ON THE ASYMPTOTIC BEHAVIOR OF FUNCTIONS HOLOMORPHIC IN THE UNIT DISC

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An *asymptotic value* of a function f meromorphic in $D = \{ |z| < 1\}$ is defined as a limit value α of $f(z)$ as $|z| \to 1$ on an arc γ in D. In terms of the associated Riemann surface $\mathscr{F}$ over the extended complex w-plane $\mathscr{W}$, the concept of asymptotic value has the following geometric interpretation: γ is the inverse image of a noncompact arc Γ on $\mathscr{F}$ whose projection into $\mathscr{W}$ ends at the point $w = \alpha$.

A set constitutes the *asymptotic set* of some meromorphic function f (that is, the set of asymptotic values of f) if and only if it is an analytic subset (possibly empty) of $\mathscr{W}$ (see [1], [2]).

The characterization of the asymptotic sets of holomorphic functions is more difficult, because many analytic sets in $\mathscr{W}$ must be excluded (see [4], [5]). The following theorem gives a trivial necessary condition. In the statement of the theorem, ∂G denotes the boundary of G, and the bar $^-$ indicates closure. We can easily verify the necessity of the condition by defining G as the image of D under f and using properties of the Riemann surface of f (see [4]).

THEOREM. *If* A *is the asymptotic set of a function* f *holomorphic in* D, *then* A *is an analytic set and there exists a domain* G *such that:*

(1) $\partial G \subset A^- \subset G^-$,

(2) *if* $\zeta \in \partial G$ *is inaccessible from* G, *then* $\zeta \notin A$,

(3) *if* $\zeta \in \partial G - A$, *then every arc in* G *to* ζ *meets* A.

The complexities of the holomorphic case are illustrated by the following example, which shows that the condition in the theorem is not sufficient. At the same time, the example answers a question posed by the author [3]. *There exists an analytic subset* A *of* $\{|w| < 1\}$ *that meets every arc in* $\{|w| < 1\}$ *ending at a point of* $\{|w| = 1\}$ *but is not the asymptotic set of any function holomorphic in* D.

To construct the example, let S be the finite domain bounded by the triangle with vertices at (0, 1/4), (0, -1/4), and (1, 0). Define

$$C_n = \{|w| = 1 - 2^{-n}\} \quad \text{and} \quad C_{n,m} = \{|w| = 1 - 2^{-n} + 2^{-m}\}.$$

Now put

$$A_n = \{C_n - S\} \cup \bigcup_{m \geq n+2} \{C_{n,m} \cap S\}$$

and

$$A = \bigcup_{n=1}^{\infty} A_n \cup \{0\}.$$

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It is clear that A is a Borel set, hence analytic [6]. Moreover, A is a subset of $\{|w| < 1\}$ that meets every arc in $\{|w| < 1\}$ ending at $\{|w| = 1\}$.

The inclusion of $\{0\}$ in the set A ensures that if A is the asymptotic set of a function f holomorphic in D, then the Riemann surface $\mathcal{F}$ of f covers $\{|w| < 1\} - A$. This implies the existence of a noncompact arc Γ on $\mathcal{F}$ whose projection is an arc in $\{|w| < 1\}$ that ends at $w = 1$. To obtain Γ, define $w_n = 1 - 2^{-n}$ and let γ_1 be an arc in $S - A$ from $w = 1/4$ to w_1. Since $w_1 \notin A$, γ_1 can be lifted completely into $\mathcal{F}$, determining there a compact arc Γ_1 ending at a point P_1 of $\mathcal{F}$ over w_1. Now $\mathcal{F}$ contains a neighborhood of P_1, and $w_2 \notin A$; consequently, some arc γ_2 joining w_1 to w_2 in S can be lifted completely into $\mathcal{F}$, determining there an arc Γ_2 beginning at P_1 and ending at a point P_2 of $\mathcal{F}$ over w_2. Since $w_n \notin A$ for any n, we can repeat the construction. The result is a noncompact arc $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots$ on $\mathcal{F}$ whose projection $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots$ is an arc in S ending at $w = 1$; this implies $1 \in A$, a contradiction. Consequently, A cannot be the asymptotic set of a function holomorphic in D.

It is interesting to note that a simple alteration in the definition of A produces an analytic set A' that has the same pathological behavior as A, but can be realized as the asymptotic set of a function holomorphic in D. Specifically, set

$$C'_{n,m} = \{|w| = 1 - 2^{-n} - 2^{-m}\},$$

and put

$$A'_n = \{C_n - S\} \cup \bigcup_{m \geq n+2} \{C'_{n,m} \cap S\}$$

and

$$A' = \bigcup_{n=1}^{\infty} A'_n .$$

Using as sheets the finite domains bounded by the A'_n, we can construct a Riemann surface over the unit disc such that A' is the asymptotic set of the corresponding holomorphic function in D.

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