HOMOTOPY EQUIVALENCE AND DIFFERENTIABLE PSEUDO-FREE CIRCLE ACTIONS ON HOMOTOPY SPHERES

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1. INTRODUCTION

This paper is concerned with differentiable pseudo-free circle actions on homotopy spheres, and the main result shows that each such action on a homotopy $(2n+1)$-sphere $(n \geq 1)$ may be mapped equivariantly, by a map of degree 1, onto a linear one on the $(2n+1)$-sphere with exactly one exceptional orbit. For the case $n = 1$, this is an easy consequence of a theorem of R. Jacoby [2], and for the case $n = 3$, it is contained in an earlier paper of Montgomery and Yang, though by a different proof [3]. The result will be used in a forthcoming paper to classify pseudo-free circle actions on spheres.

Except where it is contrarily stated, our study below is assumed to be in the differentiable category.

Let Σ^{2n+1} $(n \geq 1)$ be a homotopy $(2n+1)$-sphere on which there is a differentiable effective action of the circle group G such that all orbits are 1-dimensional. As usual, an orbit Gb in Σ^{2n+1} is called *exceptional* if the isotropy group G_b at b is not trivial. If there is at least one exceptional orbit and each exceptional orbit is isolated, the action is called *pseudo-free*. Suppose that a differentiable pseudo-free action of the circle group G on a homotopy $(2n+1)$-sphere $(n \geq 1)$ is given, and let $Gb_1, \cdots, Gb_k$ be the exceptional orbits in Σ^{2n+1}. Then for each $i = 1, \cdots, k$, the isotropy group G_{b_i} at b_i is a finite cyclic group $\mathbb{Z}q_i$ of order q_i for some integer $q_i > 1$, and the integers $q_1, \cdots, q_k$ are relatively prime to one another. In the following, we let

$$q = q_1 \cdots q_k,$$

which is an integer greater than 1.

Let G consist of complex numbers of absolute value 1, and let S^{2n+1} be the unit sphere in the unitary $(n+1)$-space $\mathbb{C}^{n+1}$. Then there exists a linear pseudo-free action of G on S^{2n+1}, given by the equation

$$g(z_0, z_1, \cdots, z_n) = (g^q z_0, gz_1, \cdots, gz_n).$$

Since $q > 1$, there exists exactly one exceptional orbit in S^{2n+1}, namely $|z_0| = 1$. The main theorem of this paper asserts the existence of an equivariant map of Σ^{2n+1} into S^{2n+1} of degree ± 1. (For the determination of the sign, see Theorem 2.) Notice that such a map induces a homotopy equivalence of the orbit space Σ^{2n+1}/G into the orbit space S^{2n+1}/G.

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Whenever G acts on a space X, we let π be the projection of X onto the orbit space X/G, and for each $A \subset X$, we let

$$A^* = \pi(A);$$

in particular, we let

$$\Sigma^* = \pi(\Sigma^{2n+1}), \qquad S^* = \pi(S^{2n+1}).$$

Let D_i be a slice at b_i that is a closed (2n)-disk of center b_i and on which $\mathbb{Z}q_i$ acts orthogonally. Then we can identify D_i with the closed unit disk in the unitary n-space $\mathbb{C}^n$, so that for some integers $r_{i,1}, \cdots, r_{i,n}$ the action of $\mathbb{Z}q_i$ on D_i is given by

$$g(z_1, \cdots, z_n) = (g^{r_{i,1}} z_1, \cdots, g^{r_{i,n}} z_n).$$

We note that each $r_{i,j}$ may be replaced by any integer congruent to $r_{i,j}$ or $-r_{i,j}$ modulo q_i. Since Gb_i is an isolated exceptional orbit,

$$r_i = r_{i,1} \cdots r_{i,n}$$

is an integer with $(q_i, r_i) = 1$. Therefore

$$\partial D_i^* = \partial D_i/\mathbb{Z}q_i = L^{2n-1}(q_i; r_{i,1}, \cdots, r_{i,n})$$

is a (2n - 1)-dimensional lens space and D_i^* is a cone of vertex b_i^* over ∂D_i^*.

In the remainder of the paper, we shall let D_i be oriented as follows. Let $\mathbb{C}^\ell$ be oriented so that its orientation is represented by the real coordinate system $(z_1 + \bar{z}_1, -\sqrt{-1}(z_1 - \bar{z}_1), \cdots, z_\ell + \bar{z}_\ell, -\sqrt{-1}(z_\ell - \bar{z}_\ell))$, and let

$$D^{2\ell} = \{(z_1, \cdots, z_\ell) \in \mathbb{C}^\ell \mid |z_1|^2 + \cdots + |z_\ell|^2 \leq 1\},$$
$$S^{2\ell-1} = \partial D^{2\ell}$$

be oriented accordingly. Then $G = \partial D^2$ is oriented. Therefore we may let D_i be so oriented that if $G \times D_i$ has the product orientation, then the local imbedding $f: G \times D_i \to \Sigma^{2n+1}$ given by $f(g, x) = gx$ is orientation-preserving. Now we require the identification of D_i with D^{2n} to be such that the orientation on D_i coincides with that on D^{2n}. Then the integer r_i, up to a congruence modulo q_i, is uniquely determined.

An important fact used in the construction of a desired equivariant map of Σ^{2n+1} into S^{2n+1} is the relation between $q_1, \cdots, q_k$ and $r_1, \cdots, r_k$ given below.

THEOREM 1. *Either*

(I) $$r_i \equiv q/q_i \bmod q_i \quad \textit{for all } i = 1, \cdots, k,$$

or

(II) $$r_i \equiv -q/q_i \bmod q_i \quad \textit{for all } i = 1, \cdots, k.$$

With Theorem 1, we obtain the following precise formulation of our main result.

THEOREM 2. *Suppose that we have an action of* G *on* Σ^{2n+1} *and an action of* G *on* S^{2n+1} *as described above. Then there exists an equivariant map of* Σ^{2n+1} *into* S^{2n+1} *that is of degree* 1 *or* -1 *according as* (I) *or* (II) *holds.*

2. PROOF OF THEOREM 1

Assume first that n is odd, say

$$n = 2m + 1,$$

where m is a positive integer. Then $\Sigma^{2n+1} = \Sigma^{4m+3}$ is a homotopy $(4m + 3)$-sphere. For the case $m = 1$, our assertion is an easy consequence of Jacoby's theorem [2]. Hence we shall assume $m > 1$.

Let G act on S^{2m+1} so that

$$g(z_0, z_1, \cdots, z_m) = (g^{q_i} z_0, gz_1, \cdots, gz_m).$$

Then there is exactly one exceptional orbit in S^{2m+1}, namely Gb with $b = (1, 0, \cdots, 0)$. Let D be a slice that is a closed (2m)-disk of center b and on which $\mathbb{Z}q_i$ acts orthogonally. As we have seen above, we may identify D with D^{2m} so that

$$\partial D^* = D/\mathbb{Z}q_i = L^{2m-1}(q_i; 1, \cdots, 1)$$

is a $(2m - 1)$-dimensional lens space. Then D^* is a cone of vertex b^* over ∂D^* and

$$A^* = (S^{2m+1})^* - \text{int } D^*$$

is a compact (2m)-manifold of boundary ∂D^*.

Let $\mathbb{Z}q_i$ act on S^{4m+1} so that

$$g(z_1, \cdots, z_{2m}, z_{2m+1}) = (gz_1, \cdots, gz_{2m}, g^{r_i} z_{2m+1}).$$

Then

$$S^{4m+1}/\mathbb{Z}q_i = L^{4m+1}(q_i; 1, \cdots, 1, r_i)$$

is a $(4m + 1)$-dimensional lens space, and there exists a homotopy equivalence

$$\phi: S^{4m+1}/\mathbb{Z}q_i \to \partial D_i^*$$

induced by a $\mathbb{Z}q_i$-equivariant map of S^{4m+1} into ∂D_i (or equivalently, mapping the preferred generator of $\pi_1(S^{4m+1}/\mathbb{Z}q_i)$ into that of $\pi_1(\partial D_i)$). Since ∂D may be naturally identified with the subset of S^{4m+1} defined by $z_{m+1} = \cdots = z_{2m+1} = 0$, and since

$$\dim \partial D_i^* = 4m + 1 = 2 \dim \partial D^* + 3,$$

$\phi \mid \partial D^*$ is homotopic to an imbedding

$$\psi: \partial D^* \to \partial D_i^*$$

that is covered by a $\mathbb{Z}q_i$-equivariant imbedding of ∂D into ∂D_i. Therefore

$$\psi: \pi_1(\partial D^*) \to \pi_1(\partial D_i^*)$$

is an isomorphism preserving the preferred generator, and

$$\psi\colon H_j(\partial D^*) \to H_j(\partial D_i^*)$$

is an isomorphism for $j < 2m - 1$ and is a surjective homomorphism for $j = 2m - 1$.

Next we assert that ψ can be extended to an imbedding

$$\bar{\psi}\colon (S^{2m+1})^* \to \Sigma^*$$

such that

(i) $\bar{\psi}$ maps D^* into D_i^* and $\bar{\psi}\colon D^* \to D_i^*$ is the natural extension of ψ,

(ii) $\bar{\psi}(A^*) \subset \Sigma^* - (\text{int } D_i^* \cup \{b_1^*, \cdots, b_k^*\})$,

(iii) $\bar{\psi}$ is covered by an equivariant imbedding

$$\tilde{\psi}\colon S^{2m+1} \to \Sigma^{4m+3} .$$

We note that since the imbedding $\bar{\psi}$ to be constructed later is differentiable everywhere except at b^*, the equivariant imbedding $\tilde{\psi}$ is expected to be differentiable everywhere except at points of Gb.

Let $\psi'\colon \partial D \to \partial D_i$ be a $\mathbb{Z}q_i$-equivariant imbedding covering ψ. Then ψ' can be uniquely extended to a G-equivariant imbedding

$$\tilde{\psi}_1\colon GD \to GD_i$$

that maps each radius of D proportionally onto a radius of D_i. Clearly, $\tilde{\psi}_1$ induces an imbedding

$$\bar{\psi}_1\colon D^* \to D_i^*$$

that is an extension of ψ. As we noted above, $\tilde{\psi}_1$ is differentiable at each point of GD - Gb, but in general it is not differentiable at each point of Gb.

If we can extend ψ to an imbedding

$$\bar{\psi}_2\colon A^* \to \Sigma^* - (\text{int } D_i^* \cup \{b_1^*, \cdots, b_k^*\})$$

so that $\bar{\psi}_2(A^*)$ intersects ∂D_i^* transversally at $\psi(\partial D^*)$ and

$$\bar{\psi}_2\colon H_j(A^*) \to H_j(\Sigma^* - (\text{int } D_i^* \cup \{b_1^*, \cdots, b_k^*\}))$$

is an isomorphism for $j \leq 2m - 2$, then we can use $\bar{\psi}_1$ and $\bar{\psi}_2$ to obtain a desired $\bar{\psi}$. In fact, there is an imbedding $\bar{\psi}\colon (S^{2m+1})^* \to \Sigma^*$ such that

$$\bar{\psi} \mid D^* = \bar{\psi}_1, \qquad \bar{\psi} \mid A^* = \bar{\psi}_2 .$$

For this $\bar{\psi}$, there may exist a corner on $\bar{\psi}((S^{2m+1})^*)$ along $\psi(\partial D^*)$; but we can round the corner by modifying $\bar{\psi}_2$. Since

$$\bar{\psi}_2\colon H_j(A^*) \to H_j(\Sigma^* - (\text{int } D_i^* \cup \{b_1^*, \cdots, b_k^*\}))$$

is an isomorphism for $j \leq 2m - 2$, $\bar{\psi}_2$ is covered by an equivariant imbedding

$$\tilde{\psi}_2\colon S^{2m+1} - \text{int } GD \to \Sigma^{4m+3} - (\text{int } GD_i \cup Gb_1 \cup \cdots \cup Gb_k),$$

and $\tilde{\psi}_2$ can be constructed so that by combining $\tilde{\psi}_1$ and $\tilde{\psi}_2$ we obtain an equivariant imbedding $\tilde{\psi}$: $S^{2m+1} \to \Sigma^{4m+3}$ covering $\bar{\psi}$. Hence we have only to construct $\bar{\psi}_2$ in order to complete the construction of $\bar{\psi}$.

For $\ell = 1, \cdots, m$, let σ_ℓ be the intersection of $A^* - \partial D^*$ with the image of

$$\{(z_0, \cdots, z_m) \in S^{2m+1} \mid z_\ell \neq 0, z_{\ell+1} = \cdots = z_m = 0\},$$

and let

$$L_\ell = \bar{\sigma}_\ell \cap \partial D^*.$$

It can be seen that for $\ell = 1, \cdots, m$ the intersection σ_ℓ is an open (2ℓ)-cell, L_ℓ is the $(2\ell - 1)$-dimensional lens space $L^{2\ell-1}(q_i; 1, \cdots, 1)$, and

$$\bar{\sigma}_\ell = \sigma_1 \cup \cdots \cup \sigma_\ell \cup L_\ell.$$

Moreover,

$$L_m = \partial D^*, \qquad \bar{\sigma}_m = A^*.$$

From the construction of ψ, it is easy to see that $\psi \mid L_1$ can be extended to an imbedding

$$\lambda_1: \bar{\sigma}_1 \to \Sigma^* - (\text{int } D_i^* \cup \{b_1^*, \cdots, b_k^*\})$$

such that $\lambda_1(\bar{\sigma}_1)$ intersects ∂D_i^* transversally at L_1 and $(\lambda_1(\bar{\sigma}_1), \lambda_1(L_1))$ represents a generator of

$$H_2(\Sigma^* - (\text{int } D_i^* \cup \{b_1^*, \cdots, b_k^*\}), \partial D_i^*).$$

Since

$$\pi_j(\Sigma^* - (\text{int } D_i^* \cup \{b_1^*, \cdots, b_k^*\})) = 0 \qquad (j = 3, \cdots, 2m-1),$$

we can construct imbeddings

$$\lambda_\ell: \bar{\sigma}_\ell \to \Sigma^* - (\text{int } D_i^* \cup \{b_1^*, \cdots, b_k^*\}) \qquad (\ell = 2, \cdots, m)$$

such that for each ℓ,

$$\lambda_\ell \mid L_\ell = \psi \mid L_\ell, \qquad \lambda_\ell \mid \bar{\sigma}_{\ell-1} = \lambda_{\ell-1},$$

and $\lambda_\ell(\bar{\sigma}_\ell)$ intersects ∂D_i^* transversally at L_ℓ. From the construction of λ_1, we know that

$$\lambda_m: H_2(A^*) \to H_2(\Sigma^* - (\text{int } D_i^* \cup \{b_1^*, \cdots, b_k^*\}))$$

is an isomorphism. Therefore we infer from the ring structure of $H^*(A^*)$ and of $H^*(\Sigma^* - (\text{int } D_i^* \cup \{b_1^*, \cdots, b_k^*\}))$ that

$$\lambda_m: H_j(A^*) \to H_j(\Sigma^* - (\text{int } D_i^* - \{b_1^*, \cdots, b_k^*\}))$$

is an isomorphism for $j \leq 2m - 2$. Hence λ_m is a desired $\bar{\psi}_2$.

Let N be a closed tubular neighborhood of $\bar{\psi}(A^*)$ $(= \bar{\psi}_2(A^*))$ in $\Sigma^* - (\text{int } D_i^* \cup \{b_1^*, \cdots, b_k^*\})$ such that $N \cap \partial D_i^*$ is a closed tubular neighborhood of $\bar{\psi}(\partial D^*)$ in ∂D_i^*. Then

$$X^* = D_i^* \cup N,$$

with the corner on ∂X^* along $\partial(N \cap D_i^*)$ rounded, is a compact $(4m + 2)$-manifold in Σ^* containing a single singularity b_i^* and having $\bar{\psi}((S^{2m+1})^*)$ as a deformation retract. Therefore

$$X = \pi^{-1}(X^*)$$

is an invariant compact $(4m + 3)$-manifold in Σ^{4m+3} having

$$\tilde{\psi}(S^{2m+1}) = \pi^{-1}(\bar{\psi}((S^{2m+1})^*))$$

as a deformation retract, and hence it is diffeomorphic to $S^{2m+1} \times D^{2m+2}$.

Let

$$Y^* = \Sigma^* - \text{int } X^*, \qquad Y = \pi^{-1}(Y^*).$$

Then Y is diffeomorphic to $D^{2m+2} \times S^{2m+1}$ and

$$X \cup Y = \Sigma^{4m+3}.$$

As we said earlier, ∂D_i is regarded as the oriented $(4m + 1)$-sphere S^{4m+1} such that the action of $\mathbb{Z}q_i$ on ∂D_i is given by

$$g(z_1, \cdots, z_{2m+1}) = (g^{r_{i,1}} z_1, \cdots, g^{r_{i,2m+1}} z_{2m+1}).$$

Similarly, ∂D is regarded as the oriented $(2m - 1)$-sphere S^{2m-1} such that the action of $\mathbb{Z}q_i$ on S^{2m-1} is given by

$$g(z_1, \cdots, z_m) = (gz_1, \cdots, gz_m).$$

Therefore, if $\mathbb{Z}q_i$ acts on the oriented $(2m + 1)$-sphere S^{2m+1} so that

$$g(z_1, \cdots, z_m, z_{m+1}) = (gz_1, \cdots, gz_m, g^{r_i} z_{m+1})$$

with $r_i = r_{i,1} \cdots r_{i,2m+1}$, then there exists a $\mathbb{Z}q_i$-equivariant map

$$\xi: S^{2m+1} \to \partial D_i$$

such that $\xi(S^{2m+1})^* \subset \partial D_i^* - N$ and the linking number of the integral singular cycles $\xi(S^{2m+1})$ and $\tilde{\psi}(\partial D)$ in ∂D_i is 1. Let D_i' be a slice at b_i that is a closed $(4m + 2)$-disk with $D_i \subset \text{int } D_i'$ and on which $\mathbb{Z}q_i$ acts orthogonally. Then $D_i'^* - \text{int } D_i^*$ is a cylinder over ∂D_i^* and

$$\dim (D_i'^* - \text{int } D_i^*) = 2 \dim S^{2m+1}/\mathbb{Z}q_i \geq 6.$$

Using Whitney's technique, we can construct an imbedding

$$\eta^*: S^{2m+1}/\mathbb{Z}q_i \to D_i'^* - \text{int } D_i^*$$

such that $\eta^*(S^{2m+1}/\mathbb{Z}q_i) \subset (D_i'^* - D_i^*) - N$, and such that η^* and the induced map

$$\xi^*: S^{2m+1}/\mathbb{Z}q_i \to \partial D_i^*$$

are homotopic as maps of $S^{2m+1}/\mathbb{Z}q_i$ into $(D_i'^* - \text{int } D_i^*) - N$. Then we have a $\mathbb{Z}q_i$-equivariant imbedding

$$\eta: S^{2m+1} \to Y$$

covering η^*. Since the linking number of the integral singular cycles $\xi(S^{2m+1})$ and $\tilde{\psi}(\partial D)$ in ∂D_i is 1 and $\xi(S^{2m+1})$ and $\eta(S^{2m+1})$ are homologous in Y, it follows that the linking number of $\eta(S^{2m+1})$ and $\tilde{\psi}(S^{2m+1})$ in Σ^{4m+3} is 1 so that the sphere

$$S = \eta(S^{2m+1})$$

is an oriented $\mathbb{Z}q_i$-invariant $(2m+1)$-sphere in Y with the property that the inclusion map of S into Y induces an isomorphism of $H_*(S)$ onto $H_*(Y)$. Hence S is a deformation retract of Y.

Let S_1 be an oriented G-invariant $(2m+1)$-sphere in Y that may be identified with S^{2m+1}, and such that the action of G on S_1 is given by

$$g(z_1, \cdots, z_{m+1}) = (gz_1, \cdots, gz_{m+1}).$$

Then the linking number of S_1 with $\tilde{\psi}(S^{2m+1})$ can be determined as follows. Let α_1 be the generator of $H^2(\Sigma^*)$ such that, if E_i is an oriented closed 2-cell in Σ^{4m+3} with $\partial E_i = Gb_i$ and $[E_i^*]$ is the element of $H_2(\Sigma^*)$ containing the 2-cycle E_i^*, then

$$\langle \alpha_1, [E_i^*] \rangle = q/q_i.$$

Then it can be seen that for $\ell = 2, \cdots, 2m+1$ there exists a generator of $H^{2\ell}(\Sigma^*)$ such that

$$\alpha_{\ell-1} \cup \alpha_1 = q\alpha_\ell.$$

Let $[\Sigma^*]$ be the generator of $H_{4m+2}(\Sigma^*)$ such that the image of $q[\Sigma^*]$ in $H_{4m+2}(\Sigma^*, \Sigma^* - \text{int } D_i^*)$ is represented by $(D_i^*, \partial D_i^*)$. Then

$$\langle \alpha_{2m+1}, [\Sigma^*] \rangle = 1 \text{ or } -1.$$

Now we make the following assertion.

LEMMA 1. *The linking number of* S_1 *with* $\tilde{\psi}(S^{2m+1})$ *in* Σ^{4m+3} *is*

$$(q/q_i) \langle \alpha_{2m+1}, [\Sigma^*] \rangle.$$

It can be seen that

$$\langle \alpha_m, [S_1^*] \rangle = q, \qquad \langle \alpha_m, [\tilde{\psi}(S^{2m+1})^*] \rangle = q/q_i.$$

Let E be an oriented, closed $(2m+2)$-cell immersed in Σ^{4m+3} such that $\partial E = S_1$. Then E^* represents an element $[E^*]$ of $H_{2m+2}(\Sigma^*)$, and

$$\langle \alpha_{m+1}, [E^*] \rangle = q.$$

Since $\alpha_{m+1} \cup \alpha_m = q\alpha_{2m+1}$, it follows that

$$[E^*]\cdot[\tilde{\psi}(S^{2m+1})^*] = (q/q_i)\left\langle \alpha_{2m+1}, [\Sigma^*]\right\rangle;$$

this means that the intersection number of E and $\tilde{\psi}(S^{2m+1})$ is

$$(q/q_i)\left\langle \alpha_{2m+1}, [\Sigma^*]\right\rangle.$$

Hence our assertion follows.

LEMMA 2. *There is a* $\mathbb{Z}q_i$*-equivariant map of* S_1 *into* S *of degree* $(q/q_i)\left\langle \alpha_{2m+1}, [\Sigma^*]\right\rangle$.

Since $\mathbb{Z}q_i$ acts freely on Y and since S is a $\mathbb{Z}q_i$-invariant deformation retract of Y, it follows that the inclusion map of $S/\mathbb{Z}q_i$ into $Y/\mathbb{Z}q_i$ is a homotopy equivalence (see [4; p. 97]). Let

$$f: Y/\mathbb{Z}q_i \to S/\mathbb{Z}q_i$$

be the inverse homotopy equivalence. Then f is covered by a $\mathbb{Z}q_i$-equivariant map

$$\tilde{f}: Y \to S$$

that is also a homotopy equivalence. Since the linking number of S with $\tilde{\psi}(S^{2m+1})$ is 1 and that of S_1 with $\tilde{\psi}(S^{2m+1})$ is $(q/q_i)\left\langle \alpha_{2m+1}, [\Sigma^*]\right\rangle$, it follows that

$$\tilde{f}\,|\,S_1: S_1 \to S$$

is a $\mathbb{Z}q_i$-equivariant map of degree $(q/q_i)\left\langle \alpha_{2m+1}, [\Sigma^*]\right\rangle$. This proves Lemma 2.

We know that

$$S/\mathbb{Z}q_i = L^{2m+1}(q_i; 1, \cdots, 1, r_i), \qquad S_1/\mathbb{Z}q_i = L^{2m+1}(q_i; 1, \cdots, 1, 1).$$

Hence Lemma 2 implies that

$$r_i \equiv (q/q_i)\left\langle \alpha_{2m+1}, [\Sigma^*]\right\rangle \bmod q_i$$

(see, for example, [1; p. 95]). This completes the proof of Theorem 1 for the case where n is odd.

Suppose next that n is even, say $n = 2m$, where m is a positive integer. Let

$$\Sigma^{4m+3} = \Sigma^{4m+1} * S^1;$$

that is, let Σ^{4m+3} be the join of $\Sigma^{4m+1} (= \Sigma^{2n+1})$ and S^1. Then Σ^{4m+3} is obtained from $\Sigma^{4m+1} \times D^2$ by identifying (x, y) with (x', y) for any $x, x' \in \Sigma^{4m+1}$ and $y \in \partial D^2$. Moreover, Σ^{4m+3} is a topological $(4m + 3)$-sphere in which $\Sigma^{4m+1} \times \text{int } D^2$ possesses a natural differentiable structure and the set

$$C = \Sigma^{4m+3} - (\Sigma^{4m+1} \times \text{int } D^2)$$

is a circle. Let G act on $\Sigma^{4m+1} \times D^2$ so that for each $g \in G$ and all $(x, y) \in \Sigma^{4m+1} \times D^2$,

$$g(x, y) = (gx, gy).$$

Then the action induces a pseudo-free action of G on Σ^{4m+3} with exceptional orbits $Gb_1, \cdots, Gb_k$, and the action is differentiable on $\Sigma^{4m+1} \times \text{int } D^2$. Therefore Σ^{4m+3}/G is a topological closed (4m + 2)-manifold with singularities $b_1^*, \cdots, b_k^*$, and there exists a natural differentiable structure on

$$\Sigma^{4m+3}/G - \{b_1^*, \cdots, b_k^*, C^*\}.$$

Let us imbed Σ^{4m+1} into Σ^{4m+3} by identifying each $x \in \Sigma^{4m+1}$ with $(x, 0) \in \Sigma^{4m+1} \times \text{int } D^2 \subset \Sigma^{4m+3}$, and study the pseudo-free circle action on Σ^{4m+3} instead. It can be seen that at each b_i, there is a slice D_i' in Σ^{4m+3} that may be identified with D^{4m+2} in such a way that $D_i = D_i' \cap \Sigma^{4m+1}$ is given by

$$z_{2m+1} = 0$$

and the action of $\mathbb{Z}q_i$ on D_i' is given by

$$g(z_1, \cdots, z_{2m}, z_{2m+1}) = (g^{r_{i,1}} z_1, \cdots, g^{r_{i,2m}} z_{2m}, g z_{2m+1}).$$

Without much difficulty one can see that our previous proof applies to this somewhat more general pseudo-free action of G on Σ^{4m+3}. It may be helpful to note that here Σ^{4m+3}/G is locally Euclidean at C^*; but in general there is no natural differentiable structure in any neighborhood of C^*. Even so, our previous proof is not affected by this situation, because the single orbit C in Σ^{4m+3} does not interfere with our argument anywhere in the proof. Hence

$$r_i = r_{i,1} \cdots r_{i,2m} \equiv (q/q_i) \langle \alpha_{2m+1}, [\Sigma^{4m+3}/G] \rangle \bmod q_i,$$

or equivalently,

$$r_i \equiv (q/q_i) \langle \alpha_{2m}, [\Sigma^*] \rangle \bmod q_i,$$

as was to be proved. This completes the proof of Theorem 1.

3. PROOF OF THEOREM 2

Suppose that we have a differentiable, pseudo-free action of the circle group G on a homotopy (2n + 1)-sphere Σ^{2n+1} and a linear pseudo-free action of G on S^{2n+1} as described in the Introduction. We recall that there are k exceptional orbits $Gb_1, \cdots, Gb_k$ in Σ^{2n+1} and that for $i = 1, \cdots, k$, $G_{b_i} = \mathbb{Z}q_i$ and there is a slice D_i at b_i which is to be identified with D^{2n} so that the action of $\mathbb{Z}q_i$ on D_i is given by

$$g(z_1, \cdots, z_n) = (g^{r_{i,1}} z_1, \cdots, g^{r_{i,n}} z_n).$$

Also, there is a single exceptional orbit Gb in S^{2n+1} with $G_b = \mathbb{Z}q$, where $q = q_1 \cdots q_k$, and there is a slice D at b, which is to be identified with D^{2n} so that the action of $\mathbb{Z}q$ on D is given by

$$g(z_1, \cdots, z_n) = (g z_1, \cdots, g z_n).$$

As seen in the theorem, if we orient $D_1, \cdots, D_k$ properly, then there is a relation between

$$r_i = r_{i,1} \cdots r_{i,n} \quad (i = 1, \cdots, k)$$

and $q_1, \cdots, q_k$. It can be assumed that the relation is

$$(1) \qquad r_i \equiv q/q_i \mod q_i \quad (i = 1, \cdots, k),$$

because we can reduce the other case to this case simply by reversing the orientation of Σ^{2n+1}.

In the case $n = 1$, the action of G on Σ^{2n+1} is linear [2], so that it is not hard to construct an equivariant map of Σ^{2n+1} into S^{2n+1} of degree 1. Hence we shall assume that $n > 1$.

Let the slices $D_1, \cdots, D_k$ be constructed so that $D_1^*, \cdots, D_k^*$ are mutually disjoint. We first assert that there exists a map

$$f': \bigcup_{i=1}^{k} D_i^* \to D^*$$

such that $f'\left(\bigcup_{i=1}^{k} \partial D_i^*\right) \subset \partial D^*$, $f': \left(\bigcup_{i=1}^{k} D_i^*, \bigcup_{i=1}^{k} \partial D_i^*\right) \to (D^*, \partial D^*)$ is of degree 1, and f' is covered by an equivariant map

$$\tilde{f}': \bigcup_{i=1}^{k} GD_i \to GD.$$

Since $q_1, \cdots, q_k$ are relatively prime to one another, there exist integers $s_1, \cdots, s_k$ such that

$$(2) \qquad \sum_{i=1}^{k} s_i(q/q_i) = 1.$$

Therefore, for $i = 1, \cdots, k$, the integer $1 - s_i(q/q_i)$ is divisible by q_i, so that for some integer t_i,

$$(3) \qquad s_i(q/q_i) + t_i q_i = 1.$$

Since

$$\partial D_i^* = L^{2n-1}(q_i; r_{i,1}, \cdots, r_{i,n})$$

and

$$\partial D/\mathbb{Z}q_i = L^{2n-1}(q_i; 1, \cdots, 1),$$

it follows from (1) and (3) that there exists a map

$$\phi_i: \partial D_i^* \to \partial D/\mathbb{Z}q_i$$

of degree s_i that is covered by a $\mathbb{Z}q_i$-equivariant map

$$\tilde{\phi}_i: \partial D_i \to \partial D.$$

Let

$$\tilde{\phi}_i': D_i \to D$$

be a $\mathbb{Z}q_i$-equivariant extension of $\tilde{\phi}_i$ that maps each radius of D_i into a radius of D. Then $\tilde{\phi}_i'$ induces an extension of ϕ_i:

$$\phi_i': D_i^* \to D/\mathbb{Z}q_i .$$

Let

$$\psi_i: D/\mathbb{Z}q_i \to D^* = D/\mathbb{Z}q$$

be the projection. Then

$$f_i' = \psi_i \phi_i': D_i^* \to D^*$$

is a map such that $f_i'(\partial D_i^*) \subset \partial D^*$, $f_i': (D_i^*, \partial D_i^*) \to (D^*, \partial D^*)$ is of degree $s_i(q/q_i)$, and f_i' is covered by an equivariant map

$$\tilde{f}_i': GD_i \to GD$$

with $\tilde{f}_i' \mid D_i = \tilde{\phi}_i'$. Let

$$f': \bigcup_{i=1}^{k} D_i^* \to D^*, \qquad \tilde{f}': \bigcup_{i=1}^{k} GD_i \to GD$$

be such that for $i = 1, \cdots, k$,

$$f' \mid D_i^* = f_i', \qquad \tilde{f}' \mid GD_i = \tilde{f}_i' .$$

Then our assertion follows. Notice that it follows from (2) that

$$f': \left(\bigcup_{i=1}^{k} D_i^*, \bigcup_{i=1}^{k} \partial D_i^* \right) \to (D^*, \partial D^*)$$

is of degree 1.

Let

$$X = \Sigma^* - \bigcup_{i=1}^{k} \operatorname{int} D_i^*, \qquad Y = S^* - \operatorname{int} D^* .$$

Let K be a triangulation of X, and for $r = 0, 1, \cdots, 2n - 1$, let X_r be the union of ∂X and all the simplexes of K of dimension at most r. We claim that there exists a map

$$f_{2n-1}: X_{2n-1} \to Y$$

with $f_{2n-1} \mid \partial X = f' \mid \partial X$. Making use of the 1-connectedness of Y, we can first extend $f' \mid \partial X$ to a map $f_1: X_1 \to Y$ and then extend f_1 to a map $f_2: X_2 \to Y$. Since Y is a 1-connected space with $\pi_2(Y) \cong \mathbb{Z}$, and since $H^3(X, \partial X; \mathbb{Z}) = 0$, it follows that the obstruction cohomology class $\gamma^3(f_2)$ (see, for example, [1; p. 180]) is equal to 0. By Eilenberg's extension theorem, there exists a map $f_3: X_3 \to Y$ with

$f_3 | X_1 = f_2 | X_1 = f_1$. For any $r = 4, \cdots, 2n - 1$, if we already have a map $f_{r-1}: X_{r-1} \to Y$, it follows from the relation $\pi_{r-1}(Y) = 0$ that the obstruction cocycle $c^r(f_{r-1}) \in Z^r(X, \partial X; \pi_{r-1}(Y))$ is equal to 0. Therefore f_{r-1} can be extended to a map $f_r: X_r \to Y$. By induction, we have a map f_{2n-1}, as desired.

Next we claim that f_{2n-1} can be extended to a map

$$f'': X \to S^*.$$

Whenever σ is a (2n)-simplex of K, the restriction $f_{2n-1} | \partial\sigma$ can be lifted, in other words, there exists a map $\tilde{f}_\sigma: \partial\sigma \to S^{2n+1}$ with $\pi\tilde{f}_\sigma = f_{2n-1} | \partial\sigma$. Clearly, $\tilde{f}_\sigma$ can be extended to a map $\tilde{f}''_\sigma: \sigma \to S^{2n+1}$. Then $f''_\sigma = \pi\tilde{f}''_\sigma: \sigma \to S^*$ is an extension of $f_{2n-1} | \partial\sigma$. Hence we obtain a desired extension f'' of f_{2n-1}, by letting $f'' | \sigma = f''_\sigma$ for every (2n)-simplex σ.

Now we let

$$f: \Sigma^* \to S^*$$

be the map such that

$$f \Big| \bigcup_{i=1}^{k} D_i^* = f', \qquad f | X = f''.$$

We assert that f is a homotopy equivalence.

Since $f': H_1(\partial X) \to H_1(\partial Y)$ is an isomorphism, we may use the homology sequences of $(X, \partial X)$ and $(Y, \partial Y)$ to show that

$$f_3: H_2(X_3) \to H_2(Y)$$

is an isomorphism. Then

$$f: H_2(\Sigma^*) \to H_2(S^*)$$

is an isomorphism. Because of this and the ring structure of the cohomology rings $H^*(\Sigma^*)$ and $H^*(S^*)$, one can show that

$$f: H^*(S^*) \to H^*(\Sigma^*)$$

is a ring isomorphism. Hence

$$f: \Sigma^* \to S^*$$

is a homotopy equivalence [5].

Let α_1 be the generator of $H^2(\Sigma^*)$ such that for each $i = 1, \cdots, k$, if E_i is an oriented closed 2-cell in Σ^{2n+1} with $\partial E_i = Gb_i$, then $\langle \alpha_1, [E_i^*] \rangle = q/q_i$. Let β_1 be the analogous generator of $H^2(S^*)$. It is not hard to show that $f(\beta_1) = \alpha_1$, so that

$$f(\beta_1^n) = \alpha_1^n.$$

By our assumption, $r_i \equiv q/q_i \bmod q_i$ $(i = 1, \cdots, k)$, or equivalently,

$$\langle \alpha_1^n, [\Sigma^*] \rangle = q^{n-1}.$$

Similarly,

$$\langle \beta_1^n, [S^*] \rangle = q^{n-1}.$$

Hence

$$f([\Sigma^*]) = [S^*].$$

In order to complete the proof of Theorem 2, we need to show that $f: \Sigma^* \to S^*$ is covered by an equivariant map $\tilde{f}: \Sigma^{2n+1} \to S^{2n+1}$ of degree 1. We know that $f': \bigcup_{i=1}^k D_i^* \to D^*$ is covered by an equivariant map $\tilde{f}': \bigcup_{i=1}^k GD_i \to GD$. If we can construct an equivariant map

$$\tilde{f}'': \pi^{-1}(X) \to S^{2n+1}$$

covering f'' and such that $\tilde{f}'' \mid \pi^{-1}(\partial X) = \tilde{f}' \mid \pi^{-1}(\partial X)$, then $\tilde{f}: \Sigma^{2n+1} \to S^{2n+1}$ defined by

$$\tilde{f} \Big| \bigcup_{i=1}^{k} GD_i = \tilde{f}', \qquad \tilde{f} \mid \pi^{-1}(X) = \tilde{f}''$$

is an equivariant map covering f. Moreover, it follows from the relation $f([\Sigma^*]) = [S^*]$ that $\tilde{f}$ is of degree 1. Hence we have only to construct a desired $\tilde{f}''$.

Let

$$f_r = f'' \mid X_r \qquad (r = 0, 1, \cdots, 2n - 1).$$

It is obvious that f_0 is covered by an equivariant map $\tilde{f}_0: \pi^{-1}(X_0) \to \pi^{-1}(Y)$. Let σ be a 1-simplex of K, and let

$$j_\sigma: \sigma \to \pi^{-1}(\sigma)$$

be a cross-section for the circle bundle $\pi: \pi^{-1}(\sigma) \to \sigma$. It is easy to construct a map

$$\tilde{f}_\sigma: j_\sigma(\sigma) \to \pi^{-1}(Y)$$

such that $\tilde{f}_\sigma \mid j_\sigma(\partial\sigma) = \tilde{f}_0 \mid j_\sigma(\partial\sigma)$ and $\pi f_\sigma j_\sigma = f'' \mid \sigma$. Therefore we have an equivariant map

$$\tilde{f}_1: \pi^{-1}(X_1) \to \pi^{-1}(Y)$$

such that $\tilde{f}_1 \mid \pi^{-1}(X_0) = \tilde{f}_0$, and such that for each 1-simplex σ of K, $\tilde{f}_1 \mid j_\sigma(\sigma) = \tilde{f}_\sigma$. Clearly, $\tilde{f}_1$ covers f_1.

The obstruction cocycle c^2 for extending $\tilde{f}_1$ to an equivariant map $\tilde{f}_2: \pi^{-1}(X_2) \to \pi^{-1}(Y)$ covering f_2 may be given as follows. Let σ be a 2-simplex of K, and let

$$j_\sigma: \sigma \to \pi^{-1}(\sigma)$$

be a cross-section. Let

$$E = \{(x, y) \in \sigma \times \pi^{-1}(y) \mid f(x) = \pi(y)\}.$$

Then there exists a free action of G on E given by

$$g(x, y) = (x, gy),$$

and we can identify the orbit space E/G with σ by setting $G(x, y) = x$. Let

$$\phi: \partial\sigma \to E$$

be defined by

$$\phi(x) = (x, \tilde{f}_1 j_\sigma(x)).$$

Then ϕ followed by the projection of E into G is a map of $\partial\sigma$ into G whose degree is $c^2(\sigma)$.

The obstruction cocycle c^2 is actually a coboundary. In fact, if S^2 is a 2-sphere in X representing a generator of $H_2(X)$, then $\pi^{-1}(S^2)$ is a 3-sphere on which G acts freely, and $f'' \mid S^2$ is covered by an equivariant map of $\pi^{-1}(S^2)$ into S^{2n+1}. Therefore the value of c^2 at S^2 is equal to 0, and hence c^2 is a coboundary.

As in Eilenberg's extension theorem, we can modify $\tilde{f}_1$ so that $c^2 = 0$. Therefore $\tilde{f}_1$ can be extended to an equivariant map $\tilde{f}_2: \pi^{-1}(X) \to \pi^{-1}(Y)$ covering f_2.

For any $r = 3, \cdots, 2n - 1$, if we already have an equivariant map $\tilde{f}_{r-1}: \pi^{-1}(X_{r-1}) \to \pi^{-1}(Y)$ covering f_{r-1}, we can extend $\tilde{f}_{r-1}$ to an equivariant map $\tilde{f}_r: \pi^{-1}(X_r) \to \pi^{-1}(Y)$ covering f_r, just as we extended $\tilde{f}_0$ to $\tilde{f}_1$. Hence we have an equivariant map

$$\tilde{f}_{2n-1}: \pi^{-1}(X_{2n-1}) \to \pi^{-1}(Y)$$

covering f_{2n-1}.

From the construction of f'', it is clear that $\tilde{f}_{2n-1}$ can be extended to an equivariant map $\tilde{f}'': \pi^{-1}(X) \to S^{2n+1}$ covering f''. Hence the proof is complete.

Remark. Since $f': \bigcup_{i=1}^{k} D_i^* \to D^*$ is of degree 1 at b^* and since $f([\Sigma^*]) = [S^*]$, it follows that $f'': X \to S^*$ is of degree 0 at b^*. Therefore we may assume that

$$f''(X) \subset Y.$$

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