EXTREMAL PROBLEMS IN ARBITRARY DOMAINS

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1. INTRODUCTION

Let U be a domain in the extended complex plane $\mathbb{C}^*$, and let $H^\infty(U)$ be the uniform algebra of bounded analytic functions on U. Let L, $L_1, \cdots, L_m$ be linear functionals on $H^\infty(U)$, and let $b_1, \cdots, b_m$ be complex numbers. We are interested in the following "Pick-Nevanlinna" extremal problem:

(*) To maximize $\Re L(f)$, among all functions $f \in H^\infty(U)$ that satisfy the conditions $\|f\| \leq 1$ and $L_j(f) = b_j$ $(1 \leq j \leq m)$.

1.1 THEOREM. *Suppose* L, $L_1, \cdots, L_m$ *are continuous with respect to the norm* $\|\cdot\|_K$ *of uniform convergence on* K, *for some compact subset* K *of* U *that does not separate* ∂U. *Suppose also that* L *is not a linear combination of* $L_1, \cdots, L_m$, *and that there exists at least one competing function for* (*). *Then there exists a unique extremal function* G *for* (*). *The extremal function* G *has modulus* 1 *on the Shilov boundary of* $H^\infty(U)$, *and it can be extended analytically across each free analytic boundary arc of* U.

There is an extensive literature on extremal problems for analytic functions, in the case where U is bounded by analytic curves. For early references, see Z. Nehari's expository article [10]. The paper of A. J. Macintyre and W. W. Rogosinski [9] has a good introduction and bibliography, covering the case in which U is the unit disc, while the paper of H. L. Royden [11] deals with finite bordered Riemann surfaces. Arbitrary domains have been treated by S. Ya. Havinson [7] and S. D. Fisher [2], [3], and in spirit our work is based on that of Fisher.

The existence assertion of Theorem 1.1 follows immediately from the compactness of the family of competing functions. The uniqueness of the extremal function can be proved most easily by the technique of Fisher [2]. That the Ahlfors functions of arbitrary domains have the properties in Theorem 1.1 has already been established by Fisher [2], [3]. Our contribution is to extend these results to a more general class of extremal problems. The extension is not trivial, though, and the main point of the proof is the use of the separation theorem in Section 5 to reduce the problem (*) to a more tractable problem.

That the extremal function G has modulus 1 on the Shilov boundary can be converted into information concerning the cluster behavior of G. As a simple consequence of work in [5] and [6], we shall obtain the following corollary, which improves upon the corresponding results in [2] and [7].

1.2 COROLLARY. *If* w *is an essential boundary point of* U, *then the cluster set of the extremal function* G *at* w *either coincides with the closed unit disc, or else it lies on the unit circle.*

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2. THE MAXIMAL-IDEAL SPACE OF $H^\infty(U)$

To avoid trivialities, we shall henceforth assume that $H^\infty(U)$ contains nonconstant functions. The maximal-ideal space of $H^\infty(U)$ will be denoted by $\mathcal{M}(U)$, and the Shilov boundary of $H^\infty(U)$ by S.

There is a natural projection Z of $\mathcal{M}(U)$ onto $\overline{U}$ (see [5] for more details). If $\phi \in \mathcal{M}(U)$, then $Z(\phi)$ can be characterized as the unique point $w \in \mathbb{C}^*$ such that $\phi(f) = f(w)$ for all $f \in H^\infty(U)$ that are analytic at w. For $w \in \overline{U}$, the subset $Z^{-1}(\{w\})$ of $\mathcal{M}(U)$ is called the *fiber over* w, and we denote it by $\mathcal{M}_w$. If $w \in U$, then $\mathcal{M}_w$ consists of only one point, namely the homomorphism that evaluates functions in $H^\infty(U)$ at w. Moreover, Z maps $Z^{-1}(U)$ homeomorphically onto U. We shall always regard U as an open subset of $\mathcal{M}(U)$, by identifying $Z^{-1}(U)$ with U via Z. Then Z can be regarded as an extension of the coordinate function from U to $\mathcal{M}(U)$. In fact, if U is bounded in $\mathbb{C}$, then Z is the Gelfand transform of the coordinate function z.

The Gelfand extension of $f \in H^\infty(U)$ to $\mathcal{M}(U)$ will also be denoted by f, and we shall consider $H^\infty(U)$ as a uniform algebra on $\mathcal{M}(U)$. Rational functions of Z can also be extended to $\mathcal{M}(U)$, in the obvious way. The $H^\infty(U)$-convex hull of a subset E of $\mathcal{M}(U)$ will be denoted by $\hat{E}$.

2.1 LEMMA. *Let* E *be a closed subset of* $\mathcal{M}(U)$, *and let* $z_0 \in U$, $z_0 \neq \infty$. *Then* $z_0 \notin \hat{E}$ *if and only if* $1/(Z - z_0)$ *can be approximated uniformly on* E *by functions in* $H^\infty(U)$.

Proof. If $z_0 \notin \hat{E}$, then there is a sequence $\{f_n\}$ in $H^\infty(U)$ such that $f_n(z_0) = 1$, while $f_n \to 0$ uniformly on E. Clearly, $(1 - f_n)/(Z - z_0) \in H^\infty(U)$, and

$$(1 - f_n)/(Z - z_0) \to 1/(Z - z_0)$$

uniformly on E. Conversely, suppose $f_n \to 1/(Z - z_0)$ uniformly on E. Choose $h \in H^\infty(U)$ so that $h(z_0) = 0$ while $h'(z_0) \neq 0$. Then $h[f_n - 1/(Z - z_0)] \in H^\infty(U)$ converges uniformly to zero on E, while it assumes the value $-h'(z_0)$ at z_0. Therefore $z_0 \notin \hat{E}$.

It follows from Lemma 2.1 that $U \setminus \hat{E}$ is relatively closed in $U \setminus E$. Since $\hat{E}$ is also relatively closed in $U \setminus E$, we obtain the following corollary.

2.2 COROLLARY. *Let* E *be a closed subset of* $\mathcal{M}(U)$, *and let* V *be a component of* $U \setminus E$. *Then either* $V \subset \hat{E}$, *or* V *is disjoint from* $\hat{E}$.

2.3 LEMMA. *Let* K *be a compact subset of* U, *and let* E *be a closed subset of* $\mathcal{M}(U)$ *such that* $Z(E) \subseteq \partial U$. *If* V *is a component of* $U \setminus K$ *such that* $V \subseteq \widehat{K \cup E}$, *then* $E \supseteq S \cap Z^{-1}(\partial V)$.

Proof. First we remark that all except finitely many components $V_1, \cdots, V_k$ of $U \setminus K$ are relatively compact in U. The sets $\partial V_j \cap \partial U$ form a partition of ∂U into disjoint open-closed subsets, which are separated by K.

Suppose that E does not contain $S \cap Z^{-1}(\partial V)$. Then there is a point $p \in S \setminus E$ such that $Z(p) \in \partial V$. Since $Z(S) \subseteq \partial U$, the component V must coincide with one of the components V_j, and Z(p) is at a positive distance from $U \setminus V$. Therefore we can find an open neighborhood N of p in $\mathcal{M}(U)$ such that N is disjoint from $E \cup K$, and such that $Z(N) \cap U \subseteq V$. By the minimality property of the Shilov boundary, we can find $f \in H^\infty(U)$ such that $\|f\| = 1$, while $|f| < 1/2$ off N. In particular, $|f| < 1/2$ on $U \setminus V$, so that $|f(z_0)| > 1/2$ for some point $z_0 \in V$. Since $|f| < 1/2$

on $E \cup K$, we see that $z_0 \notin \widehat{E \cup K}$, and V is not a subset of $\widehat{E \cup K}$. This proves the lemma.

The converse of the preceding lemma is also true, but we have no need for it.

If E is a closed subset of $\mathcal{M}(U)$ such that $\hat{E} \supseteq U$, then each f in $H^\infty(U)$ satisfies the condition $\|f\|_E = \|f\|$, so that E is a boundary for $H^\infty(U)$, and $E \supseteq S$. Combining this observation with Corollary 2.2, we obtain the following key theorem, which has been discovered independently by C. Stanton.

2.4 THEOREM. *If* ν *is a complex representing measure on* $Z^{-1}(\partial U)$ *for a point* $z_0 \in U$, *then the closed support of* ν *includes* S.

Proof. If E is the closed support of ν, then $z_0 \in \hat{E}$. By Corollary 2.2, $\hat{E} \supseteq U$. Consequently, $E \supseteq S$.

2.5 COROLLARY. *If* $f \in H^\infty(U)$ *vanishes on a nonempty, relatively open subset of* S, *then* $f \equiv 0$.

Proof. Suppose $f \in H^\infty(U)$ is not identically zero. Choose $z_0 \in U$ so that $f(z_0) \neq 0$, and let μ be a representing measure on S for z_0. Then $f\,d\mu/f(z_0)$ is a complex representing measure for z_0. By Theorem 2.4, the closed support of $f\,d\mu$ is S, so that f cannot vanish on an open subset of S.

3. A SPECIAL CASE

Let Λ be a linear functional on $H^\infty(U)$. In this section and the next, we shall consider the following extremal problem, which is formally simpler than (*).

(**) To maximize $\Re\,\Lambda(f)$, among all $f \in H^\infty(U)$ with $\|f\| \leq 1$.

In this case, competing functions always exist. If Λ is continuous with respect to the norm $\|\cdot\|_K$, for some compact subset K of U, then there is an extremal function for (**). The following is the main result of this section.

3.1 THEOREM. *Suppose that* $\Lambda \neq 0$, *and that* Λ *is continuous with respect to the norm* $\|\cdot\|_K$, *for some compact subset* K *of* U. *Then there exists a component* V *of* $U \setminus K$ *such that* $S \cap Z^{-1}(\partial V)$ *is not empty (so that* V *extends to* ∂U), *and such that each extremal function* G *for* (**) *has modulus* 1 *on* $S \cap Z^{-1}(\partial U)$.

Proof. Evidently, $\Lambda(G)$ is the norm of Λ on $H^\infty(U)$:

$$\Lambda(G) = \sup\{|\Lambda(f)| : f \in H^\infty(U),\ \|f\| \leq 1\}.$$

Let η be a measure on S such that $\Lambda(f) = \int f\,d\eta$ for all $f \in H^\infty(U)$, and $\|\eta\| = \Lambda(G)$. Then η has minimal norm, among the measures on S representing Λ.

Let E be the closed support of η. Since $\Lambda(G) = \int G\,d\eta = \|\eta\|$ and $|G| \leq 1$, we see that $|G| = 1$ on E. Since $\Lambda \neq 0$, there exists at least one component V of $U \setminus K$ such that η has some mass on $Z^{-1}(\partial V)$. The theorem will be proved once we show that $E \subseteq S \cap Z^{-1}(\partial V)$. For this, we argue by contradiction.

Suppose that E does not contain $S \cap Z^{-1}(\partial V)$. By Corollary 2.2 and Lemma 2.3, V is disjoint from $\widehat{K \cup E}$. Using Lemma 2.1 and Runge's theorem, we see that if g is any function analytic on a neighborhood of $\overline{U} \setminus V$, then the composition $g \circ Z$ can be approximated uniformly on $K \cup E$ by functions in $H^\infty(U)$. In particular, there is a sequence $\{f_n\}$ in $H^\infty(U)$ such that $f_n \to 1$ uniformly on $E \cap Z^{-1}(\partial V)$, while $f_n \to 0$ uniformly on K and on $E \setminus Z^{-1}(\partial V)$. [We use the topological fact that V separates $\partial U \cap \partial V$ from K and from $(\partial U) \setminus \partial V$.] Let μ be any measure on K that represents Λ. Then $\eta - \mu \perp H^\infty(U)$, so that also $f_n[\eta - \mu] \perp H^\infty(U)$. Passing to the limit, we find that the restriction η_0 of η to $Z^{-1}(\partial V)$ is orthogonal to $H^\infty(U)$. Consequently, $\eta - \eta_0$ is also a measure on S that represents Λ. Since

$$\|\eta - \eta_0\| = \|\eta\| - \|\eta_0\| < \|\eta\| ,$$

we have a contradiction to the minimality of the norm of η, and the theorem is established.

3.2 COROLLARY. *The extremal function* G *in Theorem* 3.1 *is unique.*

Proof. Let G_0 be another extremal function. Then convex combinations of G and G_0 are extremal, and therefore they are unimodular on $S \cap Z^{-1}(\partial V)$. This can occur only if $G = G_0$ on $S \cap Z^{-1}(\partial V)$. Now $S \cap Z^{-1}(\partial V)$ is a nonempty, open-closed subset of S on which $G - G_0$ vanishes. By Corollary 2.5, $G - G_0 \equiv 0$, and $G = G_0$.

If K does not separate ∂U, then G has modulus 1 on all of S. If however K separates ∂U, then we can say nothing about G elsewhere on S, as the following example shows.

Let U be the annulus $\{r < |z| < 1\}$, and let G be a function in $H^\infty(U)$ such that $|G| < 1$, and such that G can be extended analytically across the unit circle $\{|z| = 1\}$ and has modulus 1 on the circle. Then there is an extremal problem of type (**) for which the hypotheses of Theorem 3.1 are satisfied and for which G is the extremal function. Indeed, the functional

$$\Lambda(f) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z\,G(z)}\, dz \qquad (f \in H^\infty(U))$$

satisfies the condition $\|\Lambda\| = 1 = \Lambda(G)$. By deforming the contour of integration to a slightly smaller circle, we see that Λ is continuous with respect to the norm $\|\cdot\|_K$, for some compact $K \subseteq U$.

4. ANALYTICITY ACROSS BOUNDARY ARCS

An arc Γ is a *free boundary arc* for U if some neighborhood of one side of Γ is disjoint from U, while some neighborhood of the other side of Γ is contained in U.

4.1 THEOREM. *Let* Λ *satisfy the conditions in Theorem* 3.1, *and let* V *be the component of* $U \setminus K$ *produced in Theorem* 3.1. *If* Γ *is a free boundary arc for* U *that is contained in* ∂V, *then the extremal function* G *for* (**) *extends continuously to* Γ *and has modulus* 1 *there.*

Proof. By shrinking Γ and applying a conformal map, we may assume that U is a bounded subset of the upper half-plane H_+, that Γ is the open interval $(-1, 1)$, and that U includes the upper half of the open unit disc.

Having noted this simplification, we make a detour to prove some preliminary lemmas. Recall that the Cauchy transform of a compactly supported measure ν on $\mathbb{C}$ is the locally integrable function $\hat{\nu}$ defined by

$$\hat{\nu}(\zeta) = \int \frac{d\nu(z)}{z - \zeta} .$$

(For the basic properties of Cauchy transforms, see [4].) The following lemma is a version of Bishop's splitting lemma (see [1, p. 40]). We denote by $\Delta(z_0 ; r)$ the closed disc with center z_0 and radius r.

4.2 LEMMA. *Let* ν *be a compactly supported measure on* $\mathbb{C}$, *and fix* $z_0 \in \mathbb{C}$. *Suppose* $r_0 > 0$ *is such that the function*

$$r \to \iint_{\Delta(z_0;r)} |\hat{\nu}(z)| \, dx\, dy$$

is differentiable at r_0. *Then there exists a measure* μ *on* $\Delta(z_0 ; r_0)$ *such that*

$$\hat{\mu} = \begin{cases} \hat{\nu} & \text{on the interior of } \Delta(z_0 ; r), \\ 0 & \text{elsewhere} . \end{cases}$$

In particular, μ *coincides with* ν *on the interior of* $\Delta(z_0 ; r_0)$.

Proof. Let g_n be a smooth function, supported on $\Delta(z_0 ; r_0 + 1/n)$, such that $g_n = 1$ on $\Delta(z_0 ; r_0)$ and $\left|\frac{\partial g_n}{\partial \bar{z}}\right| \leq 2/n$. Define

$$\mu_n = g_n \nu - \frac{1}{\pi} \frac{\partial g_n}{\partial \bar{z}} \hat{\nu} \, dx\, dy ;$$

then $\hat{\mu}_n = g_n \hat{\nu}$. The differentiability condition shows that $\{\mu_n\}$ is a bounded sequence of measures. If μ is any weak-star adherent point of the sequence $\{\mu_n\}$, then μ has the desired properties.

We shall use Lemma 4.2 to establish the following standard proposition about the jump discontinuities of Cauchy integrals. While we are assuming that Γ is the interval $(-1, 1)$ on the real axis, the proof applies to much more general curves.

4.3 LEMMA. *Let* ν *be a compactly supported measure on* $\mathbb{C}$ *such that the interval* Γ *is a relatively open subset of the closed support of* ν. *Suppose* $\hat{\nu}$ *vanishes on a neighborhood of the side of* Γ *in the lower half-plane* H_-. *Then* $\hat{\nu}$ *is locally of class* H^1 *along the side of* Γ *in* H_+. *Moreover, if* g *denotes the nontangential boundary values of* $\hat{\nu}$ *on* Γ *from* H_+, *then* $2\pi i\, d\nu = g\, dx$ *on* Γ.

Proof. Since the result is local, we need only consider a fixed point $z_0 \in \Gamma$. Choose $r_0 > 0$ as in Lemma 4.2. Let W be the open disc $\{|z - z_0| < r_0\}$. We can assume that r_0 is chosen so small that $\hat{\nu} = 0$ on $W \cap H_-$, and so that the only mass of ν on W lies on Γ. By Lemma 4.2, there is a measure μ on $\overline{W}$ such that $\hat{\mu} = 0$ off $\overline{W}$, while $\hat{\mu} = \hat{\nu}$ on W. In particular, μ coincides with ν on W. Now $\hat{\mu}$ vanishes a.e. $(dx\,dy)$ on H_-, so that μ has no mass on H_-, and μ is supported by the boundary of the semidisc $W_+ = W \cap H_+$. By the version of the F. and M. Riesz theorem for W_+ [the description of the measures orthogonal to $A(W_+)$], there exists an analytic function g on W_+ of class H^1, such that μ is the boundary value measure of the analytic differential $g(z)\,dz/2\pi i$. Evidently, $\hat{\mu} = g$ on W_+. Since $\mu = \nu$ and $\hat{\mu} = \hat{\nu}$ on W, the lemma is proved.

The other main ingredient for the proof of Theorem 4.1 is the description of the piece of the Shilov boundary of $H^\infty(U)$ lying over Γ. Our results run parallel to the description of the Shilov boundary of the algebra $H^\infty(\Delta)$ given in Chapter 10 of [8]. We sketch the necessary facts.

Each function in $H^\infty(U)$ has nontangential boundary values a.e. (dx) on Γ, and these determine a function in $L^\infty(\Gamma, dx)$. The functions in $L^\infty(\Gamma, dx)$ can in turn be regarded as continuous functions on the maximal-ideal space Σ of $L^\infty(\Gamma, dx)$. Now $H^\infty(U)$ separates the points of Σ, so that Σ can be regarded as a subset of $\mathcal{M}(U)$. We are interested only in the piece Σ_0 of Σ lying over Γ:

$$\Sigma_0 = \Sigma \cap Z^{-1}(\Gamma) .$$

As in the case of the unit disc, we have the relation

$$\Sigma_0 = S \cap Z^{-1}(\Gamma) .$$

The canonical lift of the measure dx to Σ_0 will be denoted by dX. We shall use the following lemma, which is related to the abstract F. and M. Riesz theorem for $H^\infty(U)$.

4.4 LEMMA. *Let* E *be a compact subset of* Σ_0 *such that* $dX(E) = 0$. *Then there is a sequence* $\{f_n\}$ *in* $H^\infty(U)$ *such that* $\|f_n\| \le 1$, $f_n \to 1$ *pointwise on* U, *and* $f_n \to 0$ *uniformly on* E.

Proof. In fact, we can produce the required functions f_n so that they belong to $H^\infty(H_+)$. The standard construction runs as follows. Choose $u_n \in L^\infty(-\infty, \infty)$ so that $u_n \le 0$, $u_n = 0$ outside Γ, $\int u_n dx \to 0$, and $u_n \to -\infty$ uniformly on E. Extend u_n harmonically to H_+ via the Poisson integral formula, and let ${}^*u_n$ be the harmonic conjugate of u_n. Then the functions $f_n = \exp(u_n + i\,{}^*u_n)$ have the required properties.

Now we continue with the proof of Theorem 4.1, using the notation from the proof of Theorem 3.1. In particular, η is the norm-preserving extension of Λ to C(S), and μ is a measure on K that represents Λ. Let E be a compact subset of Σ_0 such that $dX(E) = 0$, let f_n be the functions from Lemma 4.4, and let F be a weak-star adherent point of the sequence $\{f_n\}$ in $L^\infty(|\eta| + |\mu|)$. Then $|F| \le 1$, $F = 1$ a.e. $(d\mu)$, and $F = 0$ on E. Since $\eta - \mu \perp H^\infty(U)$, we see that

$$f_n(\eta - \mu) \perp H^\infty(U) ,$$

so that $F(\eta - \mu) = F\eta - \mu$ is orthogonal to $H^\infty(U)$. Consequently $F\eta$ also represents Λ on $H^\infty(U)$. Since η has minimal norm, $|F| = 1$ a.e. $(d\eta)$. Hence $|\eta|(E) = 0$. It follows that the restriction of η to Σ_0 is absolutely continuous with respect to dX, and we can write $d\eta = g_0\, dX$ on Σ_0.

Now let $\nu = Z^*\eta - \mu$ be the projection of $\eta - \mu$ onto $\overline{U}$. Since $\eta - \mu \perp H^\infty(U)$, we see that

$$\int \frac{d\nu(z)}{z - \zeta} = \int \frac{d(\eta - \mu)}{Z - \zeta} = 0 \qquad (\zeta \notin \overline{U}) .$$

In particular, $\hat{\nu} = 0$ in H_-. By Lemma 4.3, the function $g = 2\pi i \hat{\nu}$ is analytic and locally of class H^1 in a neighborhood of the topside of Γ, and in terms of the boundary value function of g, we have the relation $d\nu = g\,dx$ on Γ. By the choice of V in the proof of Theorem 3.1, the closed support of η includes $S \cap Z^{-1}(\partial V) \supseteq \Sigma_0$, so that g does not vanish identically on Γ.

Now the extremal function G satisfies the condition $G\eta \geq 0$, so that $Gg_0 \geq 0$ a.e. (dX). Since $Gg_0 dX$ is the canonical lift of $Gg\,dx$ on Γ, we deduce that $Gg \geq 0$ a.e. (dx) on Γ. The remainder of the argument is classical (see [12, p. 11], for instance). The Schwarz reflection principle shows that $h = Gg$ extends analytically across Γ. The function defined to be $h(z)\overline{G(\bar{z})}$ beneath Γ is then seen to extend g analytically across Γ, because $|G| = 1$ a.e. on Γ. From this we conclude easily that G itself extends analytically across Γ.

5. REDUCTION OF (*) TO (**)

The following theorem, combined with Theorems 4.1 and 3.1, serves to complete the proof of Theorem 1.1.

5.1 THEOREM. *Suppose* $L, L_1, \cdots, L_m$ *are linear functions on* $H^\infty(U)$ *that are continuous with respect to the norm* $\|\cdot\|_K$ *for some compact subset* K *of* U, *and suppose that* L *does not depend linearly on* $L_1, \cdots, L_m$. *Then there exists a nonzero linear combination* Λ *of* $L, L_1, \cdots, L_m$ *such that each extremal function for* (*) *is also an extremal function for* (**).

Proof. Let G be an extremal function for (*). Let Q be the set of $h \in H^\infty(U)$ such that $L_j(h) = b_j$ $(1 \leq j \leq m)$ while $\Re L(h) \geq \Re L(G)$. Since L is independent of $L_1, \cdots, L_m$, it is easy to see that each extremal function for (*) has unit norm. Consequently, Q is disjoint from the open unit ball of $H^\infty(U)$. By the separation theorem for convex sets, there is a nonzero continuous linear functional Λ on $H^\infty(U)$ such that

$$a = \sup\{\Re\Lambda(f): f \in H^\infty(U),\ \|f\| < 1\} \leq \inf\{\Re\Lambda(h): h \in Q\}.$$

Evidently, $a = \|\Lambda\| > 0$. Since $G \in Q$, $\Re\Lambda(G) \geq a$. It follows that $\Re\Lambda(G) = a = \|\Lambda\|$, so that G is an extremal function for the problem (**).

Now suppose $g \in H^\infty(U)$ satisfies the condition $L_j(g) = 0$ $(1 \leq j \leq m)$, while $\Re L(G) \geq 0$. Then $G + g \in Q$, so that $\Re\Lambda(G+g) \geq a$, and $\Re\Lambda(g) \geq 0$. It follows that $\Lambda(g) = 0$ whenever $L(g) = 0$ and $L_j(g) = 0$ $(1 \leq j \leq m)$. Consequently Λ is a linear combination of $L, L_1, \cdots, L_m$. This completes the proof.

5.2 COROLLARY. *Under the hypotheses of Theorem* 5.1, *the extremal function for* (*) *is unique, whenever it exists.*

6. THE FIBER ALGEBRAS

Recall that $\mathcal{M}_w = Z^{-1}(\{w\})$ is the fiber of $\mathcal{M}(U)$ over w. It was shown in [6] that the restriction of $H^\infty(U)$ to $\mathcal{M}_w$ is a closed subalgebra of $C(\mathcal{M}_w)$ whose maximal ideal space is $\mathcal{M}_w$. Here we wish to describe the Shilov boundary S_w of the restriction algebra, and to prove Corollary 1.2.

A point $w \in \partial U$ is an *essential boundary point* for $H^\infty(U)$ if there exists a function $f \in H^\infty(U)$ that does not extend analytically across w. The essential boundary points E for U form a closed subset of ∂U. The functions in $H^\infty(U)$ extend analytically to $\overline{U} \setminus E$, so that $H^\infty(U)$ is isometrically isomorphic to $H^\infty(\overline{U} \setminus E)$.

If $w \in \overline{U}$ is not an essential boundary point, then $\mathcal{M}_w$ consists only of the evaluation homomorphism at w. On the other hand, if w is an essential boundary point for U, then $\mathcal{M}_w$ is gigantic. In this case, Theorem 6.8 of [6] asserts that each strong boundary point in S_w is contained in $S \cap \mathcal{M}_w$. Since the strong boundary points are dense in the Shilov boundary, we deduce that $S_w \subseteq S \cap \mathcal{M}_w$ whenever w is an essential boundary point. It turns out that equality holds here, but we shall not use this.

An elementary theorem concerning Banach algebras asserts that the range of an element on the Shilov boundary includes the topological boundary of its range on the maximal-ideal space. In our context, this takes the form

$$\partial f(\mathcal{M}_w) \subseteq f(S_w) \qquad (f \in H^\infty(U),\ w \in \overline{U}) .$$

Now the cluster-value theorem of [5] asserts that $f(\mathcal{M}_w)$ coincides with the cluster set $\mathrm{Cl}(f, w)$ of f at w. Our inclusion becomes

$$\partial\, \mathrm{Cl}(f, w) \subseteq f(S_w) \qquad (f \in H^\infty(U),\ w \in \overline{U}) .$$

The following theorem, which includes Corollary 1.2, is a simple consequence of these inclusions.

6.1 THEOREM. *If* w *is an essential boundary point of* U, *and if* $f \in H^\infty(U)$ *has unit modulus on the Shilov boundary of* $H^\infty(U)$, *then either* $\mathrm{Cl}(f, w)$ *coincides with the closed unit disc, or*

$$\lim_{\substack{z \to w \\ z \in U}} |f(z)| = 1 .$$

Proof. In this case we have an inclusion $\partial\, \mathrm{Cl}(f, w) \subseteq f(S \cap \mathcal{M}_w)$, and the set on the right is a subset of the unit circle.

7. YET ANOTHER EXTREMAL PROBLEM

We can apply our methods to certain extremal problems with an infinite number of side conditions. We mention one example, which yields topological information on $\mathcal{M}(U)$. Let $\{z_n\}_{n=0}^{\infty}$ be a sequence of distinct points in U. The extremal problem is the following:

(***) To maximize $\Re f(z_0)$, among all $f \in H^\infty(U)$ satisfying the conditions $\|f\| \leq 1$ and $f(z_j) = 0$ $(j \geq 1)$.

7.1 THEOREM. *Suppose there exists a nonconstant function* $g \in H^\infty(U)$ *that vanishes at each* z_j. *Then the extremal problem* (***) *has a unique extremal function* G, *and* G *has modulus* 1 *on the Shilov boundary of* $H^\infty(U)$.

Proof. By dividing g by an appropriate power of $Z - z_0$, we can find a competing function for (***) that does not vanish at z_0. Hence each extremal function G satisfies the condition $G(z_0) \neq 0$. Clearly, $G(z_0)$ is the norm of the functional "evaluation at z_0" on the ideal J of functions in $H^\infty(U)$ that vanish on $z_1, z_2, \cdots$. Hence there is a measure η on S such that $\|\eta\| = G(z_0)$, while $\int f\, d\eta = f(z_0)$ for all $f \in J$. Since $\|\eta\| = G(z_0) = \int G\,d\eta$, we see that $|G| = 1$ on the closed support of η. If $h \in H^\infty(U)$, then $hG \in J$, so that $h(z_0)G(z_0) = \int hG\,d\eta$, and $G\eta/G(z_0)$ is a complex representing measure for z_0. By Theorem 2.4, the closed support of $G\eta$ includes S, so that G must have unit modulus on S. The proof of Corollary 3.2 shows that the extremal function is unique.

7.2 COROLLARY. *If* f *is a function in* $H^\infty(U)$ *that is not identically zero, then the closure in* $\mathcal{M}(U)$ *of the sequence of zeros of* f *on* U *does not meet the Shilov boundary of* $H^\infty(U)$.

8. REMARK ON RIEMANN SURFACES

Our results extend to any domain U on a compact Riemann surface, provided there are nonconstant functions in $H^\infty(U)$. The results could be extended to more general surfaces if the following old question has an affirmative answer: If U is a Riemann surface such that $H^\infty(U)$ separates the points of U, does the natural inclusion $U \to \mathcal{M}(U)$ map U homeomorphically onto an open subset of $\mathcal{M}(U)$?

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