AN INVARIANT-SUBSPACE THEOREM

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1. INTRODUCTION

Throughout this paper, $\mathscr{H}$ will denote an infinite-dimensional, separable, complex Hilbert space, and $\mathscr{L}(\mathscr{H})$ will denote the algebra of all bounded linear operators on $\mathscr{H}$. Recall from [5] that an operator S in $\mathscr{L}(\mathscr{H})$ is *quasitriangular* if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of orthogonal projections of finite rank on $\mathscr{H}$, converging strongly to 1, such that

$$\| P_n S P_n - SP_n \| \to 0 . \tag{1}$$

Such a sequence $\{P_n\}$ satisfying (1) will be said to *implement* the quasitriangularity of S.

It is not known whether every quasitriangular operator has a nontrivial invariant subspace. The problem is important, because all quasinilpotent operators and all operators with compact imaginary part (acting on a separable space) are quasitriangular [4], [5]. Some invariant-subspace theorems concerning quasitriangular operators have been proved [1], [2], [3]. Perhaps the most interesting of these theorems is the result of W. B. Arveson and J. Feldman in [2], which asserts that if S is a quasitriangular operator in $\mathscr{L}(\mathscr{H})$ and there exists a sequence of polynomials $p_n(S)$ that converges in the norm topology to a nonzero compact operator, then S has a nontrivial invariant subspace.

The principal purpose of this note is to prove a generalization of this fundamental theorem. In order to state our main result, we introduce the following terminology and notation. If $T \in \mathscr{L}(\mathscr{H})$, we denote by $\mathscr{P}(T)$ the subalgebra of $\mathscr{L}(\mathscr{H})$ that is the uniform closure of the set of all polynomials in T. Furthermore, we denote by $\mathscr{R}(T)$ the subalgebra of $\mathscr{L}(\mathscr{H})$ that is the uniform closure of the set of all rational functions of T (a rational function of T is an operator of the form $p(T)[q(T)]^{-1}$, where p and q are polynomials). It is well known that $\mathscr{R}(T)$ coincides with the uniform closure of the algebra of all analytic functions of T. For this reason, a subspace $\mathscr{M}$ of $\mathscr{H}$ is called an *analytically invariant* subspace for T (see [6]) if it is invariant under every operator in $\mathscr{R}(T)$. In general, an invariant subspace $\mathscr{M}$ of an operator T need not be an analytically invariant subspace for T. In fact, it was shown in Lemma 2.2 of [6] that an invariant subspace $\mathscr{M}$ of T is analytically invariant for T if and only if the spectrum of $T \mid \mathscr{M}$ is contained in the spectrum of T.

Our principal result is the following theorem.

THEOREM 1.1. *Let* S *be a quasitriangular operator in* $\mathscr{L}(\mathscr{H})$, *and suppose that the algebra* $\mathscr{R}(S)$ *contains a nonzero compact operator. Then* S *has a nontrivial analytically invariant subspace.*

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The proof of this theorem requires some preliminary propositions, and thus we do not give it until Section 3. We remark now that it is an easy consequence of the maximum-modulus principle that if the spectrum of an operator T separates the plane, then the algebra $\mathscr{R}(T)$ is strictly larger than the algebra $\mathscr{P}(T)$. Thus Theorem 1.1 generalizes the fundamental theorem of Arveson and Feldman, because the compact operator involved may belong to $\mathscr{R}(T) \setminus \mathscr{P}(T)$. Of course, the second direction in which Theorem 1.1 improves the Arveson-Feldman theorem is that the invariant subspace produced by Theorem 1.1 is *analytically* invariant. In Section 4, we give a specific example that illustrates these improvements.

Finally, we mention that this paper was written in the Spring of 1971 and was presented at the Operator Theory Conference in Durham, New Hampshire, in June, 1971. Subsequently, the paper [7] of P. Meyer-Nieberg appeared, in which part of Theorem 1.1 is proved. Because of the differences in the techniques employed in our paper and in [7], we have chosen to publish the present paper in its original form.

2. INVERTIBLE QUASITRIANGULAR OPERATORS

Our program to prove Theorem 1.1 begins with a discussion of a certain class of invertible quasitriangular operators. An operator T in $\mathscr{L}(\mathscr{H})$ is said to be *triangular* (see [5]) if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite-rank projections on $\mathscr{H}$, converging strongly to 1, such that

$$P_n T P_n = T P_n \quad (n = 1, 2, \cdots) . \tag{2}$$

Such a sequence $\{P_n\}$ will be said to *implement* the triangularity of T.

Our first lemma is elementary.

LEMMA 2.1. *Let* T *be an invertible triangular operator on* $\mathscr{H}$, *and suppose that* $\{P_n\}$ *is a sequence of projections implementing the triangularity of* T. *Then* T^{-1} *is triangular, and* $\{P_n\}$ *also implements the triangularity of* T^{-1}. *Furthermore, if we set* $\mathscr{M}_n = P_n \mathscr{H}$ $(1 \leq n < \infty)$, *then for each* n, $T \mid \mathscr{M}_n$ *is invertible and* $(T \mid \mathscr{M}_n)^{-1} = T^{-1} \mid \mathscr{M}_n$.

The following theorem was discovered by P. R. Halmos in 1967, but it has thus far remained unpublished. We are grateful to Halmos for allowing us to include it here.

THEOREM 2.2. *Let* S *be a quasitriangular operator on* $\mathscr{H}$, *and let* $\{P_n\}$ *be a sequence of projections implementing the quasitriangularity of* S. *Then for every positive number* ε, *there exist a triangular operator* T, *a compact operator* C, *and a subsequence* $\{P_{n_k}\}$ *of the sequence* $\{P_n\}$ *such that* $S = T + C$, $\|C\| < \varepsilon$, *and* $\{P_{n_k}\}$ *implements the triangularity of* T. *In particular, if* S *is invertible, then* T *may be chosen to be invertible also.*

Proof. Choose a subsequence $\{P_{n_k}\}$ of the sequence $\{P_n\}$ such that

$$\sum_{k=1}^{\infty} \|S P_{n_k} - P_{n_k} S P_{n_k}\| < \varepsilon .$$

For $k = 1, 2, \cdots$, write $C_k = (1 - P_{n_k})S(P_{n_k} - P_{n_{k-1}})$, and set $P_{n_0} = 0$. Since the operator $P_{n_k} - P_{n_{k-1}}$ has finite rank, C_k has finite rank. Furthermore, if x belongs to the null space of $P_{n_{k-1}}$, then

$$C_k x = (1 - P_{n_k})S P_{n_k} x = (S P_{n_k} - P_{n_k} S P_{n_k})x ,$$

and if x belongs to the range of $P_{n_{k-1}}$, then $C_k x = 0$. It now follows easily that $\|C_k\| \leq \|S P_{n_k} - P_{n_k} S P_{n_k}\|$. Thus the series $\sum_k C_k$ converges in the uniform operator topology. If we write $C = \sum_k C_k$, then clearly $\|C\| < \varepsilon$, and since each C_k has finite rank, C is compact. Furthermore, the operator $T = S - C$ satisfies the relation $P_{n_k} T P_{n_k} = T P_{n_k}$ $(k = 1, 2, \cdots)$, and hence T is triangular. The assertion that T may be chosen to be invertible follows easily from the inequality $\|S - T\| < \varepsilon$ and the fact that the invertible operators in $\mathscr{L}(\mathscr{H})$ form an open set in the norm topology.

Before stating the next lemma, we introduce some notation that will be used several times in the sequel. Throughout the remainder of the paper, we shall frequently be concerned with a fixed increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite-rank projections that converges strongly to 1. If S is an operator in $\mathscr{L}(\mathscr{H})$ and such a sequence $\{P_n\}$ is under consideration, we shall write $S_{(n)}$ for the operator $P_n S \mid P_n \mathscr{H}$ (that is, $S_{(n)}$ is the compression of S to the subspace $P_n \mathscr{H}$). In particular, $1_{(n)}$ is the identity operator on the space $P_n \mathscr{H}$. It will also be convenient to have a notation for the operator $1 \mid (1 - P_n)\mathscr{H}$; thus we shall write $1'_{(n)} = 1 \mid (1 - P_n)\mathscr{H}$.

The following lemma is central to our purposes.

LEMMA 2.3. *Let* L *be a compact operator on* $\mathscr{H}$ *such that* $A = 1 + L$ *is invertible, and suppose that* $\{P_n\}$ *is an increasing sequence of finite-rank projections that converges strongly to* 1. *Then, for* n *sufficiently large,* $A_{(n)}$ *is invertible, and* $\|[A_{(n)}]^{-1} - [A^{-1}]_{(n)}\| \to 0$. *Furthermore, if* $\{B_n\}$ *is any sequence of invertible operators such that* $B_n \in \mathscr{L}(P_n \mathscr{H})$ $(1 \leq n < \infty)$ *and* $\|B_n - A_{(n)}\| \to 0$, *then also* $\|(B_n)^{-1} - [A_{(n)}]^{-1}\| \to 0$.

Proof. It is well known (see [3, Prop. 6.3]) that the sequence $\{1 + P_n L P_n\}$ converges to A in the uniform topology on $\mathscr{L}(\mathscr{H})$, and it follows that for n sufficiently large, $1 + P_n L P_n$ is invertible. Since $1 + P_n L P_n = A_{(n)} \oplus 1'_{(n)}$, we conclude that for n sufficiently large, $A_{(n)}$ is invertible. We have seen that $\|[A_{(n)} \oplus 1'_{(n)}] - A\| \to 0$; thus, by continuity of the inverse mapping,

$$\|[[A_{(n)}]^{-1} \oplus 1'_{(n)}] - A^{-1}\| \to 0 . \tag{3}$$

Since A^{-1} obviously has the form $A^{-1} = 1 + M$, where M is compact, a repetition of the first part of the argument above shows that

$$\|[[A^{-1}]_{(n)} \oplus 1'_{(n)}] - A^{-1}\| \to 0 . \tag{4}$$

It follows immediately from (3) and (4) that

$$\|[A_{(n)}]^{-1} - [A^{-1}]_{(n)}\| \to 0 .$$

To prove the last assertion of the lemma, observe that

$$\|[B_n \oplus 1'_{(n)}] - [A_{(n)} \oplus 1'_{(n)}]\| \to 0,$$

and thus, by continuity of the inverse mapping,

$$\|[(B_n)^{-1} \oplus 1'_{(n)}] - [[A_{(n)}]^{-1} \oplus 1'_{(n)}]\| \to 0,$$

from which it follows immediately that $\|(B_n)^{-1} - [A_{(n)}]^{-1}\| \to 0$.

The following theorem might well be regarded as the fundamental observation upon which the proof of Theorem 1.1 depends.

THEOREM 2.4. *Let* S *be an invertible quasitriangular operator on* $\mathcal{H}$, *and let* $\{P_n\}$ *be a sequence of projections implementing the quasitriangularity of* S. *Then* S^{-1} *is quasitriangular, and there exists a subsequence* $\{P_{n_k}\}$ *of the sequence* $\{P_n\}$ *with the properties*

(a) $\{P_{n_k}\}$ *implements the quasitriangularity of* S^{-1},

(b) *for* $1 \le k < \infty$, $S_{(n_k)}$ *is invertible, and*

(c) $\lim_{k\to\infty} \|[S_{(n_k)}]^{-1} - [S^{-1}]_{(n_k)}\| \to 0.$

Proof. Let $\varepsilon > 0$ be chosen so small that every R satisfying the condition $\|R - S\| < \varepsilon$ is invertible. By Theorem 2.2, there exist a subsequence $\{P_{n_k}\}$ of $\{P_n\}$ and a triangular operator T such that $\{P_{n_k}\}$ implements the triangularity of T and such that $C = S - T$ is compact and satisfies the condition $\|C\| < \varepsilon$. Since T is invertible, we may write $S = T(1 + T^{-1}C)$ and also $S^{-1} = (1 + T^{-1}C)^{-1}T^{-1}$. By Lemma 2.1, T^{-1} is triangular and $\{P_{n_k}\}$ implements its triangularity. Since $(1 + T^{-1}C)^{-1}$ is of the form $1 + M$, where M is compact, we have the relation $S^{-1} = (1 + M)T^{-1} = T^{-1} + MT^{-1}$. Thus, since MT^{-1} is compact, and since every increasing sequence of finite-rank projections tending to 1 implements the quasitriangularity of a compact operator, S^{-1} is clearly quasitriangular with implementing sequence $\{P_{n_k}\}$. Using the fact that T is triangular relative to the sequence $\{P_{n_k}\}$, we deduce that

$$S_{(n_k)} = [(1 + CT^{-1})T]_{(n_k)} = (1 + CT^{-1})_{(n_k)}T_{(n_k)}.$$

It now follows from Lemmas 2.3 and 2.1 that for sufficiently large k, $(1 + CT^{-1})_{(n_k)}$ and $T_{(n_k)}$ are invertible. Thus $S_{(n_k)}$ is invertible for sufficiently large k; by discarding the first few terms of the sequence $\{P_{n_k}\}$ and changing the notation accordingly, we complete the proof of (b).

To establish (c), we first show that

$$\|[T_{(n_k)}]^{-1}C_{(n_k)} - [T^{-1}C]_{(n_k)}\| \to 0. \tag{5}$$

To accomplish this, it suffices to note that $[T_{(n_k)}]^{-1} = [T^{-1}]_{(n_k)}$ by Lemma 2.1, and to show that

$$\|P_{n_k} T^{-1} P_{n_k} C P_{n_k} - P_{n_k} T^{-1} C P_{n_k}\| \to 0. \tag{6}$$

But the validity of (6) is an immediate consequence of the fact that

$$\|T^{-1}\| \, \|P_{n_k} C P_{n_k} - C P_{n_k}\| \to 0 .$$

Thus (5) is established. Observe now that

$$[S^{-1}]_{(n_k)} = [(1 + T^{-1}C)^{-1} T^{-1}]_{(n_k)} = [(1 + T^{-1}C)^{-1}]_{(n_k)} [T^{-1}]_{(n_k)}$$

$$= [(1 + T^{-1}C)^{-1}]_{(n_k)} [T_{(n_k)}]^{-1},$$

and that

$$[S_{(n_k)}]^{-1} = \{T_{(n_k)} [1_{(n_k)} + [T_{(n_k)}]^{-1} C_{(n_k)}]\}^{-1} = [1_{(n_k)} + [T_{(n_k)}]^{-1} C_{(n_k)}]^{-1} [T^{-1}]_{(n_k)}.$$

It follows that

$$\|[S^{-1}]_{(n_k)} - [S_{(n_k)}]^{-1}\| \le \|T^{-1}\| \, \|[(1 + T^{-1}C)^{-1}]_{(n_k)} - [1_{(n_k)} + [T_{(n_k)}]^{-1} C_{(n_k)}]^{-1}\| .$$

Thus, to complete the proof of the theorem, it suffices to show that

$$\|[(1 + T^{-1}C)^{-1}]_{(n_k)} - [1_{(n_k)} + [T_{(n_k)}]^{-1} C_{(n_k)}]^{-1}\| \to 0 ,$$

or, equivalently, by Lemma 2.3, that

$$\|[1_{(n_k)} + [T^{-1}C]_{(n_k)}]^{-1} - [1_{(n_k)} + [T_{(n_k)}]^{-1} C_{(n_k)}]^{-1}\| \to 0 . \tag{7}$$

But we know from (5) that

$$\|[1_{(n_k)} + [T^{-1}C]_{(n_k)}] - [1_{(n_k)} + [T_{(n_k)}]^{-1} C_{(n_k)}]\| \to 0 .$$

Therefore, employing Lemma 2.3 again, we conclude that (7) is valid, and thus the proof of the theorem is complete.

The following lemma has been known for some time.

LEMMA 2.5. *Let* $\{P_n\}$ *be any sequence of projections in* $\mathscr{L}(\mathscr{H})$, *and let* $Q(\{P_n\})$ *denote the set of all operators* S *on* $\mathscr{H}$ *such that* $\|(1 - P_n) S P_n\| \to 0$. *Then* $Q(\{P_n\})$ *is a uniformly closed algebra of operators.*

Proof. The only thing that is not obvious is that the product of two operators in $Q(\{P_n\})$ is again in $Q(\{P_n\})$. To establish this, let $S_1, S_2 \in Q(\{P_n\})$, and observe that

$$\|S_2 P_n S_1 P_n - S_2 S_1 P_n\| \to 0 \quad \text{and} \quad \|P_n S_2 P_n S_1 P_n - P_n S_2 S_1 P_n\| \to 0 .$$

Since $S_2 \in Q(\{P_n\})$, it follows easily that $\|(P_n S_2 P_n - S_2 P_n) S_1 P_n\| \to 0$, and hence that $\|P_n S_2 S_1 P_n - S_2 S_1 P_n\| \to 0$, as desired.

The following theorem plays a central role in the proof of Theorem 1.1.

THEOREM 2.6. *Let* S *be a quasitriangular operator in* $\mathcal{L}(\mathcal{H})$, *and let* $\{\lambda_j\}_{j=1}^{\infty}$ *be a fixed sequence of scalars in the resolvent set of* S. *Then every operator in* $\mathcal{R}(S)$ *is quasitriangular, and there exists an increasing sequence* $\{P_n\}$ *of finite-rank projections on* $\mathcal{H}$ *tending strongly to* 1, *with the properties*

(a) $\{P_n\}$ *implements simultaneously the quasitriangularity of every operator in* $\mathcal{R}(S)$,

(b) $(S - \lambda_j)_{(n)}$ *is invertible for* $n = 1, 2, \cdots$ *and* $j = 1, 2, \cdots$, *and*

(c) $\lim_{n \to \infty} \|[(S - \lambda_j)_{(n)}]^{-1} - [(S - \lambda_j)^{-1}]_{(n)}\| = 0 \quad (1 \leq j < \infty)$.

Proof. Let D be a countable dense subset of the resolvent set of S containing the sequence $\{\lambda_j\}$, and let $\{\mu_k\}_{k=1}^{\infty}$ be an enumeration of the elements of D. Since S is quasitriangular, there exists an increasing sequence $\{Q_n\}_{n=1}^{\infty}$ of finite-rank projections tending strongly to 1 that implements the quasitriangularity of S. For convenience, we write $Q_n = Q_{k_{0,n}}$ $(1 \leq n < \infty)$. Since $\{Q_{k_{0,n}}\}_{n=1}^{\infty}$ also implements the quasitriangularity of $S - \mu_1$, it follows from Theorem 2.4 that there exists a subsequence $\{Q_{k_{1,n}}\}_{n=1}^{\infty}$ of $\{Q_{k_{0,n}}\}$ possessing the properties

(a_1) $\{Q_{k_{1,n}}\}$ implements the quasitriangularity of $(S - \mu_1)^{-1}$,

(b_1) $(S - \mu_1)_{(k_{1,n})}$ is invertible for $n = 1, 2, \cdots$,

(c_1) $\lim_{n \to \infty} \|[(S - \mu_1)_{(k_{1,n})}]^{-1} - [(S - \mu_1)^{-1}]_{(k_{1,n})}\| = 0$.

Since $\{Q_{k_{1,n}}\}$ also implements the quasitriangularity of $S - \mu_2$, another application of Theorem 2.4 yields the existence of a subsequence $\{Q_{k_{2,n}}\}$ of $\{Q_{k_{1,n}}\}$ possessing properties (a), (b), and (c) of Theorem 2.4 for the operator $S - \mu_2$. Continuing in this manner, we obtain for each positive integer j a sequence $\{Q_{k_{j,n}}\}_{n=1}^{\infty}$ that is a subsequence of $\{Q_{k_{j-1,n}}\}$ and that satisfies the conditions

(a_j) $\{Q_{k_{j,n}}\}$ implements the quasitriangularity of $(S - \mu_j)^{-1}$,

(b_j) $(S - \mu_j)_{(k_{j,n})}$ is invertible for $n = 1, 2, \cdots$,

(c_j) $\lim_{n \to \infty} \|[(S - \mu_j)_{(k_{j,n})}]^{-1} - [(S - \mu_j)^{-1}]_{(k_{j,n})}\| = 0$.

Letting $\{P_n\}$ be the diagonal sequence defined by $P_n = Q_{k_{n,n}}$, we see that conditions (b) and (c) in the statement of the theorem are satisfied. Also, using Lemma 2.5 and the elementary fact that $\mathcal{R}(S)$ is the smallest uniformly closed algebra containing the operators 1, S, and $(S - \mu_k)^{-1}$ $(k = 1, 2, \cdots)$, we conclude that (a) is valid. This completes the proof of the theorem.

3. ANALYTICALLY INVARIANT SUBSPACES

There is a technique that is now standard for constructing two invariant subspaces for an arbitrary operator on $\mathcal{H}$. This technique originated with N. Aronszajn and K. T. Smith [1], and it has been refined several times--most recently in [3]. The following lemma is the cornerstone of this technique. We omit the proof of the lemma, since the arguments involved are similar to those in the proof of [3, Theorem 1].

LEMMA 3.1. *Let* $R \in \mathscr{L}(\mathscr{H})$, *and let* $\{Q_n\}$ *be a sequence of projections on* $\mathscr{H}$ *converging weakly to an operator* Q *such that* $\{Q_n R Q_n - R Q_n\}$ *tends weakly to zero. Then the subspaces* $\{x\colon Qx = x\}$ *and* $[\text{range } Q]^-$ *are invariant subspaces for* R.

The difficulty connected with using the preceding lemma to produce invariant subspaces for an arbitrary operator R on $\mathscr{H}$ is that, in general, the subspaces involved may be simply $\{0\}$ and $\mathscr{H}$. Thus, to force one of these subspaces to be nontrivial, we must make restricting assumptions on the operator R. *Henceforth, in this section,* S *will be a fixed quasitriangular operator on* $\mathscr{H}$, *and* $\{P_n\}$ *will be an increasing sequence of finite-rank projections converging strongly to* 1 *that simultaneously implements the quasitriangularity of every operator in* $\mathscr{R}(S)$. *Furthermore, we suppose (Theorem 2.6) that* $\{\lambda_i\}_{i=1}^{\infty}$ *is a dense subset of the resolvent set of* S *with the property that*

$$\lim_n \|[(S - \lambda_i)_{(n)}]^{-1} - [(S - \lambda_i)^{-1}]_{(n)}\| = 0 \qquad (1 \leq i < \infty) .$$

The above-mentioned standard technique for producing invariant subspaces now proceeds as follows. Let u and v be orthogonal unit vectors in the range of P_1 (which may be assumed to be at least two-dimensional, without loss of generality), and let ρ be the normal state on $\mathscr{L}(\mathscr{H})$ defined by the equation

$$\rho(A) = \frac{1}{2}(Au, u) + \frac{1}{2}(Av, v) .$$

An easy calculation shows that $\rho(1) = 1$ and that $\rho(E) \leq 1/2$ for every projection E of rank one in $\mathscr{L}(\mathscr{H})$. It is now easy to show that for each positive integer n there exists a subprojection Q_n of P_n satisfying the conditions $1/4 \leq \rho(Q_n) \leq 3/4$ and

$$Q_n(P_n S P_n) Q_n = (P_n S P_n) Q_n . \tag{8}$$

Furthermore, by dropping down to corresponding subsequences of the sequences $\{P_n\}$ and $\{Q_n\}$ and changing the notation, we may assume without loss of generality that the sequence $\{Q_n\}$ converges weakly to a positive operator Q. The weak continuity of ρ implies that $1/4 \leq \rho(Q) \leq 3/4$, and thus that Q is neither 0 nor 1.

The following elementary lemma now becomes pertinent.

LEMMA 3.2. *If* Q *is an operator on* $\mathscr{H}$ *such that* $0 \neq Q \neq 1$, *and if there exists some operator* $K \neq 0$ *in* $\mathscr{L}(\mathscr{H})$ *such that* $QKQ = KQ$, *then one of the subspaces* $\{x\colon Qx = x\}$ *and* $[\text{range } Q]^-$ *must be different from both* $\{0\}$ *and* $\mathscr{H}$.

Proof. Since $0 \neq Q \neq 1$, we see that $[\text{range } Q]^- \neq \{0\}$ and $\{x\colon Qx = x\} \neq \mathscr{H}$. Thus, if the range of Q is not dense in $\mathscr{H}$, then $[\text{range } Q]^-$ has the desired property. On the other hand, if the range of Q is dense in $\mathscr{H}$, then the equation $Q(KQx) = KQx$, valid for every vector x in $\mathscr{H}$, yields the fact that $\{x\colon Qx = x\} \neq \{0\}$. Thus the lemma is proved.

Our program to prove Theorem 1.1 should now be fairly transparent. We shall show that with the sequences $\{P_n\}$ and $\{Q_n\}$ and the operator Q associated with S fixed as above, we can apply Lemma 3.1 to every operator R in $\mathscr{R}(S)$. Then we shall show that $QKQ = KQ$, where K is the compact operator given in the statement of Theorem 1.1. This clearly will complete the proof of Theorem 1.1, via Lemmas 3.1 and 3.2. The somewhat stronger result that allows this program to be carried out is the following.

THEOREM 3.3. *For every operator* R *in* $\mathscr{R}(S)$, $\lim_n \|Q_n R Q_n - R Q_n\| = 0$.

Proof. We know by Lemma 2.5 that the set of operators R satisfying the condition $\|Q_n R Q_n - R Q_n\| \to 0$ is a uniformly closed algebra. Thus, to prove the present theorem it suffices to show first that

$$\|Q_n S Q_n - S Q_n\| \to 0 , \tag{9}$$

and second, that for each λ_i in the countable dense subset $\{\lambda_i\}_{i=1}^{\infty}$ of the resolvent set of S,

$$\|Q_n(S - \lambda_i)^{-1} Q_n - (S - \lambda_i)^{-1} Q_n\| \to 0 . \tag{10}$$

To establish (9), we observe that $Q_n S Q_n = P_n S Q_n$, by (8), and also that

$$\|P_n S Q_n - S Q_n\| \leq \|P_n S P_n - S P_n\| \; \|Q_n\| \to 0 .$$

This proves (9). To establish (10), let λ_i be a fixed scalar in the given countable dense subset $\{\lambda_i\}_{i=1}^{\infty}$ of the resolvent set of S, and note that since the sequence $\{P_n\}$ implements the quasitriangularity of $(S - \lambda_i)^{-1}$, we have the relation

$$\|P_n(S - \lambda_i)^{-1} Q_n - (S - \lambda_i)^{-1} Q_n\| \leq \|P_n(S - \lambda_i)^{-1} P_n - (S - \lambda_i)^{-1} P_n\| \; \|Q_n\| \to 0;$$

thus, to establish (10) it suffices to show that

$$\|Q_n(S - \lambda_i)^{-1} Q_n - P_n(S - \lambda_i)^{-1} Q_n\| \to 0 . \tag{11}$$

To see that (11) is valid, recall that we chose the sequence $\{P_n\}$ so that

$$\lim_n \|[(S - \lambda_i)_{(n)}]^{-1} - [(S - \lambda_i)^{-1}]_{(n)}\| = 0 \qquad (1 \leq i < \infty) ,$$

and then observe that

$$\begin{aligned}
\|(P_n - Q_n)(S - \lambda_i)^{-1} Q_n\| &= \|(P_n - Q_n) P_n(S - \lambda_i)^{-1} P_n Q_n\| \\
&= \|(P_n - Q_n)[(S - \lambda_i)^{-1}]_{(n)} Q_n\| \\
&\leq \|[(S - \lambda_i)^{-1}]_{(n)} - [(S - \lambda_i)_{(n)}]^{-1}\| + \|(P_n - Q_n)[(S - \lambda_i)_{(n)}]^{-1} Q_n\| .
\end{aligned}$$

Now $\|[(S - \lambda_i)^{-1}]_{(n)} - [(S - \lambda_i)_{(n)}]^{-1}\| \to 0$, as we noted above. Moreover,

$$Q_n[(S - \lambda_i)_{(n)}]^{-1} Q_n = P_n[(S - \lambda_i)_{(n)}]^{-1} Q_n ,$$

since the inverse of an operator on a finite-dimensional space is a polynomial in the operator and Q_n is an invariant projection for $P_n S P_n$. Thus (11) is established, and the proof is complete.

The proof of Theorem 1.1 is now within our grasp. We postpone it briefly to make the observation that the technique for producing invariant subspaces discussed in the present paragraph actually produces *analytically* invariant subspaces for quasitriangular operators.

THEOREM 3.4. *Let* S *be a quasitriangular operator, and let* $\{P_n\}$, $\{Q_n\}$, *and* Q *be as in the discussion preceding Lemma* 3.2. *Then* $\{x: Qx = x\}$ *and* $[\text{range } Q]^-$ *are (perhaps trivial) analytically invariant subspaces of* S.

This is an immediate consequence of Theorem 3.3 and Lemma 3.1.

Proof of Theorem 1.1. Let S be the prescribed quasitriangular operator, and let $\{P_n\}$, $\{Q_n\}$, and Q be as described in the discussion preceding Lemma 3.2. Also, let K be a nonzero compact operator in $\mathscr{R}(S)$. By virtue of Theorem 3.4, we need only show that one of the subspaces $\{x: Qx = x\}$ and $[\text{range } Q]^-$ is a proper subspace. From Theorem 3.3 we know that $\|Q_n K Q_n - K Q_n\| \to 0$. Since the sequence $\{KQ_n\}$ converges weakly to KQ, and since the sequence $\{Q_n K Q_n\}$ converges weakly to QKQ (see [3, Corollary 2.2]), we have the equation $QKQ = KQ$. Thus the proof is completed via Lemma 3.2.

It is worth remarking that our proof uses the compactness of K only in establishing the weak convergence of the sequence $\{Q_n K Q_n\}$ to QKQ.

Just as [3, Theorem 2] is slightly stronger than the fundamental theorem of Arveson and Feldman, the following result is slightly stronger than Theorem 1.1.

THEOREM 1.1'. *Let* S *be a quasitriangular operator on* $\mathscr{H}$, *and suppose that there exist sequences* $\{R_n\}$ *and* $\{K_n\}$ *in* $\mathscr{L}(\mathscr{H})$ *such that*

(a) *each* R_n *belongs to* $\mathscr{R}(S)$ *and* $\|R_n - K_n\| \to 0$, *and*

(b) *each* K_n *is compact and the sequence* $\{K_n\}$ *converges weakly to a nonzero operator.*

Then S *has a nontrivial analytically invariant subspace.*

We can easily obtain the proof of this theorem by blending the proof of Theorem 1.1 and the proof of [3, Theorem 2]; we leave the details to the interested reader.

4. SOME EXAMPLES

In this section, we make some remarks and give some examples pertinent to Theorem 1.1. The spectrum of an operator T will be denoted by $\Lambda(T)$.

We note first that if T is an operator such that $\Lambda(T)$ is not connected, then T obviously has nontrivial hyperinvariant subspaces. On the other hand, it is an immediate consequence of Runge's theorem that if $\Lambda(T)$ is connected, then $\Lambda(T)$ does not separate the plane if and only if $\mathscr{P}(T) = \mathscr{R}(T)$. Thus, in order that $\mathscr{R}(T) \setminus \mathscr{P}(T)$ be nontrivial, it is necessary that $\Lambda(T)$ separate the plane. Moreover, if $\Lambda(T)$ is connected and $\mathscr{R}(T)$ contains a nonzero compact operator K, then K is necessarily quasinilpotent. (Otherwise, $\Lambda(K)$ would be disconnected, and it would follow from [8, Theorem 3] that $\Lambda(r(T))$ is disconnected for some rational function r(T) of T sufficiently close to K; via the spectral mapping theorem, this would contradict the connectedness of $\Lambda(T)$.)

Our remarks show that it is worthwhile to exhibit a quasitriangular operator S whose spectrum is connected (and necessarily separates the plane) with the property that $\mathscr{P}(S)$ contains no nonzero compact operator but $\mathscr{R}(S)$ does contain a nonzero compact operator.

Example 4.1. Let M be a nonzero compact operator on $\mathscr{H}$ such that $M^2 = 0$, and let N be any normal operator on $\mathscr{H}$ such that $\Lambda(N) = \{\lambda \in \mathbf{C}: 1 \leq |\lambda| \leq 2\}$. Then define $S \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ to be the operator $S = (1 + M) \oplus N$. Since S is the sum of a normal operator $(1 \oplus N)$ and a compact operator $(M \oplus 0)$, it follows (see [5]) that S is quasitriangular. Furthermore, it is easily seen that $\Lambda(S) = \Lambda(N)$ and that $S^{-1} = (1 - M) \oplus N^{-1}$. Thus, for every positive integer n, $S^{-n} = (1 - nM) \oplus N^{-n}$. It follows immediately that

$$\lim_n \|S^{-n}/(-n) - (M \oplus 0)\| = 0,$$

and hence that the nonzero compact operator $M \oplus 0$ belongs to $\mathscr{R}(S)$. On the other hand, if K is a compact operator in $\mathscr{P}(S)$, then K must be of the form $K = K_1 \oplus K_2$, where K_1 and K_2 are compact operators on $\mathscr{H}$ and K_2 is normal. Moreover, if K_2 were different from 0, then the spectrum of K_2 would be disconnected, which would contradict an earlier remark (since $K_2 \in \mathscr{P}(N)$ and $\Lambda(N)$ is connected). Thus $K_2 = 0$. Now let $\{p_n(\lambda)\}$ be any sequence of polynomials such that $\|p_n(S) - K\| \to 0$, and observe that $\|p_n(N) - K_2\| = \|p_n(N)\| \to 0$. It follows from the spectral-mapping theorem and the maximum-modulus principle that the sequence $\{p_n(\lambda)\}$ converges uniformly to zero on the disc $\{\lambda: |\lambda| \leq 2\}$. Furthermore, elementary complex analysis shows that in this situation the sequence $\{p_n'(\lambda)\}$ converges to zero uniformly on every disc $\{\lambda: |\lambda| \leq 2 - \varepsilon\}$ $(\varepsilon > 0)$. In particular, the sequence $\{p_n'(1)\}$ tends to zero. This is pertinent, because an easy calculation shows that for every positive integer n, $p_n(S) = (p_n(1) + p_n'(1)M) \oplus p_n(N)$, and hence that $\|p_n(S)\| \to 0$. Thus $K = 0$, and we have shown that the operator S has all the properties attributed to it above.

It is not completely obvious that an operator T with the property that $\mathscr{R}(T)$ contains a nonzero compact operator can fail to be quasitriangular. However, this turns out to be the case, as the next example shows. In other words, in Theorem 1.1 we can not light-heartedly discard the hypothesis of quasitriangularity.

Example 4.2. Let $U \in \mathscr{L}(\mathscr{H})$ be the unilateral shift of multiplicity 1, and let M be as in Example 4.1. Then $T = (1 + M) \oplus U$ is not quasitriangular, since it is a Fredholm operator of negative index [4, Theorem 1]. However, it is easy to see that $\|(T^n/n) - (M \oplus 0)\| \to 0$, and hence the nonzero compact operator $M \oplus 0$ belongs to $\mathscr{P}(T)$.

It is not difficult to see that if $\mathscr{K}$ is a nonseparable complex Hilbert space, then each T in $\mathscr{L}(\mathscr{K})$ has a nontrivial analytically invariant subspace. One needs only observe that each cyclic subspace generated by the algebra $\mathscr{R}(T)$ is a separable subspace of $\mathscr{K}$, since $\mathscr{R}(T)$ is itself a separable Banach space.

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