ON A CHARACTERIZATION OF AW*-MODULES AND A REPRESENTATION OF GELFAND TYPE OF NONCOMMUTATIVE OPERATOR ALGEBRAS

Hideo Takemoto

1. INTRODUCTION

The direct integral decomposition of a von Neumann algebra $\mathfrak{A}$ has been obtained only for the case where $\mathfrak{A}$ is generated by its center $\mathcal{Z}$ and a countable family of other elements (see [2, Chapter 2], for example). As far as we know, the literature does not contain a successful treatment of the problem for nonseparable algebras. In the present paper, we investigate from the viewpoint of continuous-reduction theory an extension of the Gelfand representation of commutative C*-algebras to noncommutative von Neumann algebras. That is, we show that noncommutative von Neumann algebras can be represented as continuous fields of von Neumann algebras; but we can not show that each component of continuous fields of von Neumann algebras is a factor. Before considering the continuous fields of von Neumann algebras, we shall introduce a continuous field of Hilbert spaces, to which AW*-modules are equivalent; thus we can reconstruct AW*-modules from the Hilbert-space representation, and we believe that this technique is new. By using the continuous field of Hilbert spaces, we shall show that every von Neumann algebra has a representation of Gelfand type.

I. Kaplansky ([7], [8], and [9]) has discussed the theory of AW*-modules and AW*-algebras. In particular, he has shown [9] that if $\mathcal{A}$ is a commutative AW*-algebra and M is a faithful AW*-module over $\mathcal{A}$, then the set of all bounded $\mathcal{A}$-module homomorphisms on M is an AW*-algebra of type I. Conversely, an AW*-algebra of type I with the center $\mathcal{Z}$ is *-isomorphic to the AW*-algebra of all bounded $\mathcal{Z}$-module homomorphisms on a faithful AW*-module over $\mathcal{Z}$.

Now AW*-modules have many properties of Hilbert space, and they can be considered as a Banach space of all vector-valued continuous functions on a Stonean space; hence we shall have a continuous field of Hilbert spaces. Furthermore, by using this continuous field of Hilbert spaces, we can show that each von Neumann algebra can be represented as a continuous field of von Neumann algebras.

In Section 3, we define a continuous field of Hilbert spaces over a compact Hausdorff space Ω, which is different from the one by J. Dixmier [3, Section 10]. Then we obtain a result, similar to Riesz's theorem in Hilbert space, which Kaplansky has established in the case of an AW*-module, but not in the case of a C*-module. Further, any bounded module homomorphism on our continuous field of Hilbert spaces has a bounded adjoint operator, and the set of all bounded module homomorphisms on our continuous field of Hilbert spaces is a C*-algebra. In particular, let Ω be a Stonean space; then our continuous field of Hilbert spaces over Ω is an AW*-module (see Section 4). Conversely, each AW*-module is representable

Received June 26, 1972.

The author expresses his indebtedness to the referee for rewriting the statement of Theorem 4.5.

Michigan Math. J. 20 (1973).

as a continuous field of Hilbert spaces over a Stonean space. Therefore we can decompose a bounded module homomorphism on an AW*-module to a field of bounded operators on Hilbert space.

Let $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ be a continuous field of Hilbert spaces over a Stonean space Ω, let A be a bounded $C(\Omega)$-module homomorphism, and let $A = \mathscr{C}_{\Omega}^{\oplus} A(\omega)$ be the representation obtained by the decomposition of A; then the function $\omega \to \|A(\omega)\|$ is continuous on Ω. Thus $\mathfrak{A} = \mathscr{C}_{\Omega}^{\oplus} \widetilde{\mathfrak{A}(\omega)}$, where $\mathfrak{A}$ is a von Neumann algebra and $\widetilde{\mathfrak{A}(\omega)}$ is a von Neumann algebra for every $\omega \in \Omega$.

2. NOTATION AND PRELIMINARIES

Let Ω be a compact Hausdorff space, and let $\mathscr{A} = C(\Omega)$ be the algebra of all complex-valued continuous functions on Ω. For a field $\{H(\omega): \omega \in \Omega\}$ of Hilbert spaces, the elements $\xi = \{\xi(\omega)\}_{\omega\in\Omega}$ of $\prod_{\omega\in\Omega} H(\omega)$ are called *vector fields.* If $f \in C(\Omega)$ and ξ is a vector field, $f\xi = \{f(\omega)\,\xi(\omega)\}_{\omega\in\Omega}$. If ξ and η are vector fields, then (ξ, η) is the function $\omega \to (\xi(\omega) \mid \eta(\omega))$, and $|\xi|$ is the function $\omega \to \|\xi(\omega)\|$. Let B(K) be the algebra of all bounded operators on a Hilbert space K; then each element of $\prod_{\omega\in\Omega} B(H(\omega))$ is called an *operator field.* If $A \in \prod_{\omega\in\Omega} B(H(\omega))$ and $\xi \in \prod_{\omega\in\Omega} H(\omega)$, then $A\xi = \{A(\omega)\,\xi(\omega)\}_{\omega\in\Omega}$.

Let $\mathscr{A}$ be a commutative C^*-algebra with unit, and let H be an $\mathscr{A}$-module in the ordinary algebraic sense. Suppose there is defined on H an inner product taking values in $\mathscr{A}$ and satisfying the conditions

(1) $(\xi, \eta) = (\eta, \xi)^*$,

(2) $(\xi, \xi) \geq 0$ and $(\xi, \xi) = 0$ only for $\xi = 0$,

(3) $(a\xi + \xi_1, \eta) = a(\xi, \eta) + (\xi_1, \eta)$

for all ξ, ξ_1, η in H and all a in $\mathscr{A}$. We use the notation $|\xi| = (\xi, \xi)^{1/2}$, $\|\xi\| = \|(\xi, \xi)\|^{1/2}$, where on the right we mean the usual positive square root and norm in $\mathscr{A}$. We see that $\|\xi\|$ is the norm of $|\xi|$ in the algebra $\mathscr{A}$. Therefore H is a normed space with respect to the norm $\|\cdot\|$ defined above. In particular, if H is already complete, we shall call it a C^*-module over $\mathscr{A}$.

An AW*-algebra $\mathfrak{A}$ is a C^*-algebra satisfying the following two conditions.

(1) In the set of projections, each collection of orthogonal projections has a least upper bound.

(2) Each maximal commutative self-adjoint subalgebra is generated by its projections.

Let Ω be a compact Hausdorff space. Ω is called a *Stonean space* if the closure of every open set is open [1]. If Ω is a Stonean space, then the algebra $C(\Omega)$ of all complex-valued continuous functions on Ω is a commutative AW*-algebra. Conversely, let $\mathscr{A}$ be a commutative AW*-algebra; then there exists a Stonean space Ω such that $\mathscr{A}$ is *-isomorphic to $C(\Omega)$.

Let $\mathscr{A}$ be a commutative AW*-algebra. We say that H is an AW*-module over $\mathscr{A}$ if it is a C^*-module over $\mathscr{A}$ and has the following two additional properties.

(1) Let $\{e_i\}$ be orthogonal projections in $\mathscr{A}$ with sup $e_i = e$, and suppose ξ is an element of H with $e_i \xi = 0$ for all i; then $e\xi = 0$.

(2) Let $\{e_i\}$ be orthogonal projections in $\mathscr{A}$ with sup $e_i = 1$, and let $\{\xi_i\}$ be a bounded subset of H; then there exists an element ξ in H with $e_i \xi = e_i \xi_i$ for all i.

By a *bounded operator* from a C^*-module H_1 over a commutative C^*-algebra $\mathscr{A}$ into a second C^*-module H_2 over $\mathscr{A}$ we mean a linear and continuous mapping of H_1 into H_2 that is also an $\mathscr{A}$-module homomorphism. The set B(H) of all bounded operators on a C^*-module H over $\mathscr{A}$ forms a Banach algebra in the usual operator norm. We shall write the element of $\mathscr{A}$ (typically f, g, ⋯, a, b, ⋯) on the left of the element of H (typically ξ, η, ⋯).

3. CONTINUOUS FIELDS OF HILBERT SPACES OVER A COMPACT HAUSDORFF SPACE AND C*-MODULES

In this section, we define a continuous field of Hilbert spaces over a compact Hausdorff space Ω. It is a C^*-module, and the set of all bounded module homomorphisms on it is a C^*-algebra.

Definition 3.1. Let Ω be a compact Hausdorff space, and let $\{H(\omega): \omega \in \Omega\}$ be a field of Hilbert spaces. A subspace H of $\prod_{\omega \in \Omega} H(\omega)$ is said to be a *continuous field of Hilbert spaces* if there exists a subspace H_0 of $\prod_{\omega\in\Omega} H(\omega)$ such that

(1) for every $\xi \in H_0$, the function $\omega \to \|\xi(\omega)\|$ is continuous on Ω,

(2) for each $\omega \in \Omega$, the subspace $\{\xi(\omega): \xi \in H_0\}$ is dense in $H(\omega)$,

(3) $H = \{\xi \in \prod_{\omega\in\Omega} H(\omega)$: for each positive number ε and each $\omega_0 \in \Omega$, there exist an element ξ_0 in H_0 and a neighborhood $U(\omega_0)$ of ω_0 such that $\|\xi(\omega) - \xi_0(\omega)\| < \varepsilon$ for every $\omega \in U(\omega_0)\}$,

(4) if ξ is a vector field such that the function $\omega \to \|\xi(\omega)\|$ is bounded, and if for each $\eta \in H_0$, the function $\omega \to (\xi(\omega) \mid \eta(\omega))$ is a continuous function on Ω, then $\xi \in H$.

Property (1) in this definition is equivalent to the following:

(1') For all $\xi, \eta \in H_0$, the function $\omega \to (\xi(\omega) \mid \eta(\omega))$ is continuous on Ω.

The following example shows that our definition is more restrictive than the usual definition of a strong continuous field of Hilbert spaces (see [3, Section 10]) with properties (1), (2), and (3).

Example. Let $\Omega = [-1, 1]$ be the closed interval in R^1, let $f_0(\omega) = |\omega|$ for every $\omega \in [-1, 1]$, and let

$$H(\omega) = \begin{cases} \mathbf{C} & \text{if } \omega \neq 0, \\ 0 & \text{if } \omega = 0. \end{cases}$$

Then the subspace $\{ff_0: f \in C(\Omega)\}$ of $\prod_{\omega\in\Omega} H(\omega)$ satisfies the conditions (1) and (2) in Definition 3.1. Let

$$f_1(\omega) = \begin{cases} 1 & \text{if } \omega \neq 0, \\ 0 & \text{if } \omega = 0\,; \end{cases}$$

then for each $f \in C(\Omega)$, the function

$$\omega \to (f_1(\omega) \mid f(\omega) f_0(\omega)) = f(\omega) f_1(\omega) f_0(\omega) = f(\omega)\,|\omega|$$

is continuous on Ω. Therefore f_1 satisfies condition (4) in Definition 3.1. But since the function $\omega \to |f_1(\omega)|$ is not continuous, f_1 does not satisfy condition (3).

Under Definition 3.1, for any ξ and η in H, the functions $\omega \to (\xi(\omega) \mid \eta(\omega))$ and $\omega \to \|\xi(\omega)\|$ are continuous on Ω. Further, we have the following result.

PROPOSITION 3.2. *If* H *is a continuous field of Hilbert spaces, and if* $\xi \in H$ *and* $\|\xi\| = \sup\{\|\xi(\omega)\|: \omega \in \Omega\}$, *then the normed space* $(H, \|\cdot\|)$ *is complete. That is,* H *is a Banach space.*

Definition 3.3. Let $(\{H(\omega): \omega \in \Omega\}, H_0, H)$ be a field of Hilbert spaces satisfying the conditions in Definition 3.1; then H is denoted by $\mathscr{C}_{\Omega}^{\oplus H_0} H(\omega)$ or $\mathscr{C}_{\Omega}^{\oplus} H(\omega)$.

Let $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ be a continuous field of Hilbert spaces over Ω; for all $\xi, \eta \in H$, we identify (ξ, η) with the continuous function $\omega \to (\xi(\omega) \mid \eta(\omega))$.

PROPOSITION 3.4. *If* Ω *is a compact Hausdorff space, then a continuous field* $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ *of Hilbert spaces over* Ω *is a* C^**-module over* $C(\Omega)$.

By Proposition 3.4, a bounded operator A on a continuous field $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ of Hilbert spaces is a mapping on H that is not only linear and continuous in the usual operator norm, but is also a $C(\Omega)$-module homomorphism on H.

We call A^* the *adjoint operator* of A if $(A\xi, \eta) = (\xi, A^*\eta)$ for all $\xi, \eta \in H$.

LEMMA 3.5 (Kaplansky [9]). *Let* H *be a* C^**-module over a commutative* C^**-algebra* $\mathscr{A}$, *and let* A *be a bounded operator on* H *with an adjoint operator* A^* *that is also a bounded operator. Then* $\|A\| = \|A^*\|$ *and* $\|A^* A\| = \|A\|^2$.

Let $\mathscr{A}$ be a commutative C^*-algebra, and let H be a C^*-module over $\mathscr{A}$; then an $\mathscr{A}$-module mapping ϕ of H into $\mathscr{A}$ will be called a *functional* of H into $\mathscr{A}$. We shall devote the present section to the proof that a continuous field $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ of Hilbert spaces over a compact Hausdorff space Ω is self-dual in the same way that Hilbert space is self-dual. The following result is analogous to a known result on AW^*-modules.

THEOREM 3.6. *Let* Ω *be a compact Hausdorff space, let* $\mathscr{A} = C(\Omega)$, *and let* $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ *be a continuous field of Hilbert spaces over* Ω. *If* ϕ *is a continuous functional of* H *into* $\mathscr{A}$, *then there exists a* $\xi_0 \in H$ *such that* $\phi(\xi) = (\xi, \xi_0)$ *for every* $\xi \in H$, *and* $\|\phi\| = \|\xi_0\|$.

Proof. Corresponding to each $\omega \in \Omega$, we define $\phi_\omega(\xi(\omega)) = \phi(\xi)(\omega)$ for every $\xi \in H$; the functional ϕ_ω is then well-defined. Indeed, suppose that $\xi \in H$, that $\omega_0 \in \Omega$, and that $\varepsilon > 0$. Since the function $\omega \to \|\xi(\omega)\|$ is continuous on Ω, the set

$$G = \{\omega \in \Omega: \|\xi(\omega)\| < \|\xi(\omega_0)\| + \varepsilon\}$$

is open and contains ω_0. Since Ω is compact, there exists an element f of $C(\Omega)$ such that $f(\omega_0) = 1$, $0 \leq f \leq 1$, and $f(\omega) = 0$ for each $\omega \in \Omega \setminus G$. We have the relations

$$\|f\xi\| = \sup\{f(\omega)\,\|\xi(\omega)\| : \omega \in \Omega\} = \sup\{f(\omega)\,\|\xi(\omega)\| : \omega \in G\} \leq \|\xi(\omega_0)\| + \varepsilon,$$

and therefore

$$\|\phi(f\xi)\| \leq \|\phi\| \cdot \|f\xi\| \leq \|\phi\|\,(\|\xi(\omega_0)\| + \varepsilon).$$

Furthermore,

$$\|\phi(f\xi)\| = \sup\{|\phi(f\xi)(\omega)| : \omega \in \Omega\} \geq |\phi(f\xi)(\omega_0)| = f(\omega_0)\,|\phi(\xi)(\omega_0)| = |\phi(\xi)(\omega_0)|.$$

Therefore $|\phi(\xi)(\omega_0)| \leq \|\phi\| \cdot (\|\xi(\omega_0)\| + \varepsilon)$. Since ε is an arbitrary positive number, $|\phi(\xi)(\omega_0)| \leq \|\phi\| \cdot \|\xi(\omega_0)\|$. Thus, if $\xi(\omega_0) = 0$, then $\phi(\xi)(\omega_0) = 0$, and consequently ϕ_{ω_0} is well-defined, for each $\omega_0 \in \Omega$. Since $\{\xi(\omega) : \xi \in H\}$ is dense in $H(\omega)$ for each $\omega \in \Omega$, ϕ_ω is a bounded linear functional on $H(\omega)$. Therefore there exists an element $\xi_0 \in \prod_{\omega \in \Omega} H(\omega)$ such that $\phi_\omega(\xi(\omega)) = (\xi(\omega) \mid \xi_0(\omega))$ for each $\omega \in \Omega$ and each $\xi \in H$. Since

$$(\xi(\omega) \mid \xi_0(\omega)) = \phi_\omega(\xi(\omega)) = \phi(\xi)(\omega)$$

for each $\omega \in \Omega$ and each $\xi \in H$, the function $\omega \to (\xi(\omega) \mid \xi_0(\omega))$ is continuous on Ω. Furthermore, since $\{\xi(\omega) : \xi \in H\}$ is dense in $H(\omega)$ and the function $\omega \to \|\xi(\omega)\|$ is continuous on Ω, we see that $|(\zeta \mid \xi_0(\omega))| \leq \|\phi\| \cdot \|\zeta\|$ for each $\zeta \in H(\omega)$, and therefore

$$\|\xi_0(\omega)\|^2 \leq \|\phi\| \cdot \|\xi_0(\omega)\| \quad \text{and} \quad \|\xi_0(\omega)\| \leq \|\phi\|$$

for every $\omega \in \Omega$. Hence the function $\omega \to \|\xi_0(\omega)\|$ is bounded on Ω. Thus $\xi_0 \in H$ and $\phi(\xi) = (\xi, \xi_0)$ for each $\xi \in H$. Furthermore, since $\|\xi_0(\omega)\| \leq \|\phi\|$ for every $\omega \in \Omega$, $\|\xi_0\| \leq \|\phi\|$. Conversely, since $|\phi(\xi)(\omega)| \leq \|\xi(\omega)\| \cdot \|\xi_0(\omega)\|$ for every $\omega \in \Omega$, $\|\phi\| \leq \|\xi_0\|$. Therefore $\|\phi\| = \|\xi_0\|$.

Using Theorem 3.6 and the theorem of Kaplansky [9], we obtain the following result.

COROLLARY 3.7. *Let* Ω *be a compact Hausdorff space, and let* $H = \mathscr{C}_\Omega^{\oplus} H(\omega)$ *be a continuous field of Hilbert spaces over* Ω. *If* A *is an operator on* H, *then* A *is a bounded operator if and only if* A *has a bounded adjoint operator* A^*. *Thus,* B(H) *is a* C^**-algebra.*

4. DECOMPOSABLE OPERATORS, THE CONTINUOUS FIELD OF HILBERT SPACES OVER A STONEAN SPACE, AND AW*-MODULES

In the previous section, we defined the continuous field of Hilbert spaces over a compact Hausdorff space. In this section, we define a decomposable operator on a continuous field of Hilbert spaces over a compact Hausdorff space; thus we shall discuss the relation between continuous fields of Hilbert spaces over a Stonean space Ω and AW*-modules over $C(\Omega)$.

Let Ω be a compact Hausdorff space, and let $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ be a continuous field of Hilbert spaces over Ω; then B(H) is a C*-algebra (see Section 3). This permits the formulation of the following definition.

Definition 4.1. Let Ω be a compact Hausdorff space, and let $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ be a continuous field of Hilbert spaces over Ω. Then an element $A \in B(H)$ is *decomposable* if for each $\omega \in \Omega$ there exists an element $A(\omega)$ of $B(H(\omega))$ such that for all $\xi, \eta \in H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ and each $\omega \in \Omega$,

$$((A\xi)(\omega) \mid \eta(\omega)) = (A(\omega)\xi(\omega) \mid \eta(\omega)).$$

We write $A = \mathscr{C}_{\Omega}^{\oplus} A(\omega)$.

Definition 4.2. In the notation of Definition 4.1, let A be an element of $\prod_{\omega \in \Omega} B(H(\omega))$ with $A = \{A(\omega)\}_{\omega \in \Omega}$. Then A is called a *bounded continuous operator field* if (1) the function $\omega \to \|A(\omega)\|$ is bounded and (2), for each $\xi \in H$, $A\xi$ is an element of H.

Next, we show that the decomposition in Definition 4.1 is unique. In fact, if $A = \mathscr{C}_{\Omega}^{\oplus} A(\omega) = \mathscr{C}_{\Omega}^{\oplus} B(\omega)$, then, for each $\xi, \eta \in H$, we have the relation

$$(A(\omega)\xi(\omega) \mid \eta(\omega)) = (B(\omega)\xi(\omega) \mid \eta(\omega))$$

for each $\omega \in \Omega$. For each $\omega \in \Omega$, the set $\{\xi(\omega): \xi \in H\}$ is dense in $H(\omega)$; therefore $A(\omega) = B(\omega)$ for each $\omega \in \Omega$.

LEMMA 4.3. *Let* $A = \{A(\omega)\}_{\omega \in \Omega} \in \prod_{\omega \in \Omega} B(H(\omega))$ *be a bounded continuous field; then* A *is an element of* B(H) *and*

$$((A\xi)(\omega) \mid \eta(\omega)) = (A(\omega)\xi(\omega) \mid \eta(\omega)),$$

for each $\omega \in \Omega$ *and all* $\xi, \eta \in H$.

Proof. The second assertion is evident by the definition. Thus, we must show that A is an element of B(H). A is a $C(\Omega)$-module mapping of H into H, by the definition of A. We show that A is a bounded operator. For each $\xi, \eta \in H$, $A\xi \in H$, and therefore

$$\begin{aligned} |((A\xi)(\omega) \mid \eta(\omega))| = |(A(\omega)\xi(\omega) \mid \eta(\omega))| &\leq (\sup\{\|A(\omega)\|: \omega \in \Omega\}) \|\xi(\omega)\| \cdot \|\xi(\omega)\| \\ &\leq (\sup\{\|A(\omega)\|: \omega \in \Omega\}) \|\xi\| \cdot \|\eta\| \end{aligned}$$

for each $\omega \in \Omega$. Thus $\|(A\xi, \eta)\| \leq (\sup\{\|A(\omega): \omega \in \Omega\}) \|\xi\| \cdot \|\eta\|$. Therefore

$$\|A\xi\| \leq \sup\{\|A(\omega)\|: \omega \in \Omega\} \cdot \|\xi\|,$$

for every $\xi \in H$; that is, A is a bounded $C(\Omega)$-module homomorphism.

PROPOSITION 4.4. *Let* Ω *be a compact Hausdorff space, let* $\mathscr{A} = C(\Omega)$, *and let* $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ *be a continuous field of Hilbert spaces over* Ω; *then each element* A *of* B(H) *is a decomposable operator.*

Proof. For all $\xi, \eta \in H$, the method used in the proof of Theorem 3.6 shows that

$$|((A\xi)(\omega) \mid \eta(\omega))| \leq \|A\| \cdot \|\xi(\omega)\| \cdot \|\eta(\omega)\|$$

for each $\omega \in \Omega$. Put $F_\omega((\xi(\omega) \mid \eta(\omega)) = ((A\xi)(\omega) \mid \eta(\omega))$ for all $\xi, \eta \in H$. Then F_ω can be extended to a bounded bilinear form on $H(\omega)$; thus there exists an element $A(\omega)$ of $B(H(\omega))$ such that

$$F_\omega((\xi(\omega) \mid \eta(\omega)) = (A(\omega)\xi(\omega) \mid \eta(\omega)).$$

Now, since $F_\omega((\xi(\omega) \mid \eta(\omega)) = ((A\xi)(\omega) \mid \eta(\omega))$, we see that

$$((A\xi)(\omega) \mid \eta(\omega)) = (A(\omega)\xi(\omega) \mid \eta(\omega))$$

for all $\xi, \eta \in H$. Therefore A is a decomposable operator.

By Lemma 4.3 and Proposition 4.4, if A is a decomposable operator with $A = \mathscr{C}_\Omega^\oplus A(\omega)$, we can identify A with $\{A(\omega)\}_{\omega\in\Omega} \in \prod_{\omega\in\Omega} B(H(\omega))$. That is, if $B = \{A(\omega)\}_{\omega\in\Omega}$, then B is a bounded continuous operator field, and

$$((A\xi)(\omega) \mid \eta(\omega)) = ((B\xi)(\omega) \mid \eta(\omega))$$

for each $\omega \in \Omega$ and all $\xi, \eta \in H$; therefore $(A\xi, \eta) = (B\xi, \eta)$ for all $\xi, \eta \in H$.

By Proposition 4.4, each element of B(H) is decomposable; thus, let A be an element of B(H) with $A = \mathscr{C}_\Omega^\oplus A(\omega)$; then for each $\omega \in \Omega$ we can define a mapping π_ω of B(H) into $B(H(\omega))$ such that $\pi_\omega(A) = A(\omega)$. It is evident that π_ω is a *-homomorphism of B(H) into $B(H(\omega))$.

In the remainder of this section, we assume that Ω is a Stonean space, so that $\mathscr{A} = C(\Omega)$ is a commutative AW*-algebra. We have shown that every continuous field of Hilbert spaces over a compact Hausdorff space is a C*-module, and we show that every AW*-module is isometrically isomorphic to a continuous field of Hilbert spaces over a Stonean space. Thus we obtain a characterization of AW*-modules over $C(\Omega)$.

THEOREM 4.5. *Let* $H = \mathscr{C}_\Omega^\oplus H(\omega)$ *be a continuous field of Hilbert spaces over a Stonean space* Ω. *Then there exist an* AW*-*module* M *over* $C(\Omega)$ *and a mapping* U *of* H *onto* M *such that*

(1) $U(f\xi + g\eta) = fU\xi + gU\eta$ *for all* $f, g \in C(\Omega)$ *and* $\xi, \eta \in H$ *and*

(2) $(U\xi, U\eta)(\omega) = (\xi(\omega) \mid \eta(\omega))$ *for all* $\xi, \eta \in H$.

Furthermore, the mapping U *induces an isometric* *-*isomorphism of the algebra of all bounded continuous operator fields onto the algebra* B(M) *of all bounded* $C(\Omega)$-*module homomorphisms on* M *such that*

(3) $(U\{A(\omega)\}U^{-1}\xi, \eta)(\omega) = (A(\omega)(U^{-1}\xi)(\omega) \mid (U^{-1}\eta)(\omega))$ *for all bounded continuous operator fields* $\{A(\omega)\}$ *and all* ξ, η *in* M.

Conversely, let M *be an* AW*-module *over* $C(\Omega)$. *For each* $\omega \in \Omega$, *let* $I_\omega = \{\xi \in M: (\xi, \xi)(\omega) = 0\}$. *Then* I_ω *is a closed submodule of* M, *and* $M - I_\omega = H(\omega)$ *is a Hilbert space with inner product induced by* M. *Furthermore, the continuous field* $\mathscr{C}_\Omega^\oplus H(\omega)$ (*defined with respect to* M) *is isometrically isomorphic to* M *under a mapping* U *that satisfies* (1), (2), (3) *of the first part of the theorem.*

Proof. Put $\mathscr{A} = C(\Omega)$. Since H is a C*-module, we show that H is an AW*-module over $C(\Omega)$. Let $U\xi = \xi$ for every $\xi \in H$; then U satisfies (1) and (2), and by Proposition 4.4, we have (3). To show that H is an AW*-module, we must show that H has the following two properties.

(α) Let $\{e_i\}_{i\in I}$ be a family of orthogonal projections in $\mathscr{A}$, with $\sup\{e_i : i \in I\} = e$, and let ξ be an element of H such that $e_i \xi = 0$ for all $i \in I$; then $e\xi = 0$.

(β) Let $\{e_i\}_{i\in I}$ be orthogonal projections in $\mathscr{A}$ with $\sup\{e_i : i \in I\} = 1$, and let $\{\xi_i\}_{i\in I}$ be a bounded subset of H; then there exists an element ξ in H with $e_i \xi = e_i \xi_i$ for all $i \in I$.

First we establish (α). We can suppose that $e = 1$. Let G_i be the closed and open set in Ω corresponding to e_i; then $\bigcup_{i\in I} G_i$ is a dense subset in Ω, and $\xi(\omega) = 0$ for each $\omega \in \bigcup_{i\in I} G_i$. The function $\omega \to \|\xi(\omega)\|$ is continuous on Ω, and therefore $\xi(\omega) = 0$ for each $\omega \in \Omega$, and $\xi = 0$.

Next we establish (β). For each element $\xi \in H$, the family $\{(\xi, \xi_i) : i \in I\}$ is bounded in $\mathscr{A}$; thus the sum $\sum_{i\in I} e_i(\xi, \xi_i)$ is in $\mathscr{A}$. Hence, for each $\omega \in \Omega$ and every $\xi \in H$, we put

$$\phi_\omega(\xi(\omega)) = \Big(\sum_{i\in I} e_i(\xi, \xi_i)\Big)(\omega).$$

Let G_i be the closed and open set in Ω corresponding to e_i; then $\bigcup_{i\in I} G_i$ is dense in Ω. For each $\omega \in G_i$, we have the relations

$$\Big(\sum_{i\in I} e_i(\xi, \xi_i)\Big)(\omega) = (\xi(\omega) \mid \xi_i(\omega))$$

and

$$\left|\Big(\sum_{i\in I} e_i(\xi, \xi_i)\Big)(\omega)\right| \le \|\xi(\omega)\| \cdot \|\xi_i(\omega)\| \le \sup\{\|\xi_i\| : i \in I\} \cdot \|\xi(\omega)\|.$$

By the continuity of the function $\omega \to \Big(\sum_{i\in I} e_i(\xi, \xi_i)\Big)(\omega)$ and the density of $\bigcup_{i\in I} G_i$, there exists for each $\omega \in \Omega$ a net $\{\omega_\alpha\}_{\alpha\in A}$ in $\bigcup_{i\in I} G_i$ such that $\lim_\alpha \omega_\alpha = \omega$. Thus

$$\left|\Big(\sum_{i\in I} e_i(\xi, \xi_i)\Big)(\omega)\right| = \lim_\alpha \left|\Big(\sum_{i\in I} e_i(\xi, \xi_i)\Big)(\omega)\right|$$
$$\le \sup\{\|\xi_i\| : i \in I\} \cdot \lim_\alpha \|\xi(\omega_\alpha)\| = \sup\{\|\xi_i\| : i \in I\} \cdot \|\xi(\omega)\|.$$

Therefore, ϕ_ω is well-defined and $|\phi_\omega(\xi(\omega))| \le \sup\{\|\xi_i\| : i \in I\} \cdot \|\xi(\omega)\|$ for each $\xi \in H$ and $\omega \in \Omega$. Hence, since $\{\xi(\omega) : \xi \in H\}$ is dense in $H(\omega)$, ϕ_ω is a bounded linear functional of $H(\omega)$ for each $\omega \in \Omega$. Thus there exists a vector field $\xi_0 \in \prod_{\omega\in\Omega} H(\omega)$ such that $\phi_\omega(\xi(\omega)) = (\xi(\omega) \mid \xi_0(\omega))$ for each $\xi \in H$ and each $\omega \in \Omega$. Furthermore, we must show that ξ_0 is an element of H. Now, since

$$(\xi(\omega) \mid \xi_0(\omega)) = \phi_\omega(\xi(\omega)) = \Big(\sum_{i\in I} e_i(\xi, \xi_i)\Big)(\omega)$$

for each $\omega \in \Omega$ and each $\xi \in H$, the function $\omega \to (\xi(\omega) \mid \xi_0(\omega))$ is continuous on Ω and $\|\xi_0(\omega)\| \leq \sup\{\|\xi_i\|: i \in I\}$. Therefore ξ_0 is an element of H. For each $\omega \in \Omega$ and each $\xi \in H$,

$$\Big(\sum_{i\in I} e_i(\xi, \xi_i)\Big)(\omega) = (\xi, \xi_0)(\omega),$$

and therefore $\sum_{i\in I} e_i(\xi, \xi_i) = (\xi, \xi_0)$ for every $\xi \in H$. Thus, $e_i(\xi, \xi_i) = e_i(\xi, \xi_0)$ for each $i \in I$, and consequently $(\xi, e_i\xi_i) = (\xi, e_i\xi)$ for each $\xi \in H$. Therefore there exists an element ξ_0 in H such that $e_i\xi_0 = e_i\xi_i$ for each $i \in I$. Thus $H = \mathscr{C}_\Omega^{\oplus} H(\omega)$ is an AW*-module over $\mathscr{A}$.

Next we show the converse. It is evident that I_ω is a closed submodule of M, and $M - I_\omega = H(\omega)$ is a Hilbert space with inner product induced by M. For each $\xi \in M$, we represent an element $\{\xi(\omega): \xi \in \Omega\}$ in $\prod_{\omega\in\Omega} H(\omega)$ by ρ_ξ, where $\xi(\omega)$ is the quotient element of ξ in $M - I_\omega = H(\omega)$, and we write $H = \{\rho_\xi: \xi \in M\}$. Then H is a subspace of $\prod_{\omega\in\Omega} H(\omega)$. Since $(\xi(\omega) \mid \eta(\omega)) = (\xi, \eta)(\omega)$ for all $\xi, \eta \in M$ and $\omega \in \Omega$, H is a Banach space with respect to the norm $\|\rho_\xi\| = \sup\{\|\xi(\omega)\|: \omega \in \Omega\}$. Thus H is an $\mathscr{A}$-moduled Banach space, and therefore we can identify H with M, in our discussion. That is, for each $\xi \in M$, we identify ρ_ξ with ξ. Thus we must show that H is a continuous field of Hilbert spaces over Ω with respect to H. To show that H has property (3) in Definition 3.1, let ξ be an element such that for each positive number ε and each $\omega_0 \in \Omega$ there exist an element $\xi' \in H$ and a closed and open set $U(\omega_0)$ containing ω_0 with $\|\xi(\omega) - \xi'(\omega)\| < \varepsilon$ for each $\omega \in U(\omega_0)$. Then, by our assumption, there exists for each $\varepsilon > 0$ a family $\{U(\omega), \xi_\omega\}$ of pairs of closed and open sets $U(\omega)$ containing ω and elements of H such that $\|\xi(\omega') - \xi_\omega(\omega')\| < \varepsilon$ for each $\omega' \in U(\omega)$. Now, since $\{U(\omega): \omega \in \Omega\}$ is an open covering of Ω, there exists a finite subcovering $\{U(\omega_i): i = 1, 2, \cdots, n\}$ of Ω. Since each $U(\omega_i)$ is a closed and open set, we can assume that the sets $U(\omega_i)$ $(i = 1, 2, \cdots, n)$ are disjoint. Let z_i be the projection in $\mathscr{A}$ corresponding to $U(\omega_i)$; then the sets $\{z_i\}$ are mutually orthogonal. Thus, let $\xi_0 = \sum_{i=1}^n z_i\xi_i$; then ξ_0 is an element of H and $\|\xi - \xi_0\| < \varepsilon$. Therefore ξ is an element of H.

Next we show that H satisfies condition (4) of Definition 3.1. Let ξ' be an element of $\prod_{\omega\in\Omega} H(\omega)$ such that, for each $\xi \in H$, the function $\omega \to (\xi(\omega) \mid \xi'(\omega))$ is continuous on Ω and the function $\omega \to \|\xi'(\omega)\|$ is bounded; for every $\xi \in H$, let $\phi(\xi) = (\xi, \xi')$, and let $\beta = \sup\{\|\xi'(\omega)\|: \omega \in \Omega\}$. Then ϕ is an $\mathscr{A}$-module mapping of H into $\mathscr{A}$, and for each $\xi \in H$,

$$\begin{aligned}\|\phi(\xi)\| = \|(\xi, \xi')\| &= \sup\{|(\xi(\omega) \mid \xi'(\omega))|: \omega \in \Omega\}\\ &\leq \sup\{\|\xi(\omega)\| \cdot \|\xi'(\omega)\|: \omega \in \Omega\} \leq \beta\|\xi\|.\end{aligned}$$

Thus ϕ is a bounded linear functional of H into $\mathscr{A}$. Since M is an AW*-module, there exists an element ξ_0 of H such that $\phi(\xi) = (\xi, \xi_0)$. Thus H is a continuous field of Hilbert spaces over Ω (defined with respect to H). We denote H by

$\mathscr{C}_{\Omega}^{\oplus} H(\omega)$. Let $U\xi = \xi$ (that is, $U\xi = \rho_\xi$); then it is evident that U satisfies (1), (2), (3) of the first part of the theorem.

5. REPRESENTATION OF GELFAND TYPE OF VON NEUMANN ALGEBRAS

In Section 4, we considered the decomposition of bounded operators on a continuous field of Hilbert spaces. In this section, we obtain a representation of Gelfand type of von Neumann algebras.

Definition 5.1. Let Ω be a compact Hausdorff space, let $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ be a continuous field of Hilbert spaces, and let $\mathfrak{A}(\omega)$ be a C^*-subalgebra of $B(H(\omega))$ for each $\omega \in \Omega$; then we put

$$\mathfrak{A} = \{A \in B(H) \colon A(\omega) \in \mathfrak{A}(\omega) \text{ for every } \omega \in \Omega, \text{ where } A = \mathscr{C}_{\Omega}^{\oplus} A(\omega)\}.$$

Thus we write $\mathfrak{A} = \mathscr{C}_{\Omega}^{\oplus} \mathfrak{A}(\omega)$.

Then $\mathfrak{A} = \mathscr{C}_{\Omega}^{\oplus} \mathfrak{A}(\omega)$ is a $C(\Omega)$-moduled C^*-subalgebra of $B(H)$. Further, let $\mathfrak{A}$ be a $C(\Omega)$-moduled C^*-subalgebra of $B(H)$, and for each $\omega \in \Omega$, let

$$\mathfrak{A}(\omega) = \{A(\omega) = \pi_\omega(A) \colon A \in \mathfrak{A}\};$$

then we must give a condition under which $\mathfrak{A} = \mathscr{C}_{\Omega}^{\oplus} \mathfrak{A}(\omega)$. We shall show that if Ω is a Stonean space, in this problem, then $\mathfrak{A} = \mathscr{C}_{\Omega}^{\oplus} \mathfrak{A}(\omega)$. This requires some preliminary considerations.

Let Ω be a compact Hausdorff space, let $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ be a continuous field of Hilbert spaces over Ω, and let A be an element of $B(H)$ with $A = \mathscr{C}_{\Omega}^{\oplus} A(\omega)$; then, by the argument in the proof of Theorem 3.6, we can show that

$$\|A\| = \sup\{\|A(\omega)\| \colon \omega \in \Omega\};$$

further, we can show that, for each $\omega \in \Omega$,

$$\|A(\omega)\| = \sup\{\|A(\omega)\xi(\omega)\| \colon \xi \in H, \ \|\xi\| \leq 1\},$$

so that the function $\omega \to \|A(\omega)\|$ is lower-semicontinuous. In particular, if Ω is a Stonean space and $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ is a faithful continuous field (that is, if there exists an element ξ_0 of H such that $|\xi_0| = 1$ in $\mathscr{A}$), then the function $\omega \to \|A(\omega)\|$ is continuous on Ω, as shall be shown in Lemma 5.3. Before Lemma 5.3, we have the following result.

LEMMA 5.2. *Let Ω be a Stonean space, let $\mathscr{A} = C(\Omega)$, and let $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ be a continuous field of Hilbert spaces over Ω. If* H *is a faithful continuous field, then the set*

$$\ker \pi_\omega = \{A \in B(H) \colon A(\omega) = 0, \textit{ where } A = \mathscr{C}_{\Omega}^{\oplus} A(\omega)\}$$

is the norm closure of

$$\left\{ \sum_{i=1}^{n} z_i A_i \colon A_i \in B(H) \textit{ and } z_i \in \mathscr{A} \textit{ with } z_i(\omega) = 0 \right\},$$

for each $\omega \in \Omega$.

The above ideal I_ω, that is, the norm closure of $\left\{\sum_{i=1}^{n} z_i A_i : A_i \in B(H)\right.$ and $\left. z_i \in \mathscr{A} \text{ with } z_i(\omega) = 0\right\}$, has been defined for von Neumann algebras by J. Glimm [4, p. 232], and it has been used by many authors. Furthermore, we must consider the analogous result, which has been established by Glimm [4, Lemma 10] for von Neumann algebras.

Proof of Lemma 5.2. Our argument follows closely the proof of Glimm's result [4, Theorem 4]. In fact, since there exists an element ξ_0 of H with $|\xi_0| = 1$, and since H is an AW*-module over $\mathscr{A}$, H is a faithful AW*-module over $\mathscr{A}$, and B(H) is an AW*-algebra of type I with the center $\mathscr{A}$ (Theorem 4.5 and [9]). Let E_0 be an Abelian projection of B(H), defined by the equation $E_0 \xi = (\xi, \xi_0)\xi_0$ for every $\xi \in H$; then the central support of E_0 is 1. Let Φ_0 be the $\mathscr{A}$-module mapping of B(H) onto $\mathscr{A}$ defined by the equation $\Phi_0(A) = E_0 A E_0$, and let ϕ_ω be the state of B(H) defined by the equation $\phi_\omega(A) = \Phi_0(A)(\omega)$ for each $\omega \in \Omega$. Further, let $(\pi'_\omega, H'(\omega))$ be the canonical representation and representative space induced by ϕ_ω. Then Φ_0 is an $\mathscr{A}$-irreducible linear mapping of B(H) (the term "$\mathscr{A}$-irreducible" is due to H. Halpern [5, p. 200]). Furthermore, by using a result of Halpern [5, Theorem 4.3], we see that (1) $H = \{A\xi_0 : A \in B(H)\}$, (2) for each $\omega \in \Omega$, ϕ_ω is a pure state of B(H). Thus π'_ω is irreducible (see [6], for example), and therefore $\pi'_\omega(B(H)) \supset C(H'(\omega))$ for each $\omega \in \Omega$, because $\pi'_\omega(B(H))$ contains the nonzero abelian projection $\pi'_\omega(E_0)$ for every $\omega \in \Omega$. Furthermore, by the argument used in the proof of Theorem 4 in [4], $I_\omega = \ker \pi'_\omega$. The assertion $\ker \pi_\omega \supset I_\omega$ is obvious; we shall show that $\ker \pi_\omega \subset I_\omega$ for each $\omega \in \Omega$. If $A(\omega) = 0$, then $(A(\omega)\xi(\omega) \mid \eta(\omega)) = 0$ for each $\xi, \eta \in H$. Now, since $H = \{A\xi_0 : A \in B(H)\}$, we have, for all $B, C \in B(H)$, the relations

$$\begin{aligned} 0 &= (A(\omega)B(\omega)\xi_0(\omega) \mid C(\omega)\xi_0(\omega)) = (C(\omega)^* A(\omega)B(\omega)\xi_0(\omega) \mid \xi_0(\omega)) \\ &= ((C^*AB)(\omega)\xi_0(\omega) \mid \xi_0(\omega)) = ((C^*AB)(\omega)E_0(\omega)\xi_0(\omega) \mid E_0(\omega)\xi_0(\omega)) \\ &= ((E_0 C^* A B E_0)(\omega)\xi_0(\omega) \mid \xi_0(\omega)) = \Phi_0(E_0 C^* A B E_0)(\omega) = \Phi_0(C^*AB)(\omega)\,. \end{aligned}$$

Therefore, $\phi_\omega(C^*AB) = 0$ for all $B, C \in B(H)$, and therefore $\pi'_\omega(A) = 0$. Thus, $\ker \pi_\omega = \ker \pi'_\omega = I_\omega$. This proves Lemma 5.2.

By Lemma 5.2, we see that for each $A \in B(H)$, $\|A(\omega)\| = \|\pi'_\omega(A)\|$ for each $\omega \in \Omega$. Glimm has shown that if B(H) is a von Neumann algebra (that is, if Ω is a hyperstonean space), then the function $\omega \to \|\pi'_\omega(A)\|$ is continuous on Ω. His argument uses topological properties, but is independent of measure theory. Thus we can show that even if Ω is a Stonean space, the function $\omega \to \|\pi'_\omega(A)\|$ is continuous, by an argument similar to the proof of Lemma 10 in [4]. These considerations yield the following result.

LEMMA 5.3. *Under the hypotheses of Lemma* 5.2, *let* A *be an element of* B(H) *with* $A = \mathscr{C}\bigoplus_\Omega A(\omega)$; *then the function* $\omega \to \|A(\omega)\|$ *is continuous.*

PROPOSITION 5.4. *Let* Ω *be a Stonean space, let* $\mathscr{A} = C(\Omega)$, *and let* $H = \mathscr{C}\bigoplus_\Omega H(\omega)$ *be a faithful continuous field of Hilbert spaces over* Ω. *Let* $\mathfrak{A}$ *be an* $\mathscr{A}$*-moduled* C*-*subalgebra of* B(H), *and for each* $\omega \in \Omega$, *let*

$$\mathfrak{A}(\omega) = \{A(\omega) \in B(H(\omega));\ A \in \mathfrak{A} \text{ and } A = \mathscr{C}\bigoplus_\Omega A(\omega)\}\ ;$$

then $\mathfrak{A} = \mathscr{C}_{\Omega}^{\oplus} \mathfrak{A}(\omega)$. *In particular, since* $\mathscr{A}(\omega) = \mathbf{C}(\omega)$ *for every* $\omega \in \Omega$, $\mathscr{A} = C(\Omega) = \mathscr{C}_{\Omega}^{\oplus} \mathbf{C}(\omega)$, *where* $\mathbf{C}(\omega)$ *is the complex number field.*

Proof. The assertion $\mathfrak{A} \subset \mathscr{C}_{\Omega}^{\oplus} \mathfrak{A}(\omega)$ is obvious. Thus, we must show that $\mathfrak{A} \supset \mathscr{C}_{\Omega}^{\oplus} \mathfrak{A}(\omega)$. If A is an element of $\mathscr{C}_{\Omega}^{\oplus} \mathfrak{A}(\omega)$, then $A(\omega) \in \mathfrak{A}(\omega)$ for each $\omega \in \Omega$; that is, for each $\omega \in \Omega$, there exists an element $B_\omega \in \mathfrak{A}$ such that $A(\omega) = B_\omega(\omega)$. For each $B \in \mathfrak{A} \subset B(H)$, the function $\omega \to \|B(\omega)\|$ is continuous, and therefore, for each positive number ε and each $\omega \in \Omega$, there exists a closed and open set $U(\omega)$ containing ω such that $\|A(\omega') - B_\omega(\omega')\| < \varepsilon$ for every $\omega' \in U(\omega)$. Since Ω is compact, there exists a finite subcovering $\{U(\omega_i): i = 1, 2, \cdots, n\}$ of $\{U(\omega): \omega \in \Omega\}$, and since $U(\omega)$ is closed and open, we can assume that the sets $U(\omega_i)$ $(i = 1, 2, \cdots, n)$ are disjoint. Let z_i be the projection in $\mathscr{A}$, corresponding to each closed and open set $U(\omega_i)$, and let $B = \sum_{i=1}^{n} z_i B_{\omega_i}$. Then $B \in \mathfrak{A}$ and $\|A(\omega) - B(\omega)\| < \varepsilon$ for each $\omega \in \Omega$. Thus, by the comment preceding Lemma 5.2,

$$\|A - B\| = \sup\{\|A(\omega) - B(\omega)\|: \omega \in \Omega\} < \varepsilon,$$

and consequently $A \in \mathfrak{A}$. Therefore $\mathfrak{A} \supset \mathscr{C}_{\Omega}^{\oplus} \mathfrak{A}(\omega)$. Thus we have proved Proposition 5.4.

COROLLARY 5.5. *Let* $\mathfrak{A}$ *be an* AW^**-algebra of type* I *with the center* $\mathscr{A} = C(\Omega)$; *then there exists a faithful continuous field* $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ *of Hilbert spaces over* Ω *such that* $\mathfrak{A}$ *is* **-isomorphic to* $\mathscr{C}_{\Omega}^{\oplus} \mathfrak{A}(\omega)$, *where* $\mathfrak{A}(\omega)$ *is an irreducible* C^**-subalgebra of* $B(H(\omega))$ *containing* $C(H(\omega))$.

Let Ω be a Stonean space, let $\mathscr{A} = C(\Omega)$, let $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ be a faithful continuous field of Hilbert spaces over Ω, let $\mathfrak{A}$ be an $\mathscr{A}$-moduled C^*-subalgebra of $B(H)$, and denote the set

$$\{A \in B(H): AB = BA \text{ for all } B \in \mathfrak{A}\}$$

by $\mathfrak{A}'$. Then we have the following result.

LEMMA 5.6. *Let* Ω *be a Stonean space, let* $\mathscr{A} = C(\Omega)$, *let* $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ *be a faithful continuous field of Hilbert spaces over* Ω, *and let* $\mathfrak{A}$ *be an* $\mathscr{A}$*-moduled* C^**-subalgebra of* $B(H)$ *with* $\mathfrak{A} = \mathfrak{A}''$. *Then* $\mathfrak{A} = \mathscr{C}_{\Omega}^{\oplus} \widetilde{\mathfrak{A}(\omega)}$, *where* $\widetilde{\mathfrak{A}(\omega)}$ *is the weak closure of* $\mathfrak{A}(\omega) = \{A(\omega): A \in \mathfrak{A},\ A = \mathscr{C}_{\Omega}^{\oplus} A(\omega)\}$.

Proof. It is evident that $\mathfrak{A} \subset \mathscr{C}_{\Omega}^{\oplus} \widetilde{\mathfrak{A}(\omega)}$. Thus we show that $\mathfrak{A} \supset \mathscr{C}_{\Omega}^{\oplus} \widetilde{\mathfrak{A}(\omega)}$. In Proposition 5.4, we have shown that $\mathfrak{A} = \mathscr{C}_{\Omega}^{\oplus} \mathfrak{A}(\omega)$ and $\mathfrak{A}' = \mathscr{C}_{\Omega}^{\oplus} \mathfrak{A}'(\omega)$. If A_0 is an element of $\mathscr{C}_{\Omega}^{\oplus} \widetilde{\mathfrak{A}(\omega)}$, then, for each $A \in \mathfrak{A}'$, $\omega \in \Omega$, and all $\xi, \eta \in H$,

$$(A_0(\omega) A(\omega) \xi(\omega) \mid \eta(\omega)) = (A(\omega) A_0(\omega) \xi(\omega) \mid \eta(\omega)).$$

Thus $A(\omega) A_0(\omega) = A_0(\omega) A(\omega)$ for each $\omega \in \Omega$, because $\{\xi(\omega): \xi \in H\}$ is dense in $H(\omega)$, and therefore $A_0 A = A A_0$. Since $\mathfrak{A} = \mathfrak{A}''$, A_0 is an element of $(\mathfrak{A}')' = \mathfrak{A}'' = \mathfrak{A}$. Thus $\mathfrak{A} = \mathscr{C}_{\Omega}^{\oplus} \widetilde{\mathfrak{A}(\omega)}$.

Let $\mathscr{B}$ be a von Neumann algebra with the center $\mathscr{A} = C(\Omega)$; then the commutant $\mathfrak{A}$ of $\mathscr{A}$ is a von Neumann algebra of type I. Therefore there exists a faithful continuous field $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ of Hilbert spaces over Ω such that $\mathfrak{A}$ is *-isomorphic to $\mathscr{C}_{\Omega}^{\oplus} B(H(\omega)) = B(H)$. Furthermore, by Lemma 5.6 and the fact that $\mathscr{B} = \mathscr{B}''$, $\mathscr{B}$ is *-isomorphic to $\mathscr{C}_{\Omega}^{\oplus} \widetilde{\mathscr{B}(\omega)}$, where $\widetilde{\mathscr{B}(\omega)}$ is a von Neumann algebra in $B(H(\omega))$. Hence we have the following result.

THEOREM 5.7. *Let* $\mathfrak{A}$ *be a von Neumann algebra with center* $\mathscr{A} = C(\Omega)$; *then there exists a faithful continuous field* $H = \mathscr{C}_{\Omega}^{\oplus} H(\omega)$ *of Hilbert spaces such that* $\mathfrak{A}$ *is* **-isomorphic to* $\mathscr{C}_{\Omega}^{\oplus} \widetilde{\mathfrak{A}(\omega)}$, *where* $\widetilde{\mathfrak{A}(\omega)}$ *is a von Neumann algebra in* $B(H(\omega))$. *This* **-isomorphism extends the isomorphism of* $\mathscr{A}$ *with* $C(\Omega)$.

REFERENCES

1. J. Dixmier, *Sur certains espaces considérés par M. H. Stone.* Summa Brasil. Math. 2 (1951), 151-182.
2. ———, *Les algebres d'opérateurs dans l'espace hilbertien.* Gauthier-Villars, Paris, 1957.
3. ———, *Les* C*-*algèbres et leurs représentations.* Gauthier-Villars, Paris, 1964.
4. J. G. Glimm, *A Stone-Weierstrass theorem for* C*-*algebras.* Ann. of Math. (2) 72 (1960), 216-244.
5. H. Halpern, *Irreducible module homomorphisms of a von Neumann algebra into its center.* Trans. Amer. Math. Soc. 140 (1969), 195-221.
6. R. V. Kadison, *Irreducible operator algebras.* Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 273-276.
7. I. Kaplansky, *Projections in Banach algebras.* Ann. of Math. (2) 53 (1951), 235-249.
8. ———, *Algebras of type I.* Ann. of Math. (2) 56 (1952), 460-472.
9. ———, *Modules over operator algebras.* Amer. J. Math. 75 (1953), 839-858.

Tôhoku University
Sendai, Japan