APPLICATIONS OF MAPPING THEOREMS TO SCHWARTZ SPACES AND PROJECTIONS

Joseph Diestel and Robert H. Lohman

1. INTRODUCTION

S. A. Saxon [13] has shown that if E is a locally convex linear topological space (henceforth: an LCLT space) that is nuclear and X is an infinite-dimensional Banach space, then there exists an embedding of E into some power X^I of X. Saxon's theorem improves a result of A. Grothendieck (see [14, p. 102]) who had proved the theorem in the cases where $X = \ell_p$ for some p $(1 \leq p \leq \infty)$. This raises the question as to what LCLT spaces are likewise embeddable in a sufficiently high power of each infinite-dimensional Banach space.

In the first part of this paper, we show that each LCLT space E that is embeddable in a sufficiently high power of every infinite-dimensional Banach space is a Schwartz space. Actually, we do not need the full strength of embeddability in powers of every Banach space. In fact, embeddability in powers of any of a large number of pairs of Banach spaces (see Proposition 2 for a listing of the pairs) is sufficient to imply that E is a Schwartz space. Since c_0 occurs as one of the members of such a pair, and since every Schwartz space is embeddable in a sufficiently high power of c_0 (see D. J. Randtke [10]), our result may be near to characterizing Schwartz spaces. It remains open, for example, to determine whether for every p $(1 \leq p < \infty)$ every Schwartz space is embeddable in some power of ℓ_p.

The result on Schwartz spaces depends on Proposition 1, which concerns the factorization of mappings of subspaces of a product space into a normed linear space. Proposition 3 concerns normed linear subspaces of products of LCLT spaces, and it leads to some results on projections of LCLT spaces onto Banach spaces (see the next paragraph for a sketch, Section 3 for details).

It is often desirable to determine whether a subspace of a normed linear space X is the image of a continuous projection on X. This is important because the identification of such subspaces usually provides structural information about the space. A. Sobczyk's classical result in this direction states that if X is a separable normed linear space and Y is a linear subspace of X topologically isomorphic to c_0, then there exists a continuous projection of X onto Y (see [15]). A more recent result concerning projections, proved by H. P. Rosenthal in [11], asserts that if X and Y are closed, totally incomparable linear subspaces of a Banach space, then X + Y is closed. It then follows that X is complemented in X + Y.

The problem of determining the subspaces of an LCLT space that are images of continuous projections is more difficult. However, if the subspace in question is a Banach space, some interesting results can be obtained. We show that the results of Sobczyk and Rosenthal hold in a more general setting, namely, where the underlying space is an LCLT space. For instance, we prove that if E is a separable LCLT space and F is a linear subspace of E topologically isomorphic to c_0, then there exists a continuous projection of E onto F (Theorem 2). We also prove that if F is a Banach space such that F is complemented in every Banach space that

Received April 24, 1972.

Michigan Math. J. 20 (1973).

contains F, then for each LCLT space E that contains F, there is a continuous projection of E onto F (Theorem 3). Finally, it is shown that if F and G are totally incomparable Banach spaces contained in the LCLT space E, then F + G is a Banach space (Theorem 4).

Notation and Definitions. All LCLT spaces in this paper are assumed to be Hausdorff spaces. An LCLT space E is a *Schwartz space* (see [7]) whenever, corresponding to each Banach space F and each linear continuous map u: E → F, there exists a neighborhood V of 0 in E such that $\overline{u(V)}$ is compact in F. If E is a subspace of the product $\prod_{i \in I} E_i$ of LCLT spaces and σ is a finite subset of I, we denote by π_σ the natural projection of E into $\prod_{i \in \sigma} E_i$. If $E_i = G$ for all $i \in I$, we denote $\prod_{i \in I} E_i$ by G^I. If F is a Banach space, we write $F \in \mathscr{P}$ to say that F is complemented in every Banach space that contains F. For example, $\ell_\infty \in \mathscr{P}$. Also, if $F \in \mathscr{P}$ and G is topologically isomorphic to F, then $G \in \mathscr{P}$, by a result of A. Pełczyński [8, Proposition 1]. Finally, the LCLT spaces F and G are said to be *totally incomparable* if they have no topologically isomorphic infinite-dimensional subspaces.

2. SCHWARTZ SPACES

PROPOSITION 1. *Let* E *be an* LCLT *space, let* F *be a normed linear space, and let* u: E → F *be a continuous linear mapping. Suppose that* E *is a subspace of some power* G^I *of the* LCLT *space* G. *Then there exist a finite subset* σ *of* I, *a linear subspace* S *of* G^σ, *and continuous linear mappings* $u_1: E \to S$ *and* $u_2: S \to F$ *such that* $u = u_2 u_1$.

Proof. Let U denote the unit ball of F. There exists a family $\{U_i\}$ $(i \in I)$ of neighborhoods of 0 in G such that $U_i = G$ for all i not in some finite subset σ of I with $u\left(E \cap \left(\prod_{i \in I} U_i\right)\right) \subset U$. It is easily seen that if $x, y \in E$ and $\pi_\sigma(x) = \pi_\sigma(y)$, then $u(x) = u(y)$. Therefore, the linear mapping $u_2: \pi_\sigma(E) \to F$ defined by the equation $u_2(\pi_\sigma(x)) = u(x)$ is well-defined and continuous. We complete the proof by letting $S = \pi_\sigma(E)$ and $u_1 = \pi_\sigma$.

Remark. If in Proposition 1 we assume F and G to be Banach spaces, then we can assume S to be closed in the Banach space G^σ.

PROPOSITION 2. *Let* σ *and* τ *be finite sets. If* S *is a closed linear subspace of* X^σ, *then every continuous linear operator* u *from* S *to* Y^τ *is compact for the following pairs of Banach spaces* X, Y:

(1) $X = \ell_p$, $Y = \ell_q$ $(1 \le q < p < \infty)$;

(2) $X = L_p$, $Y = \ell_q$ $(1 \le q < p < \infty,\ 1 \le q < 2)$;

(3) $X = \ell_p$, $Y = L_q$ $(1 \le q < p < \infty,\ 2 < p)$;

(4) $X = c_0$; Y—*any Banach space containing no isomorphic copy of* c_0 ;

(5) $X = c_0$; Y—*any quasi-reflexive or weakly sequentially complete or reflexive Banach space;*

(6) $X = c_0$, $Y = \ell_p$ $(1 \le p < \infty)$;

(7) $X = c_0$, $Y = L_p$ $(1 \le p < \infty)$;

(8) X—*any Banach space each of whose separable subspaces has a separable dual space;* $Y = \ell_1$;

(9) $X = C(K)$, *where* K *denotes a compact, dispersed, topological Hausdorff space;* $Y = \ell_1$;

(10) X—*any reflexive Banach space;* $Y = \ell_1$.

Proof. The reason that $u: S \to Y^\tau$ is compact for the pair X, Y of (n) above is listed under (n) in the following arguments:

(1) Each of X and Y is isomorphic to its own square. Thus we may assume that $S \subset X$ and $Y^\tau = Y$. Now apply Theorem A2 of [12].

(2) and (3) imply the conclusion of compactness in the same manner as (1).

(4) Since X is isomorphic to its own square, we may assume that $S \subset X$. On the other hand, using results of [2], we can easily verify that Y^τ has the property of containing no subspace isomorphic to c_0. That (4) implies the conclusion now follows from Remark 4 on page 212 of [12].

(5) follows the argument of (4) and uses the basic facts about quasi-reflexive (see [3]), weakly sequentially complete (see [6]) or reflexive (see [6]) Banach spaces, none of which can contain isomorphs of c_0.

(6) and (7) imply the compactness of u because of the weak sequential completeness of the spaces Y, and by the same reasoning as in (5).

(8) implies compactness because of the following facts: If X is a Banach space each of whose separable subspaces has a separable dual, then (a) every closed subspace S of every finite power of X also has this property, and (b) every bounded sequence possesses a weak Cauchy subsequence; continuous linear maps between Banach spaces are weakly uniformly continuous (hence preserve weak Cauchy sequences), and weak Cauchy sequences in ℓ_1 are norm convergent (see [1]).

(9) implies compactness because of the main theorem of [9] and the fact that (8) implies compactness.

(10) is a consequence of (8) and the fact that every separable subspace of a reflexive Banach space has a separable dual space.

THEOREM 1. *Let* E *be an* LCLT *space that is topologically isomorphic to a subspace of both* X^I *and* Y^J, *where* X *and* Y *constitute one of the pairs of* (1) *to* (10) *of Proposition* 2. *Then* E *is a Schwartz space.*

Proof. We may assume that E is a subspace of both X^I and Y^J. Let u be a continuous linear map from E into the Banach space F. By two applications of Proposition 1, we can verify that there exist finite subsets σ and τ of I and J, respectively, and closed linear subspaces S_1 and S_2 of the Banach spaces X^σ and Y^τ, respectively, such that u is factorable in the form $u = u_3 u_2 u_1$, where

$$u_1: E \to S_1, \qquad u_2: S_1 \to S_2, \qquad u_3: S_2 \to F$$

are continuous linear mappings. If X and Y constitute one of the pairs of (1) to (10), then, by Proposition 2, u_2 is compact. Consequently, u is compact, so that E is a Schwartz space.

Remark. Theorem 1 shows that an LCLT space that is embeddable in a sufficiently high power of every infinite-dimensional Banach space is a Schwartz space. It is still an open question, however, whether an LCLT space with this property is nuclear.

3. PROJECTIONS ONTO BANACH SPACES

The following result of Diestel, Morris, and Saxon [5, Theorem 4.1] is useful in the study of the structure of varieties of locally convex spaces (see also [4]). It is essential to the remainder of this paper.

PROPOSITION 3. *If a normed space* E *is a subspace of the product* $\prod_{i \in I} E_i$ *of* LCLT *spaces, then there exists a finite subset* σ *of* I *such that the mapping* π_σ *is a topological isomorphism of* E *into* $\prod_{i \in \sigma} E_i$.

Proof. Let U denote the unit ball of E. Then U contains a set of the form $E \cap \left(\prod_{i \in I} U_i \right)$, where each U_i is a neighborhood of 0 in E_i and $U_i = E_i$ for all i not in some finite subset σ of I. We claim that π_σ has the property in the conclusion of Proposition 3. Since π_σ is linear and continuous, we need only show that π_σ is one-to-one and relatively open. If $\pi_\sigma(x) = 0$, then x is in every multiple of U, so that $x = 0$. On the other hand, $\left(\prod_{i \in \sigma} U_i \right) \cap \pi_\sigma(E) \subset \pi_\sigma(U)$, so that π_σ is relatively open.

Remark. It follows from the proof of the preceding proposition that if τ is a finite subset of I with $\tau \supset \sigma$, then π_τ is still a topological isomorphism of E into $\prod_{i \in \tau} E_i$.

THEOREM 2. *Let* E *be a separable* LCLT *space, and let* F *be a subspace of* E *such that* F *is topologically isomorphic to* c_0. *Then there exists a continuous projection of* E *onto* F.

Proof. As a separable LCLT space, E is topologically isomorphic to a subspace of a product $\prod_{i \in I} E_i$ of separable Banach spaces. Therefore, we may assume that E is a subspace of $\prod_{i \in I} E_i$. By Proposition 3, there exists a finite subset σ of I such that the mapping $\pi_\sigma : E \to \prod_{i \in \sigma} E_i$ is a topological isomorphism into $\prod_{i \in \sigma} E_i$ when restricted to F. Now $\prod_{i \in \sigma} E_i$ is a separable normed linear space, and $\pi_\sigma(F)$ is topologically isomorphic to c_0. Consequently, there exists a continuous projection P of $\prod_{i \in \sigma} E_i$ onto $\pi_\sigma(F)$. If $\pi_F = \pi_\sigma \mid F$, we can easily see that $Q = \pi_F^{-1} P \pi_\sigma$ is a continuous projection of E onto F.

Since every LCLT space is topologically isomorphic to a subspace of a product of normed linear spaces, we can use an argument similar to the proof of Theorem 2 to prove an analogue of Theorem 2, under the assumption that E is an LCLT space and $F \in \mathscr{P}$.

THEOREM 3. *Let* E *be an* LCLT *space, and let* F *be a subspace of* E *such that* $F \in \mathscr{P}$. *Then there exists a continuous projection of* E *onto* F.

As a final application of Proposition 3, we show that Rosenthal's theorem can be extended to the case where the underlying space is an arbitrary LCLT space.

THEOREM 4. *Let* E *be an* LCLT *space, and let* F *and* G *be subspaces of* E *such that* F *and* G *are Banach spaces. If* F *and* G *are totally incomparable, then* $F + G$ *is a Banach space.*

Proof. Since F and G are totally incomparable, $F \cap G$ is finite-dimensional. Hence we may assume that $F \cap G = \{0\}$. As before, we may assume that E is a subspace of a product $\prod_{i \in I} E_i$ of Banach spaces. By Proposition 3, there exist finite subsets σ_F and σ_G of I such that π_{σ_F} and π_{σ_G} are topological isomorphisms of F and G into $\prod_{i \in \sigma_F} E_i$ and $\prod_{i \in \sigma_G} E_i$, respectively. Let $\sigma = \sigma_F \cup \sigma_G$. By the remark following Proposition 3, the mapping $\pi_\sigma: E \to \prod_{i \in \sigma} E_i$ has the property that $\pi_\sigma \mid F$ and $\pi_\sigma \mid G$ are topological isomorphisms into $\prod_{i \in \sigma} E_i$. It follows that $\pi_\sigma(F)$ and $\pi_\sigma(G)$ are totally incomparable subspaces of the Banach space $\prod_{i \in \sigma} E_i$. By Rosenthal's result for Banach spaces, $\pi_\sigma(F) + \pi_\sigma(G)$ is a Banach space. Since $\pi_\sigma(F)$ and $\pi_\sigma(G)$ are totally incomparable, $\pi_\sigma(F) \cap \pi_\sigma(G)$ is finite-dimensional. Therefore, there exists a continuous projection P of $\pi_\sigma(F) + \pi_\sigma(G)$ onto $\pi_\sigma(F)$ that annihilates a closed, complementary subspace of $\pi_\sigma(F) \cap \pi_\sigma(G)$ in $\pi_\sigma(G)$. If $R = \pi_\sigma \mid F + G$, we can easily verify that $Q = \pi_F^{-1} PR$ is a continuous projection of $F + G$ onto F. It follows that $F + G$ is topologically isomorphic to the product $F \times G$, so that $F + G$ is a Banach space.

Acknowledgment. The authors wish to express their thanks to Professor S. A. Saxon for his comments and suggestions.

REFERENCES

1. S. Banach, *Théorie des opérations linéaires.* Monografje Matematyczne, Warsaw, 1932.
2. C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces.* Studia Math. 17 (1958), 151-164.
3. P. Civin and B. Yood, *Quasi-reflexive spaces.* Proc. Amer. Math. Soc. 8 (1957), 906-911.
4. J. Diestel, S. A. Morris, and S. A. Saxon, *Varieties of locally convex topological vector spaces.* Bull. Amer. Math. Soc. 77 (1971), 799-803.
5. ———, *Varieties of linear topological spaces.* Trans. Amer. Math. Soc. (to appear).
6. N. Dunford and J. T. Schwartz, *Linear operators. Part I: General theory.* Interscience Publishers, New York, 1958.
7. J. Horváth, *Topological vector spaces and distributions. Vol. I.* Addison-Wesley, Reading, Mass., 1966.
8. A. Pełczyński, *Projections in certain Banach spaces.* Studia Math. 19 (1960), 209-228.
9. A. Pełczyński and Z. Semadeni, *Spaces of continuous functions. III. Spaces* $C(\Omega)$ *for* Ω *without perfect subsets.* Studia Math. 18 (1959), 211-222.
10. D. J. Randtke, *A structure theorem for Schwartz spaces* (to appear).
11. H. P. Rosenthal, *On totally incomparable Banach spaces.* J. Functional Analysis 4 (1969), 167-175.
12. ———, *On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from* $L^p(\mu)$ *to* $L^r(\nu)$. J. Functional Analysis 4 (1969), 176-214.

13. S. A. Saxon, *Embedding nuclear spaces in products of an arbitrary Banach space*. Proc. Amer. Math. Soc. 34 (1972), 138-140.

14. H. H. Schaefer, *Topological vector spaces*. Macmillan, New York, 1966.

15. A. Sobczyk, *Projection of the space* (m) *on its subspace* (c_0). Bull. Amer. Math. Soc. 47 (1941), 938-947.

Kent State University
Kent, Ohio 44242