TOEPLITZ FORMS AND THE GRUNSKY-NEHARI INEQUALITIES

Heinz Leutwiler and Glenn Schober

1. INTRODUCTION

The importance of the Grunsky-Nehari inequalities [4] and their extension given by M. Schiffer and O. Tammi [6] is well established in the theory of bounded univalent functions. It is the purpose of this article to show that these inequalities also contain sufficient information to prove classical results on analytic functions with positive real part. In particular, we shall obtain the conditions that O. Toeplitz [7] gave as an algebraic characterization of C. Carathéodory's [1] geometric description of the coefficient region for functions with positive real part. In addition, we shall show that the finite Toeplitz conditions are equivalent to a strengthened form with fewer free variables.

2. GRUNSKY-NEHARI INEQUALITIES

Before deriving the Toeplitz conditions, we shall write the Grunsky-Nehari-Schiffer-Tammi inequalities in a slightly different form, which for our purposes is more convenient. Let $S(b_1)$ be the class of functions $f(z) = \sum_{n=1}^{\infty} b_n z^n$ that are univalent in the unit disk, with $|f(z)| \leq 1$, and normalized so that $b_1 > 0$. We first state the inequalities as given by Schiffer and Tammi in [6].

Let $x_0, x_1, \cdots, x_N$ *be complex numbers, with the restriction that* x_0 *is real. Then, for* $f \in S(b_1)$,

$$(1) \qquad \Re \left\{ \sum_{m,n=0}^{N} A_{mn} x_m x_n + \sum_{m,n=1}^{N} B_{mn} x_m \bar{x}_n \right\} \leq \sum_{m=1}^{N} \frac{|x_m|^2}{m},$$

where the coefficients A_{mn} *and* B_{mn} *are defined by the power series*

$$(2) \qquad \log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=0}^{\infty} A_{mn} z^m \zeta^n$$

and

$$(3) \qquad -\log[1 - f(z)\overline{f(\zeta)}] = \sum_{m,n=1}^{\infty} B_{mn} z^m \bar{\zeta}^n$$

in the bicylinder $|z| < 1$, $|\zeta| < 1$.

Received May 10, 1972.
This work was supported in part by grant GP33024 from the National Science Foundation.

Michigan Math. J. 20 (1973).

We now show that by changing slightly the generating functions (2) and (3), we can remove the requirement that x_0 is real in (1).

LEMMA 1. *Let* f *be in the class* $S(b_1)$, *and let* $\tilde{A}_{mn}$ *and* $\tilde{B}_{mn}$ *be defined by the power series*

$$\log b_1 \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=0}^{\infty} \tilde{A}_{mn} z^m \zeta^n \tag{4}$$

and

$$\log \frac{1 - f(z)\overline{f(\zeta)}}{1 - z\bar{\zeta}} \frac{z\bar{\zeta}}{f(z)\overline{f(\zeta)}} = \sum_{m,n=0}^{\infty} \tilde{B}_{mn} z^m \bar{\zeta}^n \tag{5}$$

in the bicylinder $|z| < 1$, $|\zeta| < 1$. *Then*

$$\left| \sum_{m,n=0}^{N} \tilde{A}_{mn} x_m x_n \right| \leq \sum_{m,n=0}^{N} \tilde{B}_{mn} x_m \bar{x}_n \qquad \forall\, x_0, x_1, \cdots, x_N \in \mathbb{C} . \tag{6}$$

Proof. Let $x_0, x_1, \cdots, x_N$ be complex numbers. Applying the inequalities (1) to $2\Re x_0, x_1, \cdots, x_N$, we find that

$$\begin{aligned} &\Re \left\{ 4A_{00}(\Re x_0)^2 + 4(\Re x_0) \sum_{m=1}^{N} A_{m0} x_m + \sum_{m,n=1}^{N} A_{mn} x_m x_n \right\} \\ &\qquad \leq \sum_{m=1}^{N} \frac{|x_m|^2}{m} - \sum_{m,n=1}^{N} B_{mn} x_m \bar{x}_n . \end{aligned} \tag{7}$$

Using the relations $(\Re x_0)^2 = (|x_0|^2 + \Re x_0^2)/2$, $\Re x_0 = (x_0 + \bar{x}_0)/2$, and the fact that $A_{00} = \log b_1$ is real, we can write this inequality in the form

$$\begin{aligned} &\Re \left\{ A_{00} x_0^2 + \sum_{m,n=0}^{N} A_{mn} x_m x_n \right\} \\ &\qquad \leq \sum_{m=1}^{N} \frac{|x_m|^2}{m} - \sum_{m,n=1}^{N} B_{mn} x_m \bar{x}_n - 2\Re \sum_{m=0}^{N} A_{m0} x_m \bar{x}_0 . \end{aligned} \tag{8}$$

But according to the definition of the coefficients $\tilde{A}_{mn}$ and $\tilde{B}_{mn}$, inequality (8) is simply the relation

$$\Re \sum_{m,n=0}^{N} \tilde{A}_{mn} x_m x_n \leq \sum_{m,n=0}^{N} \tilde{B}_{mn} x_m \bar{x}_n . \tag{9}$$

Since $x_0, \cdots, x_N$ are arbitrary, we obtain (6) from (9) by replacing each x_n by $x_n e^{i\phi}$, where $\phi = -\frac{1}{2} \arg \sum_{m,n=0}^{N} \tilde{A}_{mn} x_m x_n$.

Remark. The infinite matrices $(\tilde{A}_{mn})$ and $(\tilde{B}_{mn})$ remain symmetric and Hermitean, respectively.

3. TOEPLITZ CONDITIONS

We now obtain the Toeplitz conditions [7] on the basis of Lemma 1.

THEOREM 1. *Let* $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *be analytic in* $|z| < 1$ *and* $\Re p > 0$. *Then*

$$\left| \sum_{m,n=0}^{N} c_{m+n} x_m x_n \right| \leq \sum_{m,n=0}^{N} c_{m-n} x_m \bar{x}_n \quad \forall\, x_0, \cdots, x_N \in \mathbb{C}, \tag{10}$$

where by definition $c_0 = 2$ *and* $c_{-n} = \bar{c}_n$ $(n = 1, 2, \cdots)$.

Proof. The function

$$f(z) = z \exp \int_0^z \frac{1 - p(z)}{z\, p(z)}\, dz \tag{11}$$

satisfies the condition $\Re z f'(z)/f(z) > 0$. Therefore f is a normalized univalent mapping that is starlike with respect to the origin. Consequently, the function

$$F_t(z) = f^{-1}[(1 - t) f(z)] = (1 - t) z + \cdots \qquad (0 \leq t < 1) \tag{12}$$

belongs to the class $S(1 - t)$. To apply Lemma 1 to the function F_t, we introduce the coefficients $\tilde{A}_{mn}(t)$ and $\tilde{B}_{mn}(t)$ defined by the power series

$$\log (1 - t) \frac{F_t(z) - F_t(\zeta)}{z - \zeta} = \sum_{m,n=0}^{\infty} \tilde{A}_{mn}(t) z^m \zeta^n \tag{13}$$

and

$$\log \frac{[1 - F_t(z) \overline{F_t(\zeta)}] z \bar{\zeta}}{(1 - z \bar{\zeta}) F_t(z) \overline{F_t(\zeta)}} = \sum_{m,n=0}^{\infty} \tilde{B}_{mn}(t) z^m \bar{\zeta}^n . \tag{14}$$

Then, for arbitrary complex numbers $x_0, \cdots, x_N$, we have the inequality

$$\left| \sum_{m,n=0}^{N} \tilde{A}_{mn}(t) x_m x_n \right| \leq \sum_{m,n=0}^{N} \tilde{B}_{mn}(t) x_m \bar{x}_n . \tag{15}$$

Note that $\tilde{A}_{mn}(0) = 0$ and $\tilde{B}_{mn}(0) = 0$, since $F_0(z) = z$. Moreover, the one-sided derivatives $A'_{mn}(0)$ and $B'_{mn}(0)$ exist. Dividing (15) by t and letting $t \to 0$, we find that these derivatives satisfy the inequality

$$\left| \sum_{m,n=0}^{N} \tilde{A}'_{mn}(0) x_m x_n \right| \leq \sum_{m,n=0}^{N} \tilde{B}'_{mn}(0) x_m \bar{x}_n . \tag{16}$$

This inequality is precisely the inequality (10). Indeed, since

$$\left.\frac{\partial F_t(z)}{\partial t}\right|_{t=0} = -\frac{f(z)}{f'(z)} = -z\,p(z), \tag{17}$$

we conclude by differentiating the generating functions (13) and (14) that

$$-1 - \frac{z\,p(z) - \zeta\,p(\zeta)}{z - \zeta} = \sum_{m,n=0}^{\infty} \tilde{A}'_{mn}(0)\, z^m \zeta^n \tag{18}$$

and

$$\frac{p(z) + \overline{p(\zeta)}}{1 - z\bar{\zeta}} = \sum_{m,n=0}^{\infty} \tilde{B}'_{mn}(0)\, z^m \bar{\zeta}^n . \tag{19}$$

Comparing coefficients, we find that $\tilde{A}'_{mn}(0) = -c_{m+n}$ and $\tilde{B}'_{mn}(0) = c_{m-n}$.

Remark. We emphasize that the classical Toeplitz conditions

$$0 \le \sum_{m,n=0}^{N} c_{m-n}\, x_m \bar{x}_n \qquad \forall\, x_0, \cdots, x_N \in \mathbb{C}, \tag{20}$$

which follow as a corollary of Theorem 1, in their totality already characterize the coefficients of a function with positive real part, so that from this point of view one does not gain more by adding the symmetric form on the left side of (10). We should mention that there are alternate proofs of (20) by E. Fischer [3], F. Riesz [5], and others. It is not surprising that, in fact, Fischer's proof can very easily be extended to a proof of the inequality (10). Our point of view has been to establish the connection with the Grunsky-Nehari-Schiffer-Tammi inequalities.

4. APPLICATIONS

For applications, the inequalities (10) are more useful if we take the extreme with respect to x_0. It is not difficult to see that the optimal choice is $x_0 = -(c_1 x_1 + \cdots + c_n x_n)/2$. The following is then an immediate consequence of Theorem 1.

THEOREM 2. *Under the assumptions of Theorem* 1,

$$\left| \sum_{m,n=1}^{N} \left(c_{m+n} - \frac{1}{2} c_m c_n \right) x_m x_n \right| \le \sum_{m,n=1}^{N} \left(c_{m-n} - \frac{1}{2} c_m \bar{c}_n \right) x_m \bar{x}_n \qquad \forall\, x_1, \cdots, x_N \in \mathbb{C} . \tag{21}$$

By choosing special values for $x_1, \cdots, x_N$ in Theorem 2, we obtain some classical results for functions with positive real part. The choice $x_k = 1$, $x_n = 0$ $(n \ne k)$ gives the following proposition.

COROLLARY 1. *Let the function* $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ *be analytic in* $|z| < 1$ *and* $\Re p > 0$. *Then*

$$|c_{2k} - \tfrac{1}{2}c_k^2| \leq 2 - \tfrac{1}{2}|c_k|^2. \tag{22}$$

The inequality (22) actually defines the disk of values for c_{2k}, for each preassigned c_k. Indeed, the boundary functions are

$$p(z) = \frac{1 + \frac{1}{2}(c_k + \eta \bar{c}_k) z^k + \eta z^{2k}}{1 - \frac{1}{2}(c_k - \eta \bar{c}_k) z^k - \eta z^{2k}} \qquad \text{for } |\eta| = 1. \tag{23}$$

Note that the inequality $|c_k| \leq 2$ is also a consequence of (22).

The choice $x_n = z^n$ in Theorem 2 gives the following, in the limit as $N \to \infty$.

COROLLARY 2. *Let* p *be analytic in* $|z| < 1$ *and* $\Re p > 0$. *Then*

$$\left|p'(z) - \frac{p^2(z) - 1}{2z}\right| \leq \frac{2|z| \Re p(z)}{1 - |z|^2} - \frac{|p(z) - 1|^2}{2|z|}. \tag{24}$$

Inequality (24) defines the correct disk of values for $p'(z)$, for each preassigned $p(z)$. Moreover, if we introduce the function $p(z) = [1 + zF(z)]/[1 - zF(z)]$, where F is an arbitrary analytic function in $|z| < 1$ bounded by 1, then the inequality (24) transforms into

$$|F'(z)| \leq \frac{1 - |F(z)|^2}{1 - |z|^2}, \tag{25}$$

the invariant form of Schwarz's lemma.

5. REDUCTION OF FREE VARIABLES

For fixed N, the conditions (20) characterize the finite sequences $c_1, \cdots, c_N$ that are the leading coefficients of a function

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \qquad (|z| < 1) \tag{26}$$

with positive real part. Since the strengthened conditions (10) involve not only $c_1, \cdots, c_N$ but also $c_{N+1}, \cdots, c_{2N}$, it is natural to ask whether the inequalities (10) for fixed N are actually equivalent to the Toeplitz conditions (20) with $2N$ in place of N. The following theorem shows that they are, in fact, equivalent.

THEOREM 3. *Let* $c_1, \cdots, c_{2N} \in \mathbb{C}$, $c_{-n} = \bar{c}_n$, *and* $c_0 = 2$. *Then*

$$\left|\sum_{m,n=0}^{N} c_{m+n} x_m x_n\right| \leq \sum_{m,n=0}^{N} c_{m-n} x_m \bar{x}_n \qquad \forall\, x_0, \cdots, x_N \in \mathbb{C} \tag{27}$$

if and only if

(28) $$0 \leq \sum_{m,n=0}^{2N} c_{m-n}\xi_m\bar{\xi}_n \qquad \forall\, \xi_0, \cdots, \xi_{2N} \in \mathbf{C}.$$

Remark. Note that the conditions (27) reflect a substantial reduction in the number of free variables over the equivalent conditions (28).

Proof. Denote by $\mathfrak{C}_{2N}$ and $\mathfrak{K}_{2N}$ the sets of points $(c_1, \cdots, c_{2N}) \in \mathbf{C}^{2N}$ satisfying (27) and (28), respectively. It is then sufficient to show that $\mathfrak{C}_{2N} = \mathfrak{K}_{2N}$. In a letter to Carathéodory, Toeplitz [7] proved that condition (28) is sufficient for $c_1, \cdots, c_{2N}$ to be the leading coefficients of a function (26) with positive real part. But then Theorem 1 implies that (27) is satisfied. Hence $\mathfrak{C}_{2N} \supset \mathfrak{K}_{2N}$.

Carathéodory [1] observed that $\mathfrak{K}_{2N}$ is a closed convex body for which the origin is an interior point, and he proved that each boundary point is of the form $\sum_{j=1}^{2N} \lambda_j(2e^{-i\theta_j}, 2e^{-2i\theta_j}, \cdots, 2e^{-2Ni\theta_j})$ corresponding to the function

(29) $$\sum_{j=1}^{2N} \lambda_j \frac{e^{i\theta_j}+z}{e^{i\theta_j}-z},$$

where $\lambda_j \geq 0$, $\sum_{j=1}^{2N} \lambda_j = 1$, and $0 \leq \theta_1 < \theta_2 < \cdots < \theta_{2N} < 2\pi$. If $\mathfrak{C}_{2N} - \mathfrak{K}_{2N}$ contains a point, it must therefore be of the form

(30) $$\alpha \sum_{j=1}^{2N} \lambda_j(2e^{-i\theta_j}, 2e^{-2i\theta_j}, \cdots, 2e^{-2Ni\theta_j})$$

for some $\alpha > 1$. Since this point belongs to $\mathfrak{C}_{2N}$, (27) implies that

(31) $$\left|2\alpha \sum_{j=1}^{2N} \lambda_j\left(\sum_{n=0}^{N} x_n e^{-in\theta_j}\right)^2 + 2(1-\alpha)x_0^2\right| \leq 2\alpha \sum_{j=1}^{2N} \lambda_j \left|\sum_{n=0}^{N} x_n e^{-in\theta_j}\right|^2 + 2(1-\alpha)\sum_{n=0}^{N} |x_n|^2$$

for all $x_0, \cdots, x_N \in \mathbf{C}$. The finite Fourier series

(32) $$\mathfrak{I} \sum_{n=0}^{N} x_n e^{-in\theta}$$

on $[0, 2\pi)$ may be used to interpolate $2N+1$ values (see [2], for example). We choose $x_0, \cdots, x_N$ so that (32) vanishes at the $2N$ points $\theta_1, \cdots, \theta_{2N}$, but is not identically zero. Then

(33) $$\sum_{n=1}^{N} |x_n|^2 > 0.$$

Application of the triangle inequality to (31) yields the relation

$$-|2(1-\alpha)x_0^2| \leq 2(1-\alpha)\sum_{n=0}^{N}|x_n|^2 . \tag{34}$$

Since $\alpha > 1$, (33) and (34) are incompatible. Consequently, $\mathfrak{C}_{2N} = \mathfrak{K}_{2N}$.

REFERENCES

1. C. Carathéodory, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen.* Math. Ann. 64 (1907), 95-115.
2. P. J. Davis, *Interpolation and approximation.* Blaisdell Publ. Co., New York, 1963.
3. E. Fischer, *Über das Carathéodory'sche Problem, Potenzreihen mit positivem reellen Teil betreffend.* Rend. Circ. Mat. Palermo 32 (1911), 240-256.
4. Z. Nehari, *Some inequalities in the theory of functions.* Trans. Amer. Math. Soc. 75 (1953), 256-286.
5. F. Riesz, *Über ein Problem des Herrn Carathéodory.* J. Reine Angew. Math. 146 (1915), 83-87.
6. M. Schiffer and O. Tammi, *On the coefficient problem for bounded univalent functions.* Trans. Amer. Math. Soc. 140 (1969), 461-474.
7. O. Toeplitz, *Über die Fourier'sche Entwicklung positiver Funktionen.* Rend. Circ. Mat. Palermo 32 (1911), 191-192.

Indiana University
Bloomington, Indiana 47401