ASYMPTOTIC PATHS FOR POWER SERIES WITH EXPONENTS OF THE FORM n^p

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Corresponding to each function f analytic in the unit disk D, we denote by $M(r) = M(r, f)$ the maximum modulus of f on the circle $|z| = r$ $(0 \leq r < 1)$, and we define the *order* $\alpha(f)$ by the formula

$$\alpha(f) = \limsup_{r \to 1} \frac{\log \log M(r)}{\log (1 - r)}.$$

In [4], we showed implicitly that if p is an integer $(p \geq 2)$, and if f is defined in D by a Taylor series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^{n^p} \tag{1}$$

and has order $\alpha(f) > 1/(p - 1)$, then f is unbounded on every continuous curve that begins at the origin and tends to the unit circle. Using known results on Abel summability, we shall now show that the conclusion need not hold if $\alpha(f) = 1/(p - 1)$.

THEOREM. *For each integer* p $(p \geq 2)$, *there exists a function of the form* (1) *that has order* $1/(p - 1)$ *and possesses a finite radial limit at the point* $z = 1$.

Proof. In [1], B. Kuttner remarks that an earlier theorem of E. G. Phillips [3] implies that for $p > 1$, the series $\sum_{n=0}^{\infty} b^n$ is summable (A, n^p) (that is,

$$\lim_{\sigma \to 0^+} \sum_{n=0}^{\infty} b^n e^{-\sigma n^p}$$

exists and is finite), for each value $b = re^{i\theta}$ in the region defined by the inequality

$$r < \exp\{|\theta| \tan(\pi/(2p))\} \qquad (-\pi \leq \theta \leq \pi).$$

Thus, if we take $\theta = \pi$, $K = (\pi/2)\tan(\pi/(2p))$, and $r = \exp K$, then

$$\lim_{\sigma \to 0^+} \sum_{n=0}^{\infty} e^{nK}(-1)^n e^{-\sigma n^p}$$

exists and is finite. Hence, if we define f in D by the formula

Received January 15, 1972.
This research was supported in part by Grant GP16548 from the National Science Foundation.

Michigan Math. J. 19 (1972).

$$f(z) = \sum_{n=0}^{\infty} e^{nK}(-1)^n z^{n^p},$$

then f is of the form (1) and possesses a finite radial limit at $z = 1$. To see that f has the required order, we note that by a result of G. R. MacLane [2, p. 38],

$$\alpha(f) = \limsup_{n\to\infty} \frac{\log^+ \log^+ |a_n|}{\log n^p - \log^+ \log^+ |a_n|}$$

$$= \limsup_{n\to\infty} \frac{\log n + \log K}{(p-1)\log n - \log K} = 1/(p-1).$$

REFERENCES

1. B. Kuttner, *A theorem on Abel summability.* J. London Math. Soc. 37 (1962), 123-125.

2. G. R. MacLane, *Asymptotic values of holomorphic functions.* Rice Univ. Studies 49 (1963), no. 1, 83 pp.

3. E. G. Phillips, *On the region of summability* (A, λ, α) $(\alpha > 1)$ *and* $(A, \lambda \log \lambda, 1)$ *of Dirichlet's series of type* λ_n. Proc. London Math. Soc. (2) 40 (1935), 436-448.

4. L. R. Sons, *Value distribution and power series with moderate gaps.* Michigan Math. J. 13 (1966), 425-433.

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