OVERRINGS AND DIVISORIAL IDEALS OF RINGS OF THE FORM D + M

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1. INTRODUCTION

Let V be a valuation ring of the form K + M, where K is a field and M is the maximal ideal of V. If D is a subring of K, we denote by D_1 the subring D + M of V. Domains of this kind arise frequently in the literature, especially in connection with the construction of examples; see, for example, [35, p. 670], [46, p. 328], [38], [45, p. 604], [32, Chapter 4], [24, Section 5], [13, p. 252], [14, p. 500], [5, p. 305], [11, p. 280], [18, Section 4], and [6]. We compile in Theorem 2.1 the results that appear in [15] concerning domains of the form D + M. In Section 3 (Theorem 3.1), we determine the set of overrings of D_1. The theorem leads to numerous results involving special conditions on the set of overrings of domains that have been considered in the literature; these include GQR-domains [28], QQR-domains [18], domains for which each overring is an ideal transform [6], domains satisfying the transform formula for ideals [21], and domains for which the set of overrings is closed under addition [22].

In Section 4, we determine the set of divisorial ideals of D_1, and we investigate the condition, introduced by W. Heinzer in [27], that each ideal of D_1 be divisorial. We conclude, in Section 5, by deriving an expression for the dimension of the polynomial ring $D_1[X_1, \cdots, X_n]$ in terms of the dimension of $D[X_1, \cdots, X_n]$; but the main contribution of Section 5 is the information concerning realization of a sequence $\{n_0, n_1, n_2, \cdots\}$ in the form $\{\dim R, \dim R[X_1], \dim R[X_1, X_2], \cdots\}$. Our results in Section 5 depend strongly on a theorem of J. T. Arnold in [1].

Overall, our results show again what previous results concerning D_1 have indicated, namely, that the structure of D_1 reflects properties of the valuation ring V and properties of K as a ring extension of D. The richness of properties that can be realized by a construction of this type is usually due to the freedom involved in the choice of D and K.

2. A SUMMARY OF SOME KNOWN RESULTS

Because we shall need them frequently, we list in Theorem 2.1 some known results concerning domains of the form D + M. Detailed proofs for these results can be found in [15, Appendix 2].

2.1. THEOREM. *Let* V *be a nontrivial valuation ring with quotient field* L, *and assume that* V *is of the form* K + M, *where* K *is a field and* M *is the maximal ideal of* V. *Let* D *be a domain with identity that is a proper subring of* K, *and let* $D_1 = D + M$.

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(a) D_1 *is a domain with identity, and* M *is the conductor of* D_1 *in* V. *Therefore,* D_1 *and* V *have the same complete integral closure. In particular,* D_1 *is not completely integrally closed.*

(b) *The integral closure of* D_1 *is* $D' + M$, *where* D' *is the integral closure of* D *in* K.

(c) *Each ideal of* D_1 *compares with* M *under inclusion.*

(d) *The set of ideals of* D_1 *containing* M *is* $\{A_\alpha + M\}$, *where* $\{A_\alpha\}$ *is the set of ideals of* D. *Moreover,* $D_1/(A_\alpha + M)$ *is isomorphic to* D/A_α, *so that* $A_\alpha + M$ *is maximal, prime, or* $(P_\alpha + M)$*-primary in* D_1 *if and only if* A_α *is, respectively, maximal, prime, or* P_α*-primary in* D.

(e) *If* Q *is* P*-primary in* D_1, *where* $P \subset M$, *then* Q *and* P *are ideals of* V, *and* Q *is* P*-primary in* V. *If* M, *as an ideal of* V, *is unbranched, then* M *is also unbranched as an ideal of* D_1.

(f) $\dim D_1 = \dim D + \dim V$.

(g) *If* N *is a multiplicative system in* D, *then* $(D_1)_N = D_N + M$. *If* P *is prime in* D_1 *and if* $P \subset M$, *then* $(D_1)_P = V_P$, *so that* $(D_1)_P$ *is a valuation ring.*

(h) D_1 *is a valuation ring if and only if* D *is a valuation ring with quotient field* K.

(i) D_1 *is a Prüfer domain if and only if* D *is a Prüfer domain with quotient field* K.

(j) *The valuative dimension of* D_1 *is equal to* $k + \dim V$, *where* k *is the supremum of the set* $\{\dim W \mid W$ *is a valuation ring on* K *containing* $D\}$.

(k) *The finitely generated ideals of* D_1 *that properly contain* M *are those of the form* $A_\alpha + M$, *where* A_α *is a nonzero finitely generated ideal of* D. *Each finitely generated ideal* A *of* D_1 *contained in* M *can be obtained as follows: let* W *be a nonzero finitely generated* D*-submodule of* K *containing* D, *let* $m \in M - \{0\}$, *and set* $A = Wm + Mm$.

(ℓ) *If* D *is a Prüfer domain with quotient field* K, *then* D *and* D_1 *have the same class group.*

(m) D_1 *is Noetherian if and only if* V *is Noetherian,* D *is a field, and the degree of* K *over* D *is finite.*

We add one result to the statement of Theorem 2.1. Its proof is essentially the same as the proof of (k) (see [15]).

(n) *If* A *is an ideal of* D_1 *contained in* M, *then either* A *is an ideal of* V, *or* AV *is a principal ideal of* V. *If* A *is not an ideal of* V *and if* $AV = aV$, *where* $a \in A$, *then* $A = Wa + Ma$, *for some* D*-submodule* W *of* K *such that* $D \subseteq W \subset K$.

If R is a commutative ring and t is a positive integer, then R is said to have *the* t*-generator property* if each finitely generated ideal of R has a basis of t elements. In [20, pp. 148-149], R. Gilmer and W. Heinzer prove the following result, which follows essentially from (k) of Theorem 2.1.

(k') *If* t *is a positive integer, then* D_1 *has the* t*-generator property if and only if either* D *has the* t*-generator property and* K *is the quotient field of* D, *or* D *is a field and the dimension of* K *over* D *is at most* t.

The following statement is a consequence of (k') and (ℓ).

(ℓ') D_1 *is a Bezout domain if and only if* D *is a Bezout domain with quotient field* K.

3. SOME CONDITIONS ON THE SET OF OVERRINGS

Throughout the remainder of the paper, we assume that the symbols V, L, K, M, D, and D_1 are used as in Theorem 2.1. In particular, D is a nonzero proper subring of K, and D_1 is a proper subring of V. We begin with a result that determines the set of overrings of D_1. We use the term *overring of* X to mean *subring of the total quotient ring of* X *containing* X.

3.1. THEOREM. *Each* D_1-*submodule of* L *compares with* V *under inclusion. If* $\{D_\lambda\}$ *is the family of subrings of* K *containing* D, *and* $\{W_\alpha\}$ *is the family of overrings of* V, *then* $\{D_\lambda + M\} \cup \{W_\alpha\}$ *is the family of overrings of* D_1.

Proof. Let S be a D_1-submodule of L. If $S \not\subseteq V$, then $SV \supset V$, so that $V \subseteq M \cdot SV = M \cdot S \subseteq D_1 \cdot S = S$. Clearly, $\{D_\lambda + M\} \cup \{W_\alpha\}$ is contained in the set of overrings of D_1, and if S is an overring of D_1 contained in V, then by the modular law, $S = S \cap V = S \cap (K + M) = (S \cap K) + M$, where $S \cap K \in \{D_\lambda\}$. This completes the proof of Theorem 3.1.

We turn to a consideration of conditions under which the set of overrings of D_1 satisfies certain special conditions that have been studied in the literature. If R is a commutative ring, then, following [29], [2], and [8], we call a nonempty family $\mathscr{S}$ of subsets of R *a generalized multiplicative system* if $\{0\} \notin \mathscr{S}$ and $\mathscr{S}$ is closed under multiplication. *The generalized quotient ring of* R *with respect to* $\mathscr{S}$, denoted by $R_{\mathscr{S}}$, is defined to be the set of elements t of T (the total quotient ring of R) such that $tS_\alpha \subseteq R$ for some element S_α of $\mathscr{S}$; it is clear that $R_{\mathscr{S}}$ is an overring of R containing the identity element of T. Generalized quotient rings (without the nomenclature) were originally considered by W. Krull in [36], then later by D. Kirby in [34] and by L. Budach in [7]. An advantage of the concept is its generality, as indicated by Arnold and J. W. Brewer in [2]; in particular, it includes the concepts of a quotient ring with respect to a regular multiplicative system, of the ideal transform of M. Nagata [37], of a large quotient ring [25], and, for a domain, of an arbitrary intersection of localizations of the domain [29, p. 154]. In considering generalized quotient rings of R, we incur no loss of generality by assuming that the elements of the generalized multiplicative systems in question are ideals of R. Our next result determines the set of generalized quotient rings of D_1.

3.2. THEOREM. *Let* $\mathscr{S} = \{I_\alpha\}$ *be a generalized multiplicative system in* D_1.

(1) *If each* I_α *in* $\mathscr{S}$ *properly contains* M, *let* $I_\alpha = A_\alpha + M$, *where* A_α *is a nonzero ideal of* D. *Then* $\mathscr{T} = \{A_\alpha\}$ *is a generalized multiplicative system in* D, *and* $(D_1)_{\mathscr{S}} = D_{\mathscr{T}} + M$.

(2) *If some* I_α *in* $\mathscr{S}$ *is contained in* M, *let*

$$\mathscr{T} = \{I_\alpha \in \mathscr{S} \mid I_\alpha \subseteq M\} \quad \textit{and} \quad \mathscr{T}' = \{I_\alpha V \mid I_\alpha \in \mathscr{T}\}.$$

Then $\mathscr{T}$ *is a generalized multiplicative system in* D_1, *and* $\mathscr{T}'$ *is a generalized multiplicative system in* V; *moreover,* $(D_1)_{\mathscr{S}} = (D_1)_{\mathscr{T}} = V_{\mathscr{T}'}$.

Proof. In (1), it is clear that $\mathscr{T}$ is a generalized multiplicative system in D, and it is likewise clear that $D_{\mathscr{T}} + M$ is contained in $(D_1)_{\mathscr{S}}$. We observe that $(D_1)_{\mathscr{S}} \subseteq V$. Thus, if $t \in (D_1)_{\mathscr{S}}$ and $tI_\alpha \subseteq D_1$, then $tA_\alpha \subseteq V$ and each nonzero element of A_α is a unit of V; consequently, $t \in V$. By Theorem 3.1, $(D_1)_{\mathscr{S}}$ is of the form $D_\lambda + M$, where D_λ is a subring of K containing D. It follows immediately from the definition of $(D_1)_{\mathscr{S}}$, however, that $D_\lambda = D_{\mathscr{T}}$.

To prove (2), we establish the equality $(D_1)_{\mathscr{S}} = (D_1)_{\mathscr{T}} = V_{\mathscr{T}'}$. The inclusion $(D_1)_{\mathscr{S}} \supseteq (D_1)_{\mathscr{T}}$ is clear. If $t \in V_{\mathscr{T}'}$ and $tI_\alpha \subseteq V$, then $tI_\alpha^2 \subseteq I_\alpha \subseteq M \subseteq D_1$, where $I_\alpha^2 \in \mathscr{T}$ and $t \in (D_1)_{\mathscr{T}}$. Hence $V_{\mathscr{T}'} \subseteq (D_1)_{\mathscr{T}}$. It follows from (1) that if $t \in (D_1)_{\mathscr{S}}$ and $tI_\alpha \subseteq D_1$, where $I_\alpha \in \mathscr{S}$, then $t \in V \subseteq V_{\mathscr{T}'}$ provided $I_\alpha \supset M$—that is, provided $I_\alpha \notin \mathscr{T}$. And if $I_\alpha \in \mathscr{T}$, then it is clear that $t \in V_{\mathscr{T}'}$. Hence, in every case, $(D_1)_{\mathscr{S}} \subseteq V_{\mathscr{T}'}$, and our proof of the equality in (2) is complete.

In [28], Heinzer defines a GQR-*domain* to be an integral domain J with identity such that each overring of J is a generalized quotient ring of J; Heinzer proves that an integrally closed domain J is a GQR-domain if and only if J is a Prüfer domain. Using Theorems 3.1 and 3.2, we are able to determine necessary and sufficient conditions in order that D_1 be a GQR-domain.

3.3. THEOREM. *The following conditions are equivalent.*

(1) D_1 *is a* GQR-*domain.*

(2) *Either* (a) D *is a* GQR-*domain with quotient field* K, *or* (b) D *is a field, no proper subfield of* K *properly contains* D, *and* M *is idempotent.*

Before proving Theorem 3.3, we remark that a significant step in the proof is that of determining conditions under which V can be represented as a generalized quotient ring of D_1. In this connection, we shall need the following lemma; it is a generalization of Lemma 2.8 of [6].

3.4. LEMMA. *Let* W *be a valuation ring, let* $\mathscr{S} = \{I_\lambda\}$ *be a generalized multiplicative system in* W *containing a proper ideal* I_{λ_0}, *and let* $P = \bigcap_\lambda I_\lambda$. *Then* P *is a prime ideal of* W *and* $W_{\mathscr{S}} = W_P$.

Proof. If x and y are in W - P, then there exist I_σ and I_τ in $\mathscr{S}$ such that $I_\sigma \subset (x)$ and $I_\tau \subset (y)$. Thus $I_\sigma I_\tau \subseteq (x) I_\tau \subset (xy)$, and $xy \notin P$, since $xy \notin I_\sigma I_\tau \in \mathscr{S}$. Therefore P is prime in W.

Since W is a valuation ring, $W_{\mathscr{S}}$ is a quotient ring of W, and in fact, $W_{\mathscr{S}} = W_N$, where $N = \{n \in W \mid n^{-1} \in W_{\mathscr{S}}\}$. For $n \in W$ $(n \neq 0)$ the relation $n^{-1} \in W_{\mathscr{S}}$ holds if and only if $n^{-1} I_\lambda \subseteq V$ for some λ. Therefore

$$N = \{n \in W \mid I_\lambda \subseteq (n) \text{ for some } I_\lambda \in \mathscr{S}\} = W - P,$$

and $W_{\mathscr{S}} = W_P$, as we asserted.

Proof of Theorem 3.3. We prove first that if (1) holds and (2a) fails, then (2b) is valid. Let k be the quotient field of D, and let J be an overring of D. If J = k, then J is a quotient ring of D, and if $J \subset k$, then Theorem 3.2 implies that J is a generalized quotient ring of D. Hence (1) implies that D is a GQR-domain; therefore, if (2a) fails, then $k \subset K$. Since D_1 is a GQR-domain, each subring $T \neq K$ of K containing D is contained in k. This implies that D = k, for otherwise D admits a nontrivial valuation overring V_1, and V_1 admits an extension W_1 to K; thus W_1 is a proper subring of K containing D, but W_1 is not contained in k, contrary to what we have previously shown. Therefore D = k, and there is no field E with $k \subset E \subset K$. By Theorem 3.2, there is a generalized multiplicative system $\mathscr{T}'$ in V consisting of ideals contained in M such that $V_{\mathscr{T}'} = V$. But Lemma 3.4 then implies that $\mathscr{T}' = \{M\}$, and M is idempotent. Thus our proof that (1) implies (2) is complete.

That (2) implies (1) follows from Theorem 3.1, Theorem 3.2, and our proof that (1) implies (2).

In [28], Heinzer raises the following two questions.

Is the integral closure of a GQR-domain a Prüfer domain?

If J is a quasi-local GQR-domain with integral closure J^*, does there exist a domain J' such that $J \subset J' \subset J^*$?

We remark that for the GQR-domains given by (2b) of Theorem 3.3, the answer to the first question is affirmative, while the answer to the second is negative.

If J is an integral domain with identity, then Gilmer and J. Ohm in [23] say that J *has the* QR-*property* (or that J *is a* QR-*domain*) if each overring of J is a quotient ring of J. A QR-domain is a Prüfer domain, and a Prüfer domain J has the QR-property if and only if the radical of each finitely generated ideal of J is the radical of a principal ideal of J [41]. In [18], Gilmer and Heinzer generalized the notion of a QR-domain to that of a QQR-*domain;* the defining property of such domains J is that each overring of J is an intersection of quotient rings of J. A Prüfer domain is a QQR-domain [10], and the integral closure of a QQR-domain is a Prüfer domain [18]. Moreover, a QQR-domain is a GQR-domain [28]. Our next two results give conditions under which D_1 is a QQR-domain or a QR-domain.

3.5. THEOREM. *The following conditions are equivalent.*

(1) D_1 *is a* QQR-*domain.*

(2) *Either* (a) D *is a* QQR-*domain with quotient field* K, *or* (b) D *is a field, there are no proper intermediate fields between* D *and* K, *and* M *is unbranched.*

Proof. As in Theorem 3.3, we prove that if (1) holds and (2a) fails, then (2b) is valid. It is straightforward to prove that (1) implies that D is a QQR-domain. Hence D does not have quotient field K. Since a QQR-domain is a GQR-domain, Theorem 3.3 implies that D is a field, there are no fields properly between D and K, and M is idempotent. Since D_1 has the QQR-property, and since V is not a quotient ring of D_1, part (g) of Theorem 2.1 implies that $V = \bigcap_\alpha (D_1)_{P_\alpha}$ for some family $\{P_\alpha\}$ of prime ideals of D_1 (and hence of V) properly contained in M. Consequently, V is the intersection of its set of proper overrings; this implies, however, that M is unbranched, for otherwise, V_P is the unique minimal overring of V, where P is the intersection of the set of M-primary ideals of V (see [15, Theorem 14.3(e), p. 173]).

It is clear that D_1 is a QQR-domain if the conditions of (2a) are satisfied, and Theorem 3.3 of [18] implies that D_1 is a QQR-domain if the conditions of (2b) are satisfied.

3.6. THEOREM. *The domain* D_1 *has the* QR-*property if and only if* D *has the* QR-*property and* D *has quotient field* K.

Theorem 3.6 follows easily from Theorem 3.1 and the following special case of Theorem 3.2.

3.7. COROLLARY. *Let* $N = \{n_\alpha\}$ *be a multiplicative system in* D_1.

(1) *If* N *does not meet* M, *let* $n_\alpha = a_\alpha + m_\alpha$, *where* $a_\alpha \in D - \{0\}$ *and* $m_\alpha \in M$ *for each* α. *Then* $N_1 = \{a_\alpha\}$ *is a multiplicative system in* D, *and* $(D_1)_N = D_{N_1} + M$.

(2) *If* N *meets* M, *let* $N' = N \cap M$. *Then* N' *is a multiplicative system in* D_1, *and* $(D_1)_N = (D_1)_{N'} = V_{N'}$.

Since an unbranched prime ideal of a Prüfer domain is idempotent [39, Theorem 3.4], but not conversely, Theorems 3.3 and 3.5 show that the GQR-property does not imply the QQR-property. Heinzer has already made this observation in [28, p. 144].

If A is an ideal of the commutative ring R such that $A^n \neq (0)$ for each positive integer n, then $\mathcal{S} = \{A^n\}_{n=1}^{\infty}$ is a generalized multiplicative system in R. The generalized quotient ring $R_{\mathcal{S}}$ is what Nagata in [37] calls *the transform of* A; we use the notation T(A), or $T_R(A)$, for the transform of A. Nagata considered transforms in connection with Hilbert's fourteenth problem. Other authors have found the concept to be a useful tool in diverse situations; see, for example, [42], [12, p. 334], [23, p. 100], [17, pp. 282, 297], [15, Section 22]. Other recent papers [5], [21], [26], [3] have been devoted largely to the development of a general theory of ideal transforms. Our next result is a second special case of Theorem 3.2.

3.8. COROLLARY. *Let* A *be a nonzero ideal of* D_1. *If* A *properly contains* M – *say* $A = B + M$, *where* B *is a nonzero ideal of* D – *then* $T_{D_1}(A) = T_D(B) + M$. *If* $A \subseteq M$, *then* $T_{D_1}(A) = T_V(AV)$.

In [6], Brewer and Gilmer say that an integral domain J with identity has property (T) if each overring is an ideal transform; they prove that for Noetherian domains or Krull domains, property (T) is equivalent to the condition that J is a semi-local PID. Corollary 3.8 and our proofs of Theorems 3.3 and 3.5 establish the following theorem.

3.9. THEOREM. *The following conditions are equivalent.*

(1) D_1 *has property* (T).

(2) V *has property* (T), *and either* (a) D *has property* (T), *and* K *is the quotient field of* D, *or* (b) D *is a field, there are no fields properly between* D *and* K, *and* M *is idempotent.*

Brewer and Gilmer [6] raise the following three questions about an integral domain J with property (T).

1. Does each overring have property (T)?

2. Is J semi-quasi-local?

3. Is the integral closure of J a Prüfer domain?

Theorem 3.9 shows that if J is a domain of the type described in part (2b) of Theorem 3.9, then the answer to each of these three questions is affirmative.

In [21], Gilmer and J. A. Huckaba consider commutative rings R satisfying various forms of the transform formula $T_R(AB) = T_R(A) + T_R(B)$. If this formula holds for all ideals A and B of R, then R is a T_1-*ring;* if it holds for all finitely generated ideals A and B of R, then R is a T_2-*ring;* if it holds for all principal ideals A and B of R, then R is a T_3-*ring*. The motivation for considering such rings comes from the significance of this formula in the context of intersections of valuation rings, in the case where R is a Krull domain (see [37, p. 59], [21, Section 5]). A Prüfer domain is a T_2-domain, but need not be a T_1-domain. If J is a Noetherian domain with identity, then the conditions (T_1), (T_2), and (T_3) are equivalent in J, and they are satisfied if and only if the dimension of J is at most one.

If A and B are ideals of D_1 and if B is contained in M, then $T(AB) = T(A) + T(B)$. If $A \supseteq M$, this follows from Proposition (1e) of [21]; if $A \subset M$, then by Corollary 3.8, $T(AB) = T(ABV)$, $T(A) = T(AV)$, and $T(B) = T(BV)$. But $T(ABV) = T(AV) + T(BV)$, since $AV \subseteq BV$, or vice versa. It follows that D_1 is a T_i-domain, for i = 1, 2, or 3, if and only if $T((A + M)(B + M)) = T(A + M) + T(B + M)$ for all nonzero ideals, all nonzero finitely generated ideals, or all nonzero principal ideals A and B of D. Since $(A + M)(B + M) = AB + M$, and since the equalities $T(AB + M) = T(AB) + M$, $T(A + M) = T(A) + M$, and $T(B + M) = T(B) + M$ are valid by Corollary 3.8, we have proved the following theorem.

3.10. THEOREM. *Let* i = 1, 2, 3; *then* D_1 *is a* T_i-*domain if and only if* D *is a* T_i-*domain.*

In [22], Gilmer and Huckaba have passed to a more general consideration of Δ-*rings*, which are defined by the property that the set of overrings is closed under addition. In fact, if R is a subring of the ring S, then S *is a* Δ-*extension of* R if the set $\mathscr{T}$ of subrings of S containing R is closed under addition. Hence R is a Δ-ring if T, the total quotient ring of R, is a Δ-extension of R. Gilmer and Huckaba prove (Corollary 1 of [22]) that if S is a field, and if S is a Δ-extension of its subring R with identity, then either R is a field or R has quotient field S. Moreover, if R is a field, then S is a Δ-extension of R if and only if the set of intermediate fields is linearly ordered by inclusion. From these remarks and from Theorem 3.1, our next result follows easily; compare with Proposition 9 of [22].

3.11. THEOREM. *The following conditions are equivalent.*

(1) D_1 *is a* Δ-*domain.*

(2) K *is a* Δ-*extension of* D.

(3) *Either* (a) D *is a* Δ-*domain with quotient field* K, *or* (b) D *is a field and the set of subfields of* K *containing* D *is linearly ordered by inclusion.*

An integral domain J with identity is a GCD-*domain* if each pair of nonzero elements of J has a greatest common divisor in J. The terminology "GCD-domain" is that of I. Kaplansky in [33]; the terminology of N. Bourbaki [4, p. 86] is "pseudo-bezoutian" (see also [19]), and in [9], P. M. Cohn refers to such domains as "HCF-rings" (this is also the terminology of Gilmer in [16]). In [30, p. 65], P. Jaffard gives four equivalent forms of the condition "J is a GCD-domain," namely (1) each pair of nonzero elements of J has a least common multiple in J; (2) the set of principal ideals of J is closed under finite intersection; (3) each divisorial ideal of J of finite type is principal; (4) the group of divisibility of J is lattice-ordered. We determine in Theorem 3.13 conditions under which D_1 is a GCD-domain. Our proof of Theorem 3.13 requires a lemma.

3.12. LEMMA. *Let* x *be a nonzero element of* K, *and let* m *be an element of* M. *Then* $\{x + m\} D_1 = xD_1 = xD + M$.

Proof. Clearly, $(x + m)/x = 1 + (m/x)$, where $m/x \in M$. Hence $(x + m)/x$ is a unit of D_1, and $\{x + m\} D_1 = xD_1$. Since x is a unit of V, $xM = M$. Therefore $xD_1 = x(D + M) = xD + xM = xD + M$.

3.13. THEOREM. *The following conditions are equivalent.*

(1) D_1 *is a* GCD-*domain.*

(2) D *is a* GCD-*domain with quotient field* K.

Proof. We observe that if J is an integral domain with identity and with quotient field k, then J is a GCD-domain if and only if the fractional ideal $J \cap xJ$ is principal for each nonzero element x of k. This is true since

$$aJ \cap bJ = a(J \cap a^{-1}bJ)$$

for all nonzero elements a and b of k.

(1) → (2): Lemma 3.12 implies that if x is a nonzero element of K, then $xD_1 = xD + M$. Hence the ideal $xD_1 \cap D_1 = (xD \cap D) + M$ is principal, as a fractional ideal of D_1. Since $D \subset K$, part (k) of Theorem 2.1 implies that M is not a principal ideal of D_1. We conclude that $xD \cap D \neq (0)$, and this implies that x belongs to the quotient field of D. Moreover, Lemma 3.12 implies that $xD \cap D$ is a principal fractional ideal of D. Therefore, D is a GCD-domain with quotient field K.

(2) → (1): We prove that $D_1 \cap xD_1$ is principal as a fractional ideal of D_1, for each nonzero element x in L. By Theorem 3.1, xD_1 compares with V under inclusion. If xD_1 contains V, then $D_1 \cap xD_1 = D_1$ is principal. If $xD_1 \subseteq V$, then $xD_1 = yD + M$ for some nonzero element y of K, and therefore

$$D_1 \cap xD_1 = (D \cap yD) + M;$$

since D is a GCD-domain,

$$(D \cap yD) + M = zD + M = zD_1$$

for some nonzero element z of K. Therefore D_1 is a GCD-domain.

Joe Mott has pointed out to us that Theorem 3.13 also follows from results of J. Ohm [40] and Jaffard [31] concerning groups of divisibility. In fact, Theorem 3.2 of [40] implies that the group of divisibility G_1 of D_1 is a lexicographic extension of H by G, where H is the group of divisibility of V (that is, the value group of V), and G is the group of divisibility of K with respect to D (that is, the multiplicative group of nonzero principal D-submodules of K, where $Dx \leq Dy$ if and only if $y/x \in D$). This means, by definition, that there exists an exact sequence

$$0 \to G \xrightarrow{\alpha} G_1 \xrightarrow{\beta} H \to 0,$$

where α and β are order homomorphisms, such that

$$G_1^+ = \{g_1 \in G_1 \mid \beta(g_1) > 0 \text{ or } g_1 \in \alpha(G^+)\}.$$

In [31, pp. 204-205], Jaffard proves that G_1 is lattice-ordered if and only if H is totally ordered and G is lattice-ordered; Theorem 3.13 is the ring-theoretic statement of this result. Proposition 3.4 of [40] implies that G_1 is the lexicographic direct sum of G and H if and only if the following condition (ρ) is satisfied.

(ρ) There exists a set $\{x_\lambda\}_{\lambda \in \Lambda}$ of representatives of the set of nonzero fractional ideals of V in L such that if α, β, γ are in Λ and $x_\alpha x_\beta V = x_\gamma V$, then $x_\alpha x_\beta / x_\gamma$ is a unit of D_1.

4. DIVISORIAL IDEALS OF D_1

Let J be an integral domain with identity and with quotient field k, and let $\mathcal{F}$ be the family of nonzero fractional ideals of J. If $F \in \mathcal{F}$, then

$$\bigcap \{xJ \mid x \in k,\ F \subseteq xJ\}$$

is a fractional ideal of J, which we denote by F_v; we call F_v the v-*ideal* or the *divisorial ideal associated with* F. If $F = F_v$, we say that F is a v-*ideal* or a *divisorial ideal*. A divisorial ideal F is *of finite type* if $F = G_v$ for some finitely generated fractional ideal G of J. A basic development of properties of the v-operation and divisorial ideals can be found in [43, Chapter 1], [4, Section 1], and [15, Section 28]. In this section, we examine the v-operation on D_1.

4.1. THEOREM. *Let* A *be a fractional ideal of* D, *and let* $B = A + M$. *If* $A \neq (0)$, *then* $B_v = A_v + M$; *on the other hand,* M *is a divisorial ideal of* D_1 *— that is,* $M = M_v$.

Proof. By Theorem 3.1, each fractional ideal of D_1 compares with V under inclusion. Since B is contained in a principal fractional ideal of D_1 properly contained in V, it follows that $B_v = \bigcap \{xD_1 \mid B \subseteq xD_1 \subseteq V\}$. From this and Lemma 3.12, we deduce that

$$B_v = \bigcap \{xD + M \mid x \in K,\ A \subseteq xD\} = \left[\bigcap \{xD \mid A \subseteq xD\}\right] + M.$$

Since $A \neq (0)$, we observe that the inclusion $A \subseteq xD$ implies that x belongs to the quotient field k of D. Therefore,

$$B_v = \left[\bigcap \{xD \mid x \in k,\ A \subseteq xD\}\right] + M = A_v + M.$$

If D is not a field, then $\bigcap \{xD \mid x \neq 0\} = (0)$, and $M = \bigcap \{xD_1 \mid x \in D - \{0\}\}$, so that M is divisorial. If D is a field, then for $x \in K - D$, $Dx \cap D = (0)$, so that $D_1 x \cap D_1 = M$, and again M is divisorial.

4.2. LEMMA. *Let* W *be a valuation ring with quotient field* k *and maximal ideal* P, *and let* A *be a nonzero fractional ideal of* W.

(1) P *is divisorial if and only if* P *is principal.*

(2) A *is not divisorial if and only if* P *is not principal and* $A = bP$ *for some* b *in* k.

Proof. If P is principal, then P is divisorial. If P is not principal, then each principal fractional ideal of W containing P contains W, and hence $P_v = W$. This proves (1).

It is clear from (1) that A is not divisorial if it is of the form bP, where P is not principal. If A is not divisorial, then we choose b in $A_v - A$. Then $A \subset (b) \subseteq A_v \subseteq (b)$, and hence $A_v = (b)$. Since $A \subset (b)$, $A \subseteq bP$; but since there are no ideals properly between A and (b), the reverse inclusion also holds. Therefore, $A = bP$, and A is not divisorial, so that P is not principal.

The second author discussed Lemma 4.2 and its proof with William Heinzer several years ago, and (1) appears as Lemma 5.2 of [27]; but statement (2) of Lemma 4.2 is apparently not in the literature.

4.3. THEOREM. *Let* A *be a nonzero ideal of* D_1 *properly contained in* M.

(1) *If* AV *is not principal as an ideal of* V, *then* A *is divisorial as an ideal of* D_1.

(2) *If* AV = cV *is principal as an ideal of* V, *where* $c \in A$, *then* A = Wc + Mc *for some* D-*submodule* W *of* K *such that* $D \subseteq W \subset K$; *if* W *is not a fractional ideal of* D, *then* $A \subset A_v = cV$; *otherwise*, $A_v = cW_v + cM$.

Proof. In (1), we consider two cases. Note that by part (n) of Theorem 2.1, A is an ideal of V.

Case I: A is not divisorial as an ideal of V.

Lemma 4.2 shows that in this case, A = bM for some b in L. Since M is a divisorial ideal of D_1, by Theorem 4.1, it follows that A is also divisorial.

Case II: A is divisorial as an ideal of V.

Thus $A = \bigcap\{xV \mid x \in L,\ A \subseteq xV\}$, and because A is not principal as an ideal of V, the inclusion $A \subseteq xV$ implies that $A \subset xV$. Therefore $A \subseteq M \cdot xV = xM \subseteq xD_1$ for each x in L such that $A \subseteq xV$. It follows that $A = \bigcap\{xD_1 \mid x \in L,\ A \subseteq Vx\}$, and A is divisorial.

To prove (2), note that A/c = W + M and hence $A_v = c(A/c)_v = c(W + M)_v$. If W is a fractional ideal of D, it follows from Theorem 4.1 that $(W + M)_v = W_v + M$. If W is not a fractional ideal of D, let F be a principal fractional ideal of D_1 containing W. By Theorem 3.1, F either contains V or is contained in it. Lemma 3.12 implies that if $F \subseteq V$, then F = Dx + M for some x in K, and hence $D \subseteq W \subseteq Dx$. But this implies that x belongs to the quotient field of D, and hence W is a fractional ideal of D, contrary to hypothesis. We conclude that if a principal fractional ideal of D_1 contains W + M, then it also contains V. Hence $(W + M)_v = V_v$, and in order to complete the proof we need only show that $V_v = V$. Let t be an element of L - V, and let s be an element of K - D. Then $s \notin D_1$ implies that $1 \notin s^{-1}D_1$, and hence that $t \notin ts^{-1}D_1$. Moreover, $ts^{-1} \notin V$, so that $V \subseteq ts^{-1}D_1$, and $t \notin V_v$.

In [27], Heinzer has considered integral domains J with identity such that each nonzero ideal of J is divisorial. It follows easily from Theorem 4.3 that the domain D_1 has this property if and only if each D-submodule W of K satisfying the condition $D \subseteq W \subset K$ is a divisorial fractional ideal of D. Taking into account the cases where D is or is not a field, we obtain the following corollary to Theorem 4.3.

4.4. COROLLARY. *The following conditions are equivalent.*

(1) *Each nonzero fractional ideal of* D_1 *is divisorial.*

(2) *Either* (a) D *has quotient field* K *and each* D-*submodule* W *of* K *such that* $D \subseteq W \subset K$ *is a divisorial fractional ideal of* D, *or* (b) D *is a field and the degree of* K *over* D *is two.*

The condition on D-submodules of K given in (2a) is not definitive; we attempt to rectify this situation in Theorem 4.5.

4.5. THEOREM. *Let* J *be an integral domain with identity and with quotient field* $k \neq J$, *and let* $\mathcal{S} = \{W_\lambda\}$ *be the family of nontrivial valuation overrings of* J. *The following conditions are equivalent.*

(1) *Each* J-*submodule* S *of* k *such that* $J \subseteq S \subset k$ *is a fractional ideal of* J.

(2) *Each* W_λ *is a fractional ideal of* J.

(3) *The conductor of* J *in* W_λ *is nonzero for each* W_λ *in* $\mathcal{S}$.

(4) *The conductor of* J *in* W_λ *is nonzero for some* W_λ *in* $\mathcal{S}$.

Proof. The implications (1) → (2), (2) → (3), and (3) → (4) are clear. To prove that (4) implies (1), let T be a J-submodule of k such that $J \subseteq T \subset k$, and let $W \in \mathcal{S}$ be such that the conductor of J in W is nonzero. If c is a nonzero element of the conductor of J in W, then $cWT \subseteq JT \subseteq T \subset k$, and hence $WT \neq k$. If $x \in k - WT$, then $x \notin W$, since $1 \in T$. Hence $x^{-1} \in W$ and $WT \subseteq Wx$, so that $x^{-1}WT \subseteq W$, and $cx^{-1}WT \subseteq cW \subseteq J$, where cx^{-1} is a nonzero element of J. Therefore WT and T are fractional ideals of J, and (1) holds.

We remark that condition (1) of Theorem 4.5 implies that each nonzero J-submodule $S \neq k$ of k is a fractional ideal of J. Thus, if s is a nonzero element of S and if $s = a/b$, where $a, b \in J - \{0\}$, then a is a nonzero element of $S \cap J$. Since $a \circ a^{-1} S = S \neq k$, the set $a^{-1} S$ is a J-submodule of k containing J and distinct from k; therefore (1) implies that $a^{-1} S$ is a fractional ideal of J, and consequently, $S = a \circ a^{-1} S$ is also a fractional ideal of J.

Kaplansky, in [33, p. 37], departs from more common terminology and defines, for an integral domain J with identity, a fractional ideal of J to be a J-submodule S of k, the quotient field of J. Thus Kaplansky's fractional ideals are not fractional ideals in the usual sense (in which the requirement "$xS \subseteq J$ for some nonzero element x of J" is imposed), and Theorem 4.5 can be interpreted as giving conditions under which the two definitions, except for k itself, coincide. (If $J \neq k$, then k is not a fractional ideal of J under the ordinary definition.)

5. DIMENSION THEORY OF $D_1[X_1, \cdots, X_n]$

In this section, we obtain a formula for the dimension of the polynomial ring $D_1[X_1, \cdots, X_n]$ in terms of the dimension of $D[X_1, \cdots, X_n]$. To obtain our formula, we need a generalization of the following result, due to Arnold [1, Theorem 5].

5.1. THEOREM. *Let* J *be a finite-dimensional integral domain with identity and with quotient field* k, *and let* m *be a positive integer. Then*

$$\dim J[X_1, \cdots, X_m] = m + \sup\{\dim J[t_1, \cdots, t_m] \mid \{t_i\}_1^m \subseteq k\}.$$

The generalization of Arnold's theorem that we need is to the case where $\{t_i\}_1^m$ is a subset of an extension field of k.

5.2. LEMMA. *Let* J *be an integral domain with identity and with quotient field* k, *let* E *be an extension field of* k *algebraic over* k, *and let* $\{t_i\}_1^m$ *be a finite subset of* E. *If* P *is the kernel of the homomorphism* $f(X_1, \cdots, X_m) \to f(t_1, \cdots, t_m)$ *of* $J[X_1, \cdots, X_m]$ *onto* $J[t_1, \cdots, t_m]$, *then* P *has height* m *in* $J[X_1, \cdots, X_m]$.

Proof. The proof is identical to the proof of Lemma 1 of [1], with the following modifications. The ideal Q_i is nonzero because t_i is algebraic over J, and $Q_i[X_{i+1}] \subset Q_{i+1}$ because t_{i+1} is algebraic over J.

5.3. THEOREM. *Let* J *be a finite-dimensional integral domain with identity and with quotient field* k, *let* E *be an extension field of* k, *and let* m *be a positive integer. Then*

$$\sup\{\dim J[t_1, \cdots, t_m] \mid \{t_i\}_1^m \subseteq E\}$$
$$= \dim J[X_1, \cdots, X_m] - \sup\{m - \text{tr.d.}(E/k), 0\}.$$

Proof. Since $J[t_1, \cdots, t_m]$ is isomorphic to a residue class ring of $J[X_1, \cdots, X_m]$,

$$\dim J[t_1, \cdots, t_m] \leq \dim J[X_1, \cdots, X_m]$$

for each subset $\{t_i\}_1^m$ of E. Thus the formula is valid if $\text{tr.d.}(E/k) \geq m$. We establish the formula in the case where E/k is algebraic. Using Theorem 5.1, we therefore seek to prove that

$$\sup\{\dim J[t_1, \cdots, t_m] \mid \{t_i\}_1^m \subseteq E\}$$
$$\leq \sup\{\dim J[t_1, \cdots, t_m] \mid \{t_i\}_1^m \subseteq k\} = \dim J[X_1, \cdots, X_m].$$

Thus we take a subset $\{u_i\}_1^m$ of E such that the dimension of $J[u_1, \cdots, u_m]$ is as large as possible. Then $J[u_1, \cdots, u_m] \simeq J[X_1, \cdots, X_m]/P$, where P is the kernel of the homomorphism $f(X_1, \cdots, X_m) \to f(u_1, \cdots, u_m)$. By Lemma 5.2, P has height m, and hence $\dim J[u_1, \cdots, u_m] \leq \dim J[X_1, \cdots, X_m] - m$, as we wished to prove.

Finally, we consider the case where $\text{tr.d.}(E/k) = d$, with $0 < d < m$. Let $\{y_i\}_1^d$ be a transcendence basis for E over k. Then $E/k(y_1, \cdots, y_d)$ is algebraic, and hence

$$\sup\{\dim J[y_1, \cdots, y_d][t_1, \cdots, t_{m-d}] \mid \{t_i\}_1^{m-d} \subseteq E\}$$
$$= \dim J[X_1, \cdots, X_m] - (m - d) = \dim J[X_1, \cdots, X_m] - \sup\{m - d, 0\}$$
$$\leq \sup\{\dim J[u_1, \cdots, u_m] \mid \{u_i\}_1^m \subseteq E\}.$$

On the other hand, if $\{v_i\}_1^m \subseteq E$, and if the labeling is such that $\{v_i\}_1^s$ is a transcendence basis for $k(v_1, \cdots, v_m)$ over k, then

$$\dim J[v_1, \cdots, v_m] = \dim J[v_1, \cdots, v_s][v_{s+1}, \cdots, v_m]$$
$$\leq \sup\{\dim J[v_1, \cdots, v_s][w_1, \cdots, w_{m-s}] \mid \{w_i\}_1^{m-s} \subseteq k(v_1, \cdots, v_m)\}$$
$$= \dim J[X_1, \cdots, X_m] - (m - s) \leq \dim J[X_1, \cdots, X_m] - (m - d).$$

Consequently,

$$\sup\{\dim J[t_1, \cdots, t_m] \mid \{t_i\}_1^m \subseteq E\} \leq \dim J[X_1, \cdots, X_m] - (m - d),$$

and equality holds, as we wished to show.

The equality

$$\sup\{\dim J[t_1, \cdots, t_m] \mid \{t_i\}_1^m \subseteq E\} = \sup\{\dim J[t_1, \cdots, t_m] \mid \{t_i\}_1^m \subseteq k\},$$

in the case where E/k is algebraic in Theorem 5.3, follows from Theorem 3 of [1]. We have not cited Theorem 3 of [1] in our proof of Theorem 5.3, because the proof of this result, as given in [1], is incomplete (but assertion of Theorem 3 of [1] is correct). The statement in [1, p. 315] that

$$\dim J[s_1, \cdots, s_m] = \dim J[1/d_1, \cdots, 1/d_m]$$

lacks justification; since $J[1/d_1, \cdots, 1/d_m]$ need not be a subring of $J[s_1, \cdots, s_m]$, Arnold can only claim that

$$\dim J[s_1, \cdots, s_m, 1/d_1, \cdots, 1/d_m] = \dim J[s_1, \cdots, s_m],$$

and the validity of the inequality

$$\dim J[s_1, \cdots, s_m] \leq \dim J[s_1, \cdots, s_m, 1/d_1, \cdots, 1/d_m]$$

is in question.

5.4. THEOREM. *Let* m *be a positive integer, let* k *be the quotient field of* D, *and let* d *be the transcendence degree of* K *over* k (d may be infinite). *Then*

$$\dim D_1[X_1, \cdots, X_m] = \dim V + \dim D[X_1, \cdots, X_m] + \inf\{m, d\}.$$

Proof. By Theorem 5.1,

$$\dim D_1[X_1, \cdots, X_m] = m + \sup\{\dim D_1[t_1, \cdots, t_m] \mid \{t_i\}_1^m \subseteq L\},$$

and Theorem 3.1 implies that

$$\sup\{\dim D_1[t_1, \cdots, t_m] \mid \{t_i\}_1^m \subseteq L\} = \sup\{\dim D_1[t_1, \cdots, t_m] \mid \{t_i\}_1^m \subseteq K\}.$$

Moreover, if $\{t_i\}_1^m \subseteq K$, then $D_1[t_1, \cdots, t_m] = D[t_1, \cdots, t_m] + M$, and by part (f) of Theorem 2.1, $\dim(D[t_1, \cdots, t_m] + M) = \dim D[t_1, \cdots, t_m] + \dim V$. Therefore

$$\begin{aligned}\dim D_1[X_1, \cdots, X_m] &= m + \sup\{\dim D[t_1, \cdots, t_m] + \dim V \mid \{t_i\}_1^m \subseteq K\}\\ &= m + \dim V + \sup\{\dim D[t_1, \cdots, t_m] \mid \{t_i\}_1^m \subseteq K\}\\ &= m + \dim V + \dim D[X_1, \cdots, X_m] - \sup\{m - d, 0\}\\ &= m + \dim V + \dim D[X_1, \cdots, X_m] + \inf\{d - m, 0\}\\ &= \dim V + \dim D[X_1, \cdots, X_m] + \inf\{d, m\},\end{aligned}$$

and this completes the proof of Theorem 5.4.

If R is a commutative ring with identity, and if R is of finite dimension n_0, then associated with R we have the sequence $\{n_i\}_{i=0}^\infty$, where $n_i = \dim R[X_1, \cdots, X_i]$ for each i; we shall call $\{n_i\}$ *the dimension sequence for* R. We shall call the sequence $\{n_1 - n_0, n_2 - n_1, \cdots\}$ *the difference sequence for* R. Theorem 5.4 is related to the problem of determining the sequences of positive integers that are realizable as the dimension sequence of a ring. More precisely, we have the following result.

5.5. COROLLARY. *Let* $\{n_i\}_0^\infty$ *be the dimension sequence for* D, *where* $n_0 < \infty$, *and let* $\{d_i\}_1^\infty$ *be the difference sequence for* D. *Let* k *be the quotient field of* D, *and assume that the dimension of* V *has a finite value* w. *Then the dimension sequence for* D_1 *and the difference sequence for* D_1 *are, respectively,*

(1) $\{w + n_i + i\}_{i=0}^\infty$ *and* $\{d_1 + 1, d_2 + 1, \cdots\}$ *if* tr. d. (K/k) *is infinite,*

(2) $\{w + n_i\}_0^\infty$ *and* $\{d_1, d_2, \cdots\}$ *if* tr. d. (K/k) = 0,

(3) $\{w + n_0, w + n_1 + 1, \cdots, w + n_j + j, w + n_{j+1} + j, w + n_{j+2} + j, \cdots\}$ *and* $\{d_1 + 1, \cdots, d_j + 1, d_{j+1}, d_{j+2}, \cdots\}$ *if* $\text{dr.d.}(K/k) = j$, *where* $0 < j < \infty$.

Numerous conditions satisfied by the dimension sequence $\{n_i\}_0^\infty$ and the difference sequence $\{d_i\}_1^\infty$ for a ring are known. For example, A. Seidenberg showed in [44] that $n_i + 1 \leq n_{i+1} \leq 2n_i + 1$ for each i; in particular, $n_0 + 1 \leq n_1 \leq 2n_0 + 1$. Moreover, Seidenberg [45] proves that if m_0 and m_1 are nonnegative integers such that $m_0 + 1 \leq m_1 \leq 2m_0 + 1$, then there exists an integrally closed domain J such that $\dim J = m_0$ and $\dim J[X_1] = m_1$. In [32, p. 17], Jaffard proves that there are at most $t + 1$ prime ideals in a chain of primes of $R[X_1, \cdots, X_t]$ lying over a fixed prime ideal of R, and hence $t + n_0 \leq n_t \leq t(1 + n_0) + n_0$ for each positive integer t. In particular, if $n_0 = 0$, then $n_t = t$ for each t; that is, $\{0, 1, 2, \cdots\}$ is the only sequence with first term 0 that can be realized as the dimension sequence for a ring. In [32, p. 42], Jaffard proves that the difference sequence is eventually constant: $d_r = d_{r+1} = \cdots$ for some r, where $d_r \leq n_0 + 1$. Moreover, Jaffard proves that if R is an integral domain, then the difference sequence is eventually the constant 1 if and only if R has finite valuative dimension; this result also follows from Theorem 5.1 and results of [15, Section 25]. If $n_0 = 1$, then n_1 is 2 or 3. Seidenberg [44] proved that if R is an integral domain, then $n_1 = 2$ if and only if the integral closure of R is a Prüfer domain; he also proved that if $n_1 = 2$, then $n_i = i + 1$ for each positive integer i. It is easy to show that the same result is valid for a one-dimensional ring S with zero divisors — that is, if $\dim S[X_1] = 2$, then $\dim S[X_1, \cdots, X_n] = n + 1$ for positive integer n.

If J is a one-dimensional domain with identity such that $\dim J[X_1] = 3$, then $n + 2 \leq \dim D[X_1, \cdots, X_n] \leq 2n + 1$, for each positive integer n. Further, Seidenberg [45, Theorem 7] shows that if n and N are positive integers such that $n + 2 \leq N \leq 2n + 1$, then there exists a one-dimensional domain J_1 with identity such that $\dim J_1[X_1] = 3$ and $\dim J_1[X_1, \cdots, X_n] = N$. In [32, Chapitre III, Section 2], Jaffard improves Seidenberg's results by proving that if R is a one-dimensional ring with identity, then the dimension sequence $\{n_i\}$ is of one of the three forms

$$\{1, 2, 3, \cdots\}, \quad \{1, 3, 5, \cdots\}, \quad \{1, 3, 5, \cdots, 2t + 1, 2t + 2, 2t + 3, \cdots\}.$$

Moreover, it follows from Corollary 5.5 that each of these sequences can be realized as the dimension sequence for an integral domain with identity. It is clear that repeated use of Corollary 5.5 yields numerous sequences that can be realized as the dimension sequence for an integral domain J with identity; in particular, dim J and the eventual value d of the difference sequence for J can be chosen arbitrarily, subject to the condition $1 \leq d \leq \dim J + 1$. In closing, we remark that an example suggested by William Heinzer and presented by Arnold [1, p. 325] shows that for each positive integer n, the sequence $\{n, n + 1, \cdots, 2n - 1, 2n + 1, 2n + 2, 2n + 3, \cdots\}$ is the dimension sequence of an integral domain with identity. In this connection, if J is an n-dimensional domain with identity such that $\dim J[X_1, \cdots, X_t] = n + t$ for $1 \leq t \leq n$, then J has valuative dimension n, and $\dim J[X_1, \cdots, X_t] = n + t$ for each positive integer t [15, Theorem 25.10].

Addendum. Arnold and the second author have recently determined all sequences of nonnegative integers that can be realized as the dimension sequence of a ring.

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