Let* G *be a countable, cancellative, amenable semigroup. Then there exists a sequence* $\{S_n\}$ *of finite subsets of* G *such that* $S_n \subseteq S_{n+1}$ *for* $n = 1, 2, \cdots$, $G = \bigcup_{n=1}^{\infty} S_n$, *and for each* g *in* G,

$$\lim_{n \to \infty} |S_n g \cap S_n|/|S_n| = \lim_{n \to \infty} |g S_n \cap S_n|/|S_n| = 1 .$$

Proof. Since G is countable, G can be written as the union $\bigcup_{n=1}^{\infty} B_n$ of finite sets such that $B_n \subset B_{n+1}$ for $n = 1, 2, \cdots$. Select any finite set $S_1 \supseteq B_1$. Having picked S_{n-1}, we are entitled by Theorem 2.4 to choose a finite set $S_n \supseteq C_n = S_{n-1} \cup B_n$ such that

$$|S_n g \cap S_n| > (1 - 1/n)|S_n| \quad \text{and} \quad |gS_n \cap S_n| > (1 - 1/n)|S_n|$$

for all g in B_n. The sequence $\{S_n\}$ so chosen will have all the desired properties.

3. EXAMPLES

The sequence $\{\sigma_n\}$ of finite subsets of integers, where

$$\sigma_n = \{0, 1, 2, \cdots, n - 1\},$$

is a familiar summing sequence giving rise to the classical Cesàro method of summability. It is worthwhile to note that $(Z^+, +)$ has many other summing sequences as well. For example, let $F = \{0, 1, k\}$, and let F^n denote $F + F + \cdots + F$ (n summands). Then, for $n \geq k$,

$$\begin{aligned} F^n = \{&0, 1, 2, 3, \cdots, (n - k + 3)k - 2, \\ &(n - k + 3)k, (n - k + 3)k + 1, \cdots, (n - k + 4)k - 3, \\ &(n - k + 4)k, (n - k + 4)k + 1, \cdots, (n - k + 5)k - 4, \\ &\cdots \\ &(n - k + p + 3)k, (n - k + p + 3)k + 1, \cdots, [(n - k + p + 4)k - (p + 3)], \cdots \\ &\cdots \\ &(n - 1)k, (n - 1)k + 1, nk\}. \end{aligned}$$

It follows that

$$F^n = \sigma_{(n-k+3)k-1} \cup \bigcup_{i=1}^{k-2} [\sigma_i + (n - i + 1)k] \quad \text{(disjoint)}. \tag{3.1}$$

Clearly $F^n \subset F^{n+1}$, $Z^+ = \bigcup_{n=1}^{\infty} F^n$, and

$$|F^n| = (n - k + 3)k - 1 + \sum_{i=1}^{k-2} i = (n - k + 3)k - 1 + \frac{1}{2}(k - 2)(k - 1).$$

To show that $\{F^n\}$ is a summing sequence, we need merely show that if $j_0 \in Z^+$ and $0 < r < 1$, then $|(j_0 + F^n) \cap F^n| > r|F^n|$ for sufficiently large n. Referring to (3.1), we see that

$$(j_0 + F^n) \cap F^n \supset (j_0 + \sigma_m) \cap \sigma_m = j_0 + \sigma_{m-j_0},$$

where $m = (n - k + 3)k - 1$. Consequently, $|(j_0 + F^n) \cap F^n| > (n - k + 3)k - 1 - j_0$. Finally, because $0 < r < 1$, we have the inequality

$$(n - k + 3)k - 1 - j_0 > r\left\{(n - k + 3)k - 1 + \frac{1}{2}(k - 2)(k - 1)\right\} = r|F^n|$$

whenever

$$n > k - 3 + \left\{1 + j_0 - r\left[1 - \frac{1}{2}(k-2)(k-1)\right]\right\}[(1-r)k]^{-1}.$$

Consequently, $\{F^n\}$ is a summing sequence. We can use a similar argument to verify that if $F = \{0, 1, k_1, k_2, \cdots, k_s\}$, then $\{F^n\}$ is a summing sequence for $(Z^+, +)$.

In general, let $\{S_n\}$ be any summing sequence for $(Z^+, +)$. The structure of S_n for large n is revealed as follows: fix any $\varepsilon > 0$ and any $m \geq 1$. By the summing-sequence property,

$$\left|\bigcap_{j=0}^{m-1} (j + S_n)\right| > (1 - \varepsilon)|S_n| \tag{3.2}$$

for all sufficiently large n. Note that $k \in \bigcap_{j=0}^{m-1} (j + S_n)$ if and only if $\{k - (m-1), k - (m-2), \cdots, k-1, k\} \subset S_n$. Therefore

$$S_n = (i_1 + \sigma_m) \cup (i_2 + \sigma_m) \cup \cdots \cup (i_s + \sigma_m) \cup R_n \qquad \text{(disjoint)},$$

where $R_n = S_n \setminus \bigcap_{j=0}^{m-1} (j + S_n)$. To see this, let k_1 be the largest integer in $\bigcap_{j=0}^{m-1} (j + S_n)$ and put $i_1 = k_1 - m + 1$, so that $i_1 + \sigma_m \subset S_n$. Pick k_2 to be the largest integer in $\bigcap_{j=0}^{m-1} (j + S_n)$ that is less than i_1, and put $i_2 = k_2 - m + 1$. Continue this process until $\bigcap_{j=0}^{m-1} (j + S_n)$ is decomposed into the disjoint union $\bigcup_{p=1}^{s} (i_p + \sigma_m)$. Then

$$|R_n| = \left|S_n \setminus \bigcap_{j=0}^{m-1} (j + S_n)\right| = |S_n| - \left|\bigcap_{j=0}^{m-1} (j + S_m)\right| < \varepsilon |S_n|,$$

by (3.2). To recapitulate, to each $m \geq 1$ and each $\varepsilon > 0$ there corresponds an $n(m, \varepsilon) \in Z^+$ such that for each $n > n(m, \varepsilon)$ there exists a finite set $F = F(n) = \{i_1, i_2, \cdots, i_s\}$ such that

$$\left|S_n \setminus \bigcup_{p=1}^{s} (i_p + \sigma_m)\right| = |R_n| < \varepsilon |S_n| \quad \text{and} \quad |F|\,|\sigma_m| < (1 + \varepsilon)|S_n|.$$

This property motivates the following definition.

Definition 3.1. Let G be a countable, amenable semigroup with two summing sequences $\{S_n\}$ and $\{T_n\}$. We say that the sequence $\{S_n\}$ *nearly divides* the sequence $\{T_n\}$ if, to each $m \geq 1$ and each $\varepsilon > 0$, there corresponds an $n(m, \varepsilon)$ such that, for each $n \geq n(m, \varepsilon)$, there exists a finite set $F = F(n) \subset G$ such that

(i) $\left|T_n \setminus \bigcup_{g \in F} gS_m\right| < \varepsilon |T_n|$

and

(ii) $|F|\,|S_m| < (1 + \varepsilon)|T_n|$.

Clearly, (i) says that T_n is "almost covered" by left translates of S_m; a moment's reflection will suffice to confirm that (ii) says that the translates are almost disjoint. We have proved above that the classical summing sequence $\{\sigma_n\}$ nearly divides each summing sequence for $(Z^+, +)$.

Definition 3.2. A summing sequence that nearly divides itself will be called *nearly divisible.*

THEOREM 3.3. *Let* $\{S_n\}$ *and* $\{T_n\}$ *be two summing sequences for* G, *and let each nearly divide the other. Then each of the sequences is nearly divisible.*

Proof. Fix $\varepsilon > 0$, select ε' so that $0 < \varepsilon' < \sqrt{1+\varepsilon} - 1$, and choose a positive integer m. Since $\{S_n\}$ nearly divides $\{T_n\}$, there exists an $n(m, \varepsilon')$ such that $p \geq n(m, \varepsilon')$ entails the existence of a finite set $F = F(p) \subset G$ satisfying the conditions

$$\left| T_p \setminus \bigcup_{g \in F} gS_m \right| < \varepsilon' |T_p| \tag{3.3}$$

and

$$|F|\,|S_m| < (1+\varepsilon')\,|T_p|\,. \tag{3.4}$$

Fix $p \geq n(m, \varepsilon')$ and the set F. Since $\{T_n\}$ also nearly divides $\{S_n\}$, there exists an $n(p, \varepsilon')$ such that for $r \geq n(p, \varepsilon')$ we can pick a finite set $E = E(r)$ in G satisfying the conditions

$$\left| S_r \setminus \bigcup_{h \in E} hT_p \right| < \varepsilon' |S_r| \tag{3.5}$$

and

$$|E|\,|T_p| < (1+\varepsilon')\,|S_r|\,. \tag{3.6}$$

Fix $r \geq n(p, \varepsilon')$ and E. Let $D = EF$. Then

$$\bigcup_{x \in D} xS_m = \bigcup_{h \in E} h\left(\bigcup_{g \in F} gS_m\right).$$

Write

$$\bigcup_{g \in F} gS_m = (T_p \setminus A_p) \cup B_p,$$

where

$$A_p = T_p \setminus \left(\bigcup_{g \in F} gS_m\right) \quad \text{and} \quad B_p = \bigcup_{g \in F} gS_m \setminus T_p.$$

Because $h(T_p \setminus A_p) \subseteq h\left(\bigcup_{g \in F} gS_m\right)$ for all h, we have the relation

$$\bigcup_{h \in E} h(T_p \setminus A) \subseteq \bigcup_{h \in E} h\left(\bigcup_{g \in F} gS_m\right) = \bigcup_{x \in D} xS_m,$$

and consequently

$$S_r \setminus \bigcup_{x \in D} xS_m \subseteq S_r \setminus \bigcup_{h \in E} h(T_p \setminus A_p).$$

Applying (3.3), (3.5), and (3.6), we obtain the inequalities

$$\left|S_r \setminus \bigcup_{x \in D} xS_m\right| \le \left|S_r \setminus \bigcup_{h \in E} hT_p\right| + \left|\bigcup_{h \in E} hA_p\right|$$

$$< \varepsilon' |S_r| + |E|\, |A_p| < \varepsilon' |S_r| + |E|\, \varepsilon' |T_p|$$

$$< \varepsilon' |S_r| + \varepsilon'(1+\varepsilon') |S_r| = \varepsilon'(2+\varepsilon') |S_r| < \varepsilon |S_r| .$$

For the second part,

$$|D|\, |S_m| \le |E|\, |F|\, |S_m| < |E|\, (1+\varepsilon') |T_p| < (1+\varepsilon')^2 |S_r| < (1+\varepsilon) |S_r| ,$$

by (3.4) and (3.6). Hence $\{S_n\}$ is nearly divisible.

As a further example, let $G_1 = G_2 = \cdots = G_r = (Z^+, +)$, and put $G = G_1 \times G_2 \times \cdots \times G_r$. Then G is a countable, amenable semigroup. If $S_n = \sigma_n \times \sigma_n \times \cdots \times \sigma_n$ (r factors), then $\{S_n\}$ is a summing sequence for G that nearly divides each other summing sequence.

An example of a summing sequence for an amenable semigroup with an infinite set of generators is provided by the multiplicative semigroup G of positive integers. The generating set $P = \{p_1, p_2, \cdots\}$ of positive prime integers provides the elements with which we can methodically construct S_n. Specifically, for each $n = 1, 2, \cdots$, let

$$S_n = \left\{ \prod_{i=1}^{n} p_i^{k_i} : 0 \le k_i \le n \right\}.$$

We see immediately that $S_n \subset S_{n+1} \uparrow G$. The summing-sequence property is established as follows: let g be any integer in G with prime factorization $\prod_{i=1}^{s} p_i^{m_i}$, where the m_i denote nonnegative integers. When $n \ge \max\{s, m_1, m_2, \cdots, m_s\}$, we have the formula

$$gS_n \cap S_n = \left\{ \prod_{i=1}^{s} p_i^{t_i} \prod_{i=s+1}^{n} p_i^{k_i} : m_i \le t_i \le n \text{ for } i = 1, 2, \cdots, s; \right.$$

$$\left. 0 \le k_i \le n \text{ for } i = s+1, \cdots, n \right\}.$$

Hence, if $0 < r < 1$, we can pick n so that

$$n^2 + n > (1 - r)^{-1} \sum_{i=1}^{s} m_i$$

as well. Then

$$|gS_n \cap S_n| = n^2 + n - \sum_{i=1}^{s} m_i > rn(n+1) = r|S_n|.$$

Therefore, $\{S_n\}$ is a summing sequence for G. We leave it to the reader to confirm that $\{S_n\}$ nearly divides each summing sequence for G.

By reexamining the summing sequence $\{\sigma_n\}$ for $(Z^+, +)$, we shall identify a further property of some summing sequences. Suppose that $\varepsilon > 0$, that m and n are positive integers $(n > m/\varepsilon)$, and that π is a function mapping σ_n into $\{1, 2, 3, \cdots, m\}$. We shall show that we can select a finite set $F = F(n, \pi)$ such that

(i) $(k + \sigma_{\pi(k)}) \subseteq \sigma_n$ for each $k \in F$,

(ii) $(j + \sigma_{\pi(j)}) \cap (k + \sigma_{\pi(k)}) = \emptyset$ for all j and k in F $(j \neq k)$,

and

(iii) $\left|\sigma_n \setminus \bigcup_{k \in F} (k + \sigma_{\pi(k)})\right| < \varepsilon |\sigma_n|$.

To see this, let $k_1 = 0$, so that $k_1 + \sigma_{\pi(k_1)} = \{0, 1, 2, \cdots, \pi(k_1) - 1\}$. Let $k_2 = \pi(k_1)$, so that $k_2 + \sigma_{\pi(k_2)} = \{\pi(k_1), \pi(k_1) + 1, \cdots, \pi(k_1) + \pi(k_2) - 1\}$. Let $k_3 = \pi(k_1) + \pi(k_2)$. Continue the procedure, picking $k_{j+1} = \sum_{i=1}^{j} \pi(k_i)$ until $n - m < \sum_{i=1}^{t} \pi(k_i) \leq n$, and put $F = \{k_1, k_2, \cdots, k_t\}$. Then F has the required properties.

Definition 3.4. A summing sequence $\{S_n\}$ for a countable amenable semigroup G is said to be *left-uniform* if, for every $\varepsilon > 0$ and every positive integer m, there exists a positive integer $n_0 = n_0(m, \varepsilon)$ such that to every $n \geq n_0$ and every function π mapping S_n into $\{1, 2, \cdots, m\}$ there corresponds a finite set $F = F(n, \pi)$ in S_n such that

(i) $gS_{\pi(g)} \subseteq S_n$ for each g in F,

(ii) $gS_{\pi(g)} \cap hS_{\pi(h)} = \emptyset$ for all g, h in F $(g \neq h)$,

(iii) $\left|S_n \setminus \bigcup_{g \in F} gS_{\pi(g)}\right| < \varepsilon |S_n|$.

The property of being left-uniform places stringent requirements on the summing sequence; our next example provides a summing sequence (other than $\{\sigma_n\}$) that is left-uniform.

Let G be a countable, locally finite group, that is, a countable group each of whose finite subsets generates a finite subgroup. It is well known that G is amenable. To construct a summing sequence for G, let $\{F_n\}$ be an increasing sequence of finite subsets of G, with $F_1 = \{e\}$, and such that $G = \bigcup_{n=1}^{\infty} F_n$. Let S_n be the finite subgroup generated by F_n. It is easy to see that $\{S_n\}$ is a summing sequence for G.

The proof that $\{S_n\}$ is left-uniform requires some labor. Let m and n be positive integers $(n \geq m)$, and let π be any function mapping S_n into $\{1, 2, \cdots, m\}$. Note first that $gS_{\pi(g)} \subseteq S_n$ for each g in S_n, since $gS_{\pi(g)}$ is a left coset of the subgroup $S_{\pi(g)}$ of S_n. Note also that if g and h are in S_n and $\pi(g) = \pi(h)$, then either $gS_{\pi(g)} = hS_{\pi(h)}$ or $gS_{\pi(g)} \cap hS_{\pi(h)} = \emptyset$, since these sets are left cosets of the same subgroup. Now, for $k = 1, 2, \cdots, m$, define

$$A_k = \{g \in S_n \colon \pi(g) = k\},$$

so that $S_n = \bigcup_{k=1}^m A_k$ (disjoint). Let

$$\mathscr{B}_m = \{B\colon B \subseteq A_m;\ g, h \in B,\ g \neq h \Rightarrow gS_m \cap hS_m = \emptyset\},$$

and pick B_m in $\mathscr{B}_m$ so that $|B_m|$ is maximal. We observe that

$$A_m \subseteq \bigcup_{g \in B_m} gS_m;$$

for if there were a g_0 in $A_m \setminus \bigcup_{g\in B_m} gS_m$, then $B'_m = \{g_0\} \cup B_m$ would be a subset of A_m, and if h were in B_m, then the relation $hS_m = g_0 S_m$ would require that $g_0 \in hS_m \subseteq \bigcup_{g\in B_m} gS_m$, a contradiction. Therefore it can only be that $hS_m \cap g_0 S_m = \emptyset$ and B'_m is in $\mathscr{B}_m$. But this contradicts the maximality of $|B_m|$. Therefore $A_m \subseteq \bigcup_{g\in B_m} gS_m$.

Next, let

$$\mathscr{B}_{m-1} = \left\{B\colon B \subseteq A_{m-1} \setminus \bigcup_{g \in B_m} gS_m;\ g, h \in B,\ g \neq h \Rightarrow gS_{m-1} \cap hS_{m-1} = \emptyset\right\},$$

and choose a B_{m-1} in $\mathscr{B}_{m-1}$ such that $|B_{m-1}|$ is maximal.

For each g in B_m and h in B_{m-1}, the set $gS_m \cap hS_{m-1}$ must be empty, for otherwise $gS_m \cap hS_m \neq \emptyset$, so that $gS_m = hS_m$ and $h \in \bigcup_{g\in B_m} gS_m$, contrary to the definition of $\mathscr{B}_{m-1}$. Furthermore,

$$A_{m-1} \subseteq \bigcup_{i=m-1}^{m} \bigcup_{g \in B_i} gS_i,$$

as the proof above with minor modifications will show.

Continue the procedure; for $k = m-1, m-2, \cdots, 2, 1$, select a set $B_k \subseteq A_k \setminus \bigcup_{i=k+1}^{m} \bigcup_{g\in B_i} gS_i$ that satisfies the conditions

$$gS_k \cap hS_j = \emptyset \quad \text{whenever } g \in B_k,\ h \in B_j\ (g \neq h \text{ and } k \leq j \leq m)$$

and

$$A_k \subseteq \bigcup_{i=k}^{m} \bigcup_{g \in B_i} gS_i .$$

Put $F = \bigcup_{k=1}^{m} B_k$. Then

(i) $gS_{\pi(g)} \subseteq S_n$ for each g in F,

(ii) $gS_{\pi(g)} \cap hS_{\pi(h)} = \emptyset$ for all $g, h \in F$ $(g \neq h)$,

and

(iii) $S_n = \bigcup_{k=1}^{m} A_k \subseteq \bigcup_{i=1}^{m} \bigcup_{g \in B_i} gS_i = \bigcup_{g \in F} gS_{\pi(g)} \subseteq S_n$.

Consequently, equality holds and $\left|S_n \setminus \bigcup_{g \in F} gS_{\pi(g)}\right| = 0$. Therefore $\{S_n\}$ is left-uniform.

It is easy to see that the same summing sequence nearly divides each summing sequence of finite subgroups of G.

As a final example, suppose that G is a finitely generated amenable semigroup with identity e and that $\{S_n\}$ is a sequence of finite subsets of G such that $S_n \subseteq S_{n+1} \uparrow G$. It is easy to see that $\{S_n\}$ is a summing sequence for G if and only if

$$\lim_{n \to \infty} |gS_n \setminus S_n|/|S_n| = \lim_{n \to \infty} |S_n g \setminus S_n|/|S_n| = 0 ,$$

for each g in each generating set F. From this it follows immediately that if F is a finite generating set for G that contains e, then $\{F^n\}$ is a summing sequence for G if and only if

$$\lim_{n \to \infty} |F^{n+1}|/|F^n| = 1 ,$$

or equivalently, if and only if

$$\lim_{n \to \infty} |F^{n+1} \setminus F^n|/|F^n| = 0 .$$

For example, let G be the free product of two groups of order 2, and let a and b denote the generators of the two groups. This is the only nontrivial free product of groups that is amenable. Let $F = \{e, a, b\}$. Then $|F^{n+1}| = |F^n| + 2$, hence $\{F^n\}$ is a summing sequence for G.

REFERENCES

1. M. M. Day, *Amenable semigroups*. Illinois J. Math. 1 (1957), 509-544.

2. I. Namioka, *Følner's conditions for amenable semi-groups*. Math. Scand. 15 (1964), 18-28.

University of California, Davis, California
and
Southern Oregon College, Ashland, Oregon 97520