COMPACTNESS PROPERTIES OF TOPOLOGICAL GROUPS

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In a topological group, an element is called *bounded* if its conjugate class is relatively compact. The concept of bounded elements is useful in the study of the structure of locally compact groups. Many results concerning bounded elements have been established by V. I. Ušakov (see [6], [7], [8], [9]) and J. Tits [5]; we mention two of them for later use in this paper.

THEOREM A (V. I. Ušakov). *Let* G *be a totally disconnected, locally compact group. If a relatively compact subset* A *of* G *is invariant under all inner automorphisms of* G, *and if each of its elements belongs to a compact subgroup of* G, *then the closed subgroup generated by* A *is a compact, normal subgroup of* G.

THEOREM B (J. Tits). (a) *If* G *is a projective limit of Lie groups, then the set* B(G) *of all bounded elements of* G *is closed in* G.

(b) *Let* G *be an analytic group without compact, normal subgroups except the identity subgroup. Then*

(i) *the identity component* $B_0(G)$ *of* B(G) *is a vector group,*

(ii) *if* Z(G) *denotes the center of* G, *then* $B(G) = B_0(G)\,Z(G)$, *and*

(iii) *if* α *is a bounded automorphism on* G *(that is, if the set* $\{\alpha(g)\,g^{-1} \mid g \in G\}$ *is relatively compact), then there exists an element* g *in* B(G) *such that the inner automorphism* I_g *on* G *induced by* g *is equal to* α.

The set B(G) of bounded elements of a topological group G actually forms a characteristic subgroup of G; we shall call it the *bounded part* of G. A group is called an $\overline{FC}$-*group* if all its elements are bounded. In this paper, we relax the condition of boundedness in three directions: (I) B(G) is open, (II) $\overline{B(G)} = G$, and (III) $G/\overline{B(G)}$ is compact.

Locally compact groups with open bounded parts will be discussed in Section 2, where we shall prove the following two results:

(a) *In a locally compact group, the bounded part is open if and only if there exists a compact invariant neighborhood of the identity.*

(b) *In a* σ*-compact, locally compact group, the bounded part is open if and only if it is of second category.*

Sections 3 and 4 are devoted to the study of locally compact groups with dense bounded parts. There we shall generalize some results on $\overline{FC}$-groups and suggest a structure theorem. The main results are as follows.

(c) *In a locally compact group with dense bounded part, the periodic part (the set of elements that are contained in compact subgroups) forms a closed characteristic subgroup whose factor group is a direct product of a vector group and a discrete, torsion-free, abelian group.*

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(d) *Each locally compact group with dense bounded part is an extension of a compact group by a direct product of a vector group and a totally disconnected, locally compact group with dense bounded part.*

A closed subgroup H in a locally compact group G is called a *uniform subgroup* of G if G/H is a compact space. In recent years, the structures of locally compact groups with uniform vector subgroups and uniform, discrete, finitely generated, free abelian subgroups have been studied (see [1], [10], [11]). We find that both types fall into the class of groups G for which $G/\overline{B(G)}$ is compact. In Section 5, we shall establish a theorem that describes the structure of such groups.

Finally, in Section 6 we describe some examples.

1. DEFINITIONS AND NOTATION

1. The letter G will always stand for a locally compact group, unless it is otherwise specified.

2. An automorphism α on G is called a *bounded automorphism* if the set $\{\alpha(g)g^{-1} \mid g \in G\}$ is relatively compact.

3. The conjugate class determined by an element g in G is denoted by C(g), and I_g denotes the inner automorphism on G induced by g.

4. A subset of G is said to be *invariant* in G if it is invariant under all inner automorphisms on G.

5. An element g in G is called a *bounded element* in G if I_g is a bounded automorphism on G. It is easy to see that g is bounded if and only if C(g) is relatively compact.

6. The set of bounded elements in G is called the *bounded part* of G and is denoted by B(G).

7. A subset of G is called *bounded* if each of its elements is bounded. A locally compact group is called an $\overline{FC}$-*group* if each of its elements is bounded.

8. An element g in G is called a *periodic element* if it is contained in a compact subgroup of G.

9. The set of all periodic elements of G is called the *periodic part* of G and is denoted by P(G).

10. A subset of G is called *periodic* if each of its elements is periodic. In particular, if all elements of G are periodic, then G is called a *periodic group.*

11. A locally compact group is said to be *pure* if P(G) contains only the identity.

12. Two topological groups G and H are said to be *isomorphic* if there exists an algebraic isomorphism of G onto H that is also a homeomorphism. We use $G \cong H$ to denote such a relation.

13. Let A be a subset of a group. The subgroup generated by A is denoted by $\langle A \rangle$. If a is the only element in A, then $\langle A \rangle$ is written as $\langle a \rangle$.

14. Let A and B be two subsets of a group. The set of commutators $[a, b] = aba^{-1}b^{-1}$ $(a \in A,\ b \in B)$ is denoted by [A, B].

2. LOCALLY COMPACT GROUPS WITH OPEN BOUNDED PARTS

THEOREM 1. *Let* G *be a locally compact group. Then* B(G) *is open if and only if* G *has a compact invariant neighborhood of the identity.*

Proof. (a) Suppose G has a compact invariant neighborhood N of the identity of G. Then N is contained in B(G), and hence B(G) is open.

(b) Suppose B(G) is open and G has no compact invariant neighborhoods of the identity e. Let N be a compact neighborhood of e such that $N \subseteq B(G)$. Let V be an open symmetric neighborhood of e such that $\overline{V}^2 \subseteq N$. We shall construct sequences $\{x_n\}$ and $\{a_n\}$ satisfying the conditions

(i) $y_n = x_1 x_2 \cdots x_n \in V$, for all n;

(ii) $a_n x_{n+1} x_{n+2} \cdots x_m a_n^{-1} \in V$, for all n and m $(m > n)$;

(iii) the sets $N, a_1 x_1 a_1^{-1} N, a_2 x_1 x_2 a_2^{-1} N, \cdots, a_k x_1 x_2 \cdots x_k a_k^{-1} N, \cdots$ are disjoint.

First we show that the existence of such sequences is sufficient for the proof. Since $y_n \in V \subseteq N$ for each n, there exists a subnet $\{y_{n_\lambda}\}$ that converges to a point $y \in N$. For each n,

$$a_n y a_n^{-1} = \lim_\lambda (a_n x_1 x_2 \cdots x_{n_\lambda} a_n^{-1}) = \lim_\lambda (a_n x_1 \cdots x_n a_n^{-1})(a_n x_{n+1} \cdots x_{n_\lambda} a_n^{-1}),$$

and is therefore contained in $a_n x_1 x_2 \cdots x_n a_n^{-1} \overline{V}$. Suppose the net $\{a_{n_\lambda} y a_{n_\lambda}^{-1}\}$ converges to some $x \in G$. Then $a_{n_\lambda} y a_{n_\lambda}^{-1}$ eventually lies in xV, in other words, $a_{n_\lambda} y a_{n_\lambda}^{-1} = x v_\lambda$ for some $v_\lambda \in V$. We see that

$$x = a_{n_\lambda} y a_{n_\lambda}^{-1} v_\lambda^{-1} \in a_{n_\lambda} x_1 x_2 \cdots x_{n_\lambda} a_{n_\lambda}^{-1} \overline{V}^2 \subseteq a_{n_\lambda} x_1 \cdots x_{n_\lambda} a_{n_\lambda}^{-1} N .$$

This contradicts condition (iii), and hence the net $\{a_{n_\lambda} y a_{n_\lambda}^{-1}\}$ cannot converge. A similar argument shows that $\{a_{n_\lambda} y a_{n_\lambda}^{-1}\}$ has no convergent subnets. Consequently, C(y) is not relatively compact. But $y \in N \subseteq B(G)$, hence we have a contradiction.

Now let us start the construction. First we claim that there exist $a_1 \in G$ and $x_1 \in V$ such that $N \cap a_1 x_1 a_1^{-1} N = \emptyset$. Otherwise, suppose N meets each set $axa^{-1}N$, where $a \in G$ and $x \in V$; then there exist $y, z \in N$ such that $axa^{-1}y = z$ and axa^{-1} is contained in NN^{-1}. Thus, $A_0 = \bigcup_{a \in G} aVa^{-1} \subseteq NN^{-1}$, and we have a compact invariant neighborhood $\overline{A}_0$ of e; this is a contradiction to our initial assumption.

Suppose we have constructed the sets $\{x_1, \cdots, x_k\}$ and $\{a_1, \cdots, a_k\}$ $(k > 1)$ satisfying the three conditions

(1) $y_i = x_1 x_2 \cdots x_i \in V$ $(i = 1, 2, \cdots, k)$,

(2) $a_i x_{i+1} \cdots x_j a_i^{-1} \in V$ $(i = 1, 2, \cdots, k-1;\ j > i)$,

(3) the sets $N, a_1 x_1 a_1^{-1} N, \cdots, a_k x_1 \cdots x_k a_k^{-1} N$ are disjoint.

Consider the continuous mappings

$$x \to y_k x ,$$

$$x \to a_i x_{i+1} \cdots x_k x a_i^{-1} \qquad (i = 1, 2, \cdots, k-1),$$

$$x \to a_k x a_k^{-1}.$$

Since each of the mappings maps e into V, some neighborhood W of e is mapped into V by all the mappings. Let R_k be the set

$$N \cup a_1 x_1 a_1^{-1} N \cup a_2 x_1 x_2 a_2^{-1} N \cup \cdots \cup a_k x_1 \cdots x_k a_k^{-1} N.$$

Suppose $R_k \cap a x_1 x_2 \cdots x_k x a^{-1} N \neq \emptyset$, for all $x \in W$ and $a \in G$. Then

$$A_k = \bigcup_{a \in G} a(x_1 \cdots x_k W) a^{-1} \subseteq R_k R_k^{-1},$$

and $\overline{A}_k \overline{A}_k^{-1}$ is a compact invariant neighborhood of e. A contradiction! Hence, there exist $x_{k+1} \in W$ and $a_{k+1} \in G$ such that $R_k \cap a_{k+1} x_1 x_2 \cdots x_{k+1}^{-1} N = \emptyset$. The construction is complete.

COROLLARY 1. *If* G *is an* $\overline{\text{FC}}$*-group, then* G *has a compact invariant neighborhood of the identity.*

COROLLARY 2. *If* $\overline{B(G)} = G$, *then* G *is an* $\overline{\text{FC}}$*-group if and only if* G *has a compact invariant neighborhood of the identity.*

COROLLARY 3 (D. H. Lee and T. S. Wu). *If* G *is a locally compact, totally disconnected group, then* B(G) *is open if and only if* G *has a compact, normal open subgroup.*

Proof. Clearly, if G has a compact, open, normal subgroup, then B(G) is open.

Conversely, suppose B(G) is open; then there exists a compact, invariant neighborhood N of the identity. Let K be a compact, open subgroup of G contained in N, and consider the subset $\bigcup_{x \in G} xKx^{-1}$, which is relatively compact, invariant and periodic. By Theorem A, it generates a compact, open, normal subgroup of G. The proof is complete.

COROLLARY 4. *Let* G *be a locally compact, totally disconnected* $\overline{\text{FC}}$*-group, and let* K *be a compact subgroup of* G. *Then* K *is contained in a compact, open, normal subgroup of* G.

Proof. Let $k \in K$. Then the set C(k) is relatively compact, invariant, and periodic. Hence, by Theorem A, it generates a compact, normal subgroup N_k of G. Since G is an $\overline{\text{FC}}$-group, by Corollary 3, there exists a compact, open, normal subgroup N of G. Clearly, K is covered by finitely many compact, open, normal subgroups of the form NN_k. Their union is compact, invariant and periodic. Hence, again by Theorem A, it generates a compact, open, normal subgroup of G. Obviously, K is contained in this group.

The following theorem shows that if the bounded part of a σ-compact, locally compact group is not open, then it is, in a sense, rare indeed.

THEOREM 2. *Let* G *be a* σ*-compact, locally compact group. Then* B(G) *is open if and only if it is of second category.*

We thank the referee for improving the statement and the proof of this theorem.

Proof. Clearly, if B(G) is open, then it is of second category.

Conversely, suppose B(G) is not open. Let $\{U_n\}$ ($n = 1, 2, \cdots$) be a countable cover of G, where each U_n is open and relatively compact. For each n, let

$$W_n = \bigcap_{g \in G} g(\overline{U}_1 \cup \overline{U}_2 \cup \cdots \cup \overline{U}_n)g^{-1} .$$

Then W_n is a compact, invariant subset of G, and hence it lies in B(G). If $b \in B(G)$, then there exists a positive integer n such that C(b) is contained in $\overline{U}_1 \cup \overline{U}_2 \cup \cdots \cup \overline{U}_n$. This implies that $b \in W_n$. As a result, $B(G) = \bigcup_{n=1}^{\infty} W_n$. Since, by assumption, B(G) is not open, each W_n has empty interior. It follows that B(G) is of first category.

3. TOTALLY DISCONNECTED, LOCALLY COMPACT GROUPS WITH DENSE BOUNDED PARTS

In this section, we are concerned with totally disconnected, locally compact groups. First of all, let us notice that for each totally disconnected, locally compact group G, the set $B(G) \cap P(G)$ forms a subgroup. In fact, let $x, y \in B(G) \cap P(G)$; then C(x) and C(y) are relatively compact, invariant, and periodic. By Theorem A, they generate compact, normal subgroups N_x and N_y. Hence xy belongs to the compact, normal subgroup $N_x N_y$ and is again in $B(G) \cap P(G)$.

LEMMA 1. *Let* G *be a totally disconnected, locally compact group with dense bounded part. Then* $\overline{B(G) \cap P(G)}$ *is the union of all compact open subgroups of* G. *In particular,* $\overline{B(G) \cap P(G)}$ *is open and periodic.*

Proof. (a) Let K be a compact open subgroup of G. Then K is contained in P(G), and hence $B(G) \cap K$ is contained in $B(G) \cap P(G)$. Since K is open, $\overline{B(G)} \cap K \subseteq \overline{B(G) \cap K}$. Now, since $\overline{B(G)} = G$, we have the relation

$$K = G \cap K = \overline{B(G)} \cap K \subseteq \overline{B(G) \cap K} \subseteq \overline{B(G) \cap P(G)} .$$

Hence $\overline{B(G) \cap P(G)}$ contains all compact open subgroups of G.

(b) Let $F = \overline{B(G) \cap P(G)}$. Clearly, $B(G) \cap P(G)$ is contained in $B(F) \cap P(F)$, and $B(F) \cap P(F)$ is dense in F. Therefore B(F) is also dense in F. Since F is a totally disconnected, locally compact group, $B(F) \cap P(F)$ is an open (hence closed) subgroup of B(F). This implies that

$$B(F) \cap P(F) = \overline{B(F) \cap P(F)} \cap B(F) = F \cap B(F) = B(F) ,$$

and hence $B(F) \subseteq P(F)$. Let K be a compact, open subgroup of F. Then $F = B(F)K$. Let $f = bk \in F$, where $b \in B(F)$ and $k \in K$. The element b, being periodic and bounded, is contained in a compact, normal subgroup C of F, by Theorem A. Hence f belongs to the compact, open subgroup CK of F. By part (a), we know that F is open in G. Therefore CK is also open in G. Thus $\overline{B(G) \cap P(G)}$ is contained in the union of compact, open subgroups of G.

Remark. From the second half of the proof of Lemma 1, we know that $\overline{B(G) \cap P(G)}$ is always periodic, for a totally disconnected, locally compact group (the condition that $\overline{B(G)} = G$ is not required).

THEOREM 3. *Let* G *be a totally disconnected, locally compact group with dense bounded part. Then* P(G) *forms an open characteristic subgroup of* G, *and* G/P(G) *is a discrete, torsion-free, abelian group.*

Proof. We shall show that $P(G) = \overline{B(G) \cap P(G)}$. By Lemma 1, $\overline{B(G) \cap P(G)}$ is contained in P(G). Now it is sufficient to prove that $G/\overline{B(G) \cap P(G)}$ is torsion-free. Let $F = \overline{B(G) \cap P(G)}$. Let x be an element of G such that $x^k \in F$ for some integer k. By Lemma 1, x^k belongs to some compact, open subgroup K of G. Since $\overline{B(G)} = G$, there exists a net $\{x_\lambda\}$ in B(G) such that $x_\lambda \to x$. This implies that $x_\lambda^k \to x^k$. Since K is open, we can assume that $x_\lambda^k \in K$ for each λ. Hence $x_\lambda \in P(G)$ for each λ, because the cyclic group $\langle x_\lambda \rangle$ generated by x_λ cannot be infinite and discrete. Thus x lies in $\overline{B(G) \cap P(G)} = F$, and G/F is torsion-free.

Since $\overline{B(G)} = G$, we see that $\overline{B(G/F)} = G/F$. But G/F is discrete; therefore it is an FC-group (a group with finite conjugate classes). Being torsion-free, it is abelian [4]. The proof is complete.

PROPOSITION 1. *Let* G *be a totally disconnected, locally compact group with dense bounded part* B(G). *Then an element* g *is in* B(G) *if and only if the smallest closed normal subgroup containing* g *is either compact or a semidirect product of a compact group and an infinite, discrete, cyclic group.*

Proof. Let $g \in B(G)$. Consider the subset [G, C(g)]. Let $x, y, a \in G$. Then

$$[x, aga^{-1}] = (xa)g(xa)^{-1}(ag^{-1}a^{-1}),$$

and the right-hand member is contained in $C(g)C(g^{-1})$. Hence [G, C(g)] is relatively compact. Also,

$$y[x, aga^{-1}]y^{-1} = [yxy^{-1}, (ya)g(ya)^{-1}];$$

therefore [G, C(g)] is invariant. Since, by Theorem 3, G/P(G) is abelian, we see that $[G, C(g)] \subseteq P(G)$. Thus, by Theorem A, [G, C(g)] generates a compact, normal subgroup D_g of G. Now let N_g be the smallest closed normal subgroup containing g. Then $N_g = \overline{\langle C(g) \rangle}$. Clearly, N_g contains D_g. Since $[G, g] \subseteq D_g$, the coset gD_g is a central element of G/D_g, and it is not hard to see that N_g/D_g is equal to $\overline{\langle gD_g \rangle}$. If N_g/D_g is compact, then N_g is compact. If N_g/D_g is not compact, it is an infinite, discrete, cyclic group, and $D_g \cap \langle g \rangle$ is trivial. In this case, $\langle g \rangle$ is also infinite and discrete, and hence $\langle g \rangle \cong N_g/D_g$. This implies that $N_g = D_g \circledS \langle g \rangle$ (semidirect product).

Conversely, suppose N_g is compact; then obviously g is in B(G). If $N_g = B \circledS \langle a \rangle$, where B is a compact subgroup and $\langle a \rangle$ is an infinite, discrete, cyclic group. Let T be an automorphism on N_g, and let b be any element in B. Then $T(b) = xy$, for some $x \in B$ and $y \in \langle a \rangle$. Hence the element $y = x^{-1}T(b)$ belongs to B T(B), which is a compact subgroup of N_g. This implies that y is the identity. Consequently, B is a characteristic subgroup of N_g, and therefore it is normal in G. It is sufficient to show that gB is a bounded element in G/B. Since gB is an element of N_g/B and N_g/B is a cyclic, normal subgroup of G/B, gB has only a finite number of conjugates. Therefore $gB \in B(G/B)$. The proof is complete.

PROPOSITION 2. *Let* G *be a totally disconnected, locally compact group with dense bounded part. If* G *is compactly generated, then* G *is an* $\overline{FC}$*-group.*

Proof. Let K be a compact, open subgroup of G. Since G is compactly generated and $\overline{B(G)} = G$, there exist a finite number of elements $g_1, g_2, \cdots, g_n \in B(G)$ such that $G = \langle g_1, g_2, \cdots, g_n, K \rangle$. Using the same notation as in the proof of Proposition 1, we see that D_{g_i} is a compact normal subgroup of G, for $i = 1, 2, \cdots, n$, and hence $KD_{g_1} D_{g_2} \cdots D_{g_n}$ is a compact, open subgroup of G. Now we show that $KD_{g_1} D_{g_2} \cdots D_{g_n}$ is in fact normal in G. Since

$$G = \langle g_1, g_2, \cdots, g_n, K \rangle,$$

it suffices to show that the group is invariant under the actions of elements g_i $(i = 1, 2, \cdots, n)$ and $k \in K$. Obviously, $KD_{g_1} D_{g_2} \cdots D_{g_n}$ is invariant under K. Now let $k \in K$ and $d_i \in D_{g_i}$ $(i = 1, 2, \cdots, n)$. Then, for any g_i,

$$\begin{aligned} g_i k d_1 \cdots d_n g_i^{-1} &= (g_i k g_i^{-1})(g_i d_1 g_i^{-1}) \cdots (g_i d_n g_i^{-1}) \\ &= k(k^{-1} g_i k g_i^{-1})(g_i d_1 g_i^{-1}) \cdots (g_i d_n g_i^{-1}) \\ &\in KD_{g_i} D_{g_1} \cdots D_{g_n} = KD_{g_1} \cdots D_{g_n}. \end{aligned}$$

It follows that $KD_{g_1} \cdots D_{g_n}$ is contained in B(G) and B(G) is open. Thus $B(G) = G$. This completes the proof.

4. LOCALLY COMPACT GROUPS WITH DENSE BOUNDED PARTS

LEMMA 2. *Let* G *be a locally compact group with dense bounded part. Then the identity component* G_0 *of* G *is an* $\overline{FC}$-*group.*

Proof. Let G_1 be the natural inverse image of a compact, open subgroup of G/G_0 in G. Then G_1/G_0 is compact and G_1 is a projective limit of Lie groups. Since $\overline{B(G)} = G$ and G_1 is open, we see that $\overline{B(G_1)} = G_1$. But $B(G_1)$ is a closed subgroup of G_1, by Theorem B. Therefore G_1 is an $\overline{FC}$-group, and G_0, as a closed subgroup of G_1, is also an $\overline{FC}$-group.

LEMMA 3. *Let* G *be a locally compact group. Then* G *is an extension of a compact group by a locally compact group whose identity component is an analytic group without compact, normal subgroups except the identity subgroup. Furthermore, if* $\overline{B(G)} = G$, *then the analytic group is a vector group.*

Proof. Let G_0 be the identity component of G. As a connected locally compact group, G_0 contains a maximal compact normal subgroup L such that G_0/L is an analytic group [2]. The identity component of G/L is exactly G_0/L, and it contains no compact, normal subgroups other than the identity subgroup.

If, in addition, $\overline{B(G)} = G$, then, by Lemma 2, G_0 is an $\overline{FC}$-group. Hence G_0/L is also an $\overline{FC}$-group. It follows from Theorem B that G_0/L is a vector group.

LEMMA 4. *Let* G *be a locally compact group and* G_0 *its identity component. Then there exists a compact subgroup* K *of* G *such that* KG_0 *is an open subgroup of* G. *Moreover, if* G_0 *has no compact, normal subgroups other than the identity subgroup, then* $KG_0 \cong K \times G_0$ *and* K *is totally disconnected.*

Proof. Let G_1 be an open subgroup of G that contains G_0 and is a projective limit of Lie groups. There exists a compact, normal subgroup K of G_1 such that G_1/K is a Lie group. The identity component of G_1/K is KG_0/K, and KG_0 is therefore an open subgroup of G_1. Since G_1 is open in G, KG_0 is also open in G.

Now suppose G_0 has no compact, normal subgroups other than the identity subgroup $\{e\}$. Then $K \cap G_0 = \{e\}$. Since both K and G_0 are normal subgroups of G_1 and $K \times G_0$ is σ-compact, we see that $K \times G_0 \cong KG_0$. Finally, we notice that $K \cong (K \times G_0)/G_0$, and hence K is totally disconnected.

LEMMA 5. *Let* G *be a locally compact group such that* $\overline{B(G)} = G$ *and the identity component* G_0 *of* G *is a vector group. Let* $\mathrm{pr}: G \to G/G_0$ *be the natural projection. If*

$$\tilde{G} = G/G_0, \qquad F = \overline{B(G) \cap P(G)}, \qquad \tilde{F} = \overline{B(\tilde{G}) \cap P(\tilde{G})},$$

then $\mathrm{pr}^{-1}(\tilde{F}) = G_0 F \cong G_0 \times F$.

Proof. First we show that G_0 is central. Let $g \in G_0$ and $x \in B(G)$. Consider the element $[g, x] \in G_0$. Because G_0 is abelian,

$$[g, x]^2 = g(xg^{-1}x^{-1})g(xg^{-1}x^{-1}) = gg(xg^{-1}x^{-1})(xg^{-1}x^{-1}) = g^2xg^{-2}x^{-1}.$$

In general, for any positive n, $[g, x]^n = g^n x g^{-n} x^{-1}$. This implies that $\langle [g, x] \rangle$ is contained in the set $C(x)x^{-1} \cup xC(x^{-1})$, which is relatively compact. Hence $[g, x]$ is a periodic element in G_0. But G_0 is pure. Thus $[g, x]$ is equal to the identity e, that is, $gx = xg$. Since $\overline{B(G)} = G$, the component G_0 is central.

Next we show that $B(G) \cap P(G)$ forms a subgroup. Let a, b be two elements in $B(G) \cap P(G)$. Then $\mathrm{pr}(a), \mathrm{pr}(b) \in B(\tilde{G}) \cap P(\tilde{G})$. Since $\tilde{G}$ is totally disconnected, we can see, using Theorem A, that $\mathrm{pr}(a)$ is contained in a compact, normal subgroup $\tilde{N}$ of $\tilde{G}$. Let $N = \mathrm{pr}^{-1}(\tilde{N})$. Since N/G_0 is compact and G_0 is a central vector group, there exists a compact subgroup K of N such that $N = G_0 K \cong G_0 \times K$ [1]. Since $a \in N$ and G_0 is pure, we see that $a \in K$. But K is clearly a characteristic subgroup of N, and hence it is normal in G. Similarly, b is contained in a compact, normal subgroup L of G. This implies that ab is in the compact normal subgroup KL and ab is an element of $B(G) \cap P(G)$.

Now we show that $G_0 F \cong G_0 \times F$. By Lemma 4, there exists a compact subgroup K of G such that $G_0 K$ is open and $G_0 K \cong G_0 \times K$. Let $H = B(G) \cap P(G)$. Clearly, $G_0 KH$ is open and hence closed. It is easy to see that $G_0 \cap KH = \{e\}$, and since G_0 is central, both G_0 and KH are normal subgroups of $G_0 KH$. Let V be an open neighborhood of the identity in G_0, and let W be an open neighborhood of the identity in KH. Then $V(W \cap K)$ is open in $G_0 K \cong G_0 \times K$ and hence open in $G_0 KH$. This implies that $G_0 KH \cong G_0 \times KH$. It follows that KH is a closed subgroup of $G_0 KH$, and hence a closed subgroup of G. Thus $F \subseteq KH$. Now let $k \in K$. Then, as an element in the open subgroup $G_0 K$, it is the limit of a net $\{x_\lambda\}$ in $(G_0 K) \cap B(G)$. Let $x_\lambda = g_\lambda k_\lambda$ for each λ, where $g_\lambda \in G_0$ and $k_\lambda \in K$. Clearly, the projection k_λ of x_λ in K converges to k, since K is a direct factor of $G_0 K$. Since g_λ is central, $k_\lambda = g_\lambda^{-1} x_\lambda \in B(G) \cap P(G)$. It follows that k lies in F. Hence K is contained in F. Consequently, $F = KH$ and $G_0 F \cong G_0 \times F$.

Finally, we show that $\mathrm{pr}^{-1}(\tilde{F}) = G_0 F$. Obviously, $G_0 F \subseteq \mathrm{pr}^{-1}(\tilde{F})$. By Lemma 1, $\tilde{F}$ is an open subgroup, and hence $\mathrm{pr}^{-1}(\tilde{F})$ is open. Let $x \in \mathrm{pr}^{-1}(\tilde{F}) \cap B(G)$. Then, again by Lemma 1, $\mathrm{pr}(x)$ is contained in a compact subgroup $\tilde{C}$ of $\tilde{G}$. Let

$C = pr^{-1}(\tilde{C})$. Then C/G_0 is compact. Since G_0 is a vector group, there exists a compact subgroup B of C such that $C = G_0 \circledS B$ (semidirect product) [1]. Hence $x = gb$ for some $g \in G_0$ and $b \in B$. It follows that $b = g^{-1}x \in B(G) \cap P(G)$. This implies that $x \in G_0 H$ and therefore $x \in G_0 F$. Consequently, $pr^{-1}(\tilde{F})$ is contained in $G_0 F$ and thus $pr^{-1}(\tilde{F}) = G_0 F$. The lemma is proved.

LEMMA 6. *Let* G *be a locally compact group such that* $\overline{B(G)} = G$ *and the identity component* G_0 *is a vector group. Then the set of all periodic elements in* G *forms a closed characteristic subgroup of* G *whose factor group is a direct product of a vector group and a discrete, torsion-free, abelian group.*

Proof. Let $\tilde{G} = G/G_0$ and $\tilde{F} = \overline{B(\tilde{G}) \cap P(\tilde{G})}$. Let pr: $G \to \tilde{G}$ be the natural projection. Then $G/pr^{-1}(\tilde{F}) \cong \tilde{G}/\tilde{F}$ is discrete, torsion-free and abelian, by Theorem 3. Clearly, $P(G) \subseteq pr^{-1}(\tilde{F}) = pr^{-1}(P(\tilde{G}))$. By Lemma 5,

$$pr^{-1}(\tilde{F}) = G_0 F \cong G_0 \times F ,$$

where $F = \overline{B(G) \cap P(G)}$. The pureness of G_0 implies that $P(G) \subseteq F$. As we see from the proof of the lemma above, $F = KH \subseteq P(G)$. Hence $P(G) = \overline{B(G) \cap P(G)}$.

Now let $\hat{G} = G/P(G)$, and let ϕ: $G \to G/P(G)$ be the natural mapping. Then $\phi(G_0) = G_0 P(G)/P(G)$ is an open dense subgroup of the identity component $\hat{C}$ of $\hat{G}$, and hence it is identical with $\hat{C}$. Since $G_0 P(G) \cong G_0 \times P(G)$, we see that $\hat{C}$ is isomorphic to the vector group G_0. It is easy to see that $\overline{B(\hat{G})} = \hat{G}$, and we notice, as in the proof of Lemma 5, that $\hat{C}$ is central. Also, $\hat{G}/\hat{C} \cong G/G_0 P(G) \cong \tilde{G}/\tilde{F}$, and therefore $\hat{G}/\hat{C}$ is abelian. Now we show that $\hat{G}$ is also abelian. Let $x \in B(\hat{G})$ and $y \in \hat{G}$, and consider the element $[x, y]$. Since $\hat{G}/\hat{C}$ is abelian and $\hat{C}$ is central, it follows that $[x, y] \in \hat{C}$ and

$$[x, y]^2 = xyx^{-1}y^{-1}(xyx^{-1}y^{-1}) = xyx^{-1}(xyx^{-1}y^{-1})y^{-1} = xy^2x^{-1}y^{-2}.$$

In general, $[x, y]^n = xy^n x^{-1} y^{-n}$ for each positive integer n, and the subgroup generated by $[x, y]$ is contained in the compact subset $xC(x^{-1}) \cup C(x)x^{-1}$. Hence $[x, y]$ is a periodic element in $\hat{C}$. Consequently, $xyx^{-1}y^{-1} = e$ and $xy = yx$. Now, since $\overline{B(\hat{G})} = \hat{G}$, we conclude that $\hat{G}$ is abelian. As an open divisible subgroup in the abelian group $\hat{G}$, $\hat{C}$ is a topological direct factor of $\hat{G}$; that is, there exists a normal subgroup $\hat{D}$ of $\hat{G}$ such that $\hat{G} = \hat{C}\hat{D} \cong \hat{C} \times \hat{D}$. Finally, we notice that $\hat{D} \cong \hat{G}/\hat{C}$ and that it is discrete, torsion-free, and abelian. The proof is complete.

THEOREM 4. *Let* G *be a locally compact group such that* $\overline{B(G)} = G$. *Then* P(G) *is a closed characteristic subgroup of* G, *and* G/P(G) *is a direct product of a vector group and a discrete, torsion-free, abelian group.*

Proof. Let G_0 be the identity component of G. By Lemma 2, we know that G_0 is an $\overline{FC}$-group; and by Lemma 3, it is an extension of a compact group B by a vector group G_0/B. Clearly, $B = P(G_0)$.

Let $\tilde{G} = G/B$; then the identity component of $\tilde{G}$ is exactly $\tilde{G}_0 = G_0/B$, which is a vector group. By Lemma 5, the periodic part $P(\tilde{G})$ of $\tilde{G}$ forms a closed characteristic subgroup, and $\tilde{G}/P(\tilde{G})$ is a direct product of a vector group and a discrete, torsion-free, abelian group. Let pr: $G \to G/B$ be the natural mapping. Then $G/pr^{-1}(P(\tilde{G}))$ is isomorphic to $\tilde{G}/P(\tilde{G})$. Clearly, $P(G) \subseteq pr^{-1}(P(\tilde{G}))$. Now let $x \in pr^{-1}(P(\tilde{G}))$; then $pr(x) \in \tilde{K}$ for some compact subgroup $\tilde{K}$ of $\tilde{G}$. Since B is compact, $pr^{-1}(\tilde{K})$ is compact. This implies that $x \in P(G)$. Thus $P(G) = pr^{-1}(P(\tilde{G}))$, and $G/P(G)$ is a direct product of a vector group and a discrete, torsion-free, abelian group. This completes the proof.

COROLLARY 5. *Let* G *be a locally compact group with dense bounded part. Then the closure of the commutator group* G' *is periodic.*

Proof. By Theorem 4, $G/P(G)$ is abelian. Hence $\overline{G'} \subseteq P(G)$.

COROLLARY 6. *Let* G *be a locally compact group with dense bounded part. If* G *is pure, then* G *is abelian.*

The following theorem suggests that in order to study the structure of the class of locally compact groups with dense bounded parts, it is sufficient, in a sense, to study totally disconnected, locally compact groups with dense bounded parts.

THEOREM 5. *Let* G *be a locally compact group such that* $\overline{B(G)} = G$. *Then* G *is an extension of a compact group by a direct product of a vector group and a totally disconnected, locally compact group with dense bounded part. Moreover, the vector group is central.*

Proof. Let G_0 be the identity component of G. As we showed in the proof above, there exists a compact, normal subgroup B of G_0 such that G_0/B is a vector group. Let $\tilde{G} = G/B$; then the identity component of $\tilde{G}$ is $\tilde{G}_0 = G_0/B$, and hence it is a vector group. Furthermore, $\tilde{G}_0$ is central (see the proof of Lemma 5). By Lemma 6, $P(\tilde{G})$ forms a closed, characteristic subgroup of $\tilde{G}$. Let $\hat{G} = \tilde{G}/P(\tilde{G})$; then $\hat{G}$ is a direct product of its identity component $\hat{C}$ and a discrete, torsion-free, abelian group $\hat{D}$; that is, $\hat{G} = \hat{C} \times \hat{D}$. Let $\tilde{D}$ be the natural inverse image of $\hat{D}$ in $\tilde{G}$. We show that $\tilde{G} = \tilde{G}_0 \tilde{D} \cong \tilde{G}_0 \times \tilde{D}$ and that $\tilde{D}$ is totally disconnected. In order to establish the relation $\tilde{G} = \tilde{G}_0 \tilde{D}$, it suffices to show that $\tilde{D}$ is mapped naturally onto $\tilde{G}/\tilde{G}_0 P(\tilde{G})$. But $\tilde{D}$ is mapped onto $\hat{D} \subseteq \tilde{G}/P(\tilde{G})$, and $\hat{D}$ is mapped onto

$$\hat{G}/\hat{C} = [\tilde{G}/P(\tilde{G})]/[\tilde{G}_0 P(\tilde{G})/P(\tilde{G})] \cong \tilde{G}/\tilde{G}_0 P(\tilde{G}).$$

Let $\tilde{x}$ be an element of $\tilde{G}_0 \cap \tilde{D}$; then $\tilde{x}P(\tilde{G})$ is in $[\tilde{G}_0 P(\tilde{G})/P(\tilde{G})] \cap \hat{D}$. Hence $\tilde{x}P(\tilde{G}) = P(\tilde{G})$ and $\tilde{x} \in P(\tilde{G})$. But $P(\tilde{G}) \cap \tilde{G}_0 = \tilde{e}$, where $\tilde{e}$ is the identity of $\tilde{G}$. Therefore $\tilde{x} = \tilde{e}$. Finally, let $\tilde{V}$ be a neighborhood of $\tilde{e}$ in $\tilde{G}_0$, and let $\tilde{W}$ be a neighborhood of $\tilde{e}$ in $\tilde{D}$. Then $\tilde{V}(\tilde{W} \cap P(\tilde{G}))$ is a neighborhood of $\tilde{e}$ in $\tilde{G}_0 P(\tilde{G}) \cong \tilde{G}_0 \times P(\tilde{G})$. But $\tilde{G}_0 P(\tilde{G})$ is open, and hence $\tilde{V}(\tilde{W} \cap P(\tilde{G}))$ is a neighborhood of $\tilde{e}$ in $\tilde{G}$. It follows that $\tilde{G} \cong \tilde{G}_0 \times \tilde{D}$. Since $\tilde{D} \cong \tilde{G}/\tilde{G}_0$, $\tilde{D}$ is totally disconnected and locally compact and has dense bounded part. The proof of the theorem is complete.

PROPOSITION 3. *Let* G *be a locally compact group with dense bounded part. Then an element* g *belongs to* B(G) *if and only if the smallest closed normal subgroup* N_g *containing* g *is either compact or is a semidirect product of a compact group and an infinite discrete, cyclic group.*

We may run the proof of this proposition in exactly the same way as we did for Proposition 1, because, by Theorem 4, $G/P(G)$ is abelian.

PROPOSITION 4. *Let* G *be a locally compact group with dense bounded part. If* G *is compactly generated, then* G *is an* $\overline{FC}$*-group.*

Proof. By Theorem 5, G is an extension of a compact group K by $G/K \cong V \times T$, where V is a vector group and T is a totally disconnected, locally compact group such that $\overline{B(T)} = T$. Since G/K is compactly generated and $T \cong (G/K)/V$, T is also compactly generated. Therefore Proposition 2 implies that $B(T) = T$. Since $B(G/K) \cong V \times B(T)$, G/K is an $\overline{FC}$-group. It follows that G is an $\overline{FC}$-group.

5. LOCALLY COMPACT GROUPS G FOR WHICH $G/\overline{B(G)}$ IS COMPACT

LEMMA 7. *Let* G *be a locally compact group, and let* G_0 *be its identity component. If* G_0 *is an analytic group without compact, normal subgroups except the identity subgroup, then the elements of* $B(G)$ *commute with the elements of* $B(G_0)$.

Proof. Let x be an element of $B(G)$, and let I_x be the inner automorphism on G_0 induced by x. Then, by Theorem B, there exists an element $g \in B(G_0)$ such that $I_g = I_x$, where I_g is the inner automorphism on G_0 induced by g. Now let y be an element in $B(G_0)$; then $I_g(y) = I_x(y)$, that is, $gyg^{-1} = xyx^{-1}$. But $B(G_0)$ is abelian (Theorem B); thus $xyx^{-1} = y$, so that $xy = yx$.

LEMMA 8. *Let* G *be a locally compact group such that* $G/\overline{B(G)}$ *is compact, and let* G_0 *be the identity component of* G. *Then* $B(G_0)$ *is contained in* $B(G)$.

Proof. By Lemma 3, there exists a compact, normal subgroup L of G such that $L \subseteq G_0$, and the identity component G_0/L of G/L is an analytic group without compact, normal subgroups other than the identity subgroup. Because L is compact, it is sufficient to show that $B(G_0/L) \subseteq B(G/L)$. Let $\tilde{G} = G/L$ and $\tilde{G}_0 = G_0/L$. Since $G/\overline{B(G)}$ is compact, it follows that $\tilde{G}/\overline{B(\tilde{G})}$ is compact, and there exists a compact subset $\tilde{C}$ of $\tilde{G}$ such that $\tilde{G} = \tilde{C}\,\overline{B(\tilde{G})}$. Now let $y \in B(\tilde{G}_0)$ and $x = kb \in \tilde{G}$, where $k \in \tilde{C}$ and $b \in \overline{B(\tilde{G})}$. Then $xyx^{-1} = kbyb^{-1}k^{-1} \in \tilde{C}y\tilde{C}^{-1}$, since, by Lemma 7, $byb^{-1} = y$. Hence $y \in B(\tilde{G})$.

LEMMA 9. *Let* G *be a locally compact group whose identity component* G_0 *is an analytic group without compact, normal subgroups except the identity subgroup. If* $G/\overline{B(G)}$ *is compact, then* $G_0/B(G_0)$ *is also compact.*

Proof. By Lemma 4, there exists a compact subgroup K of G such that KG_0 is open and isomorphic to $K \times G_0$. Since KG_0 is σ-compact, it follows that $KG_0\overline{B(G)}/\overline{B(G)}$ is isomorphic to $KG_0/(KG_0) \cap \overline{B(G)}$. By assumption, $G/\overline{B(G)}$ is compact; therefore $KG_0\overline{B(G)}/\overline{B(G)}$, as a closed subgroup of $G/\overline{B(G)}$, is also compact. This implies that $KG_0/(KG_0) \cap \overline{B(G)}$ is compact.

Next we show that $(KG_0) \cap \overline{B(G)} \subseteq KB(G_0)$. Let x be an element of $(KG_0) \cap \overline{B(G)}$. Since $KG_0 \cong K \times G_0$, there exist unique elements k and g in K and G_0, respectively, such that $x = kg$. Since $x \in \overline{B(G)}$, there exists a net $\{x_\lambda\}$ in $B(G)$ converging to x. We can assume that all x_λ lie in KG_0, because KG_0 is open. Now, let $x_\lambda = k_\lambda g_\lambda$, where $k_\lambda \in K$ and $g_\lambda \in G_0$. Clearly, $k_\lambda \to k$ and $g_\lambda \to g$. Since each $k_\lambda g_\lambda$ lies in $(KG_0) \cap B(G)$, it follows that $k_\lambda g_\lambda \in B(KG_0)$. But $B(KG_0) = K \times B(G_0)$; thus $g_\lambda \in B(G_0)$. This implies that $x = kg \in KB(G_0)$, because $B(G_0)$ is closed (Theorem B). Hence $(KG_0) \cap \overline{B(G)} \subseteq KB(G_0)$. The compactness of $KG_0/(KG_0) \cap \overline{B(G)}$ implies the compactness of $KG_0/KB(G_0)$. Finally, we notice that

$$KG_0/KB(G_0) \cong (K/K) \times (G_0/B(G_0)) \cong G_0/B(G_0).$$

Hence $G_0/B(G_0)$ is compact.

COROLLARY 7. *Let* G *be a locally compact group whose identity component* G_0 *is an analytic group without compact, normal subgroups other than the identity subgroup. If* $G/\overline{B(G)}$ *is compact, then* $G_0\overline{B(G)}$ *is a locally compact subgroup.*

Proof. By Lemma 9, we know that $G_0/B(G_0)$ is compact. Lemma 8 tells us that $B(G_0) \subseteq B(G)$. It follows that $G_0/G_0 \cap \overline{B(G)}$ is compact, since $B(G_0) \subseteq G_0 \cap B(G)$. Because $G_0\overline{B(G)}/\overline{B(G)}$ is a continuous image of $G_0/G_0 \cap \overline{B(G)}$, it is also compact. This implies that $G_0\overline{B(G)}$ is locally compact.

Remark. Let G be a locally compact group whose identity component G_0 is an analytic group without compact, normal subgroups except the identity subgroup. If $G/\overline{B(G)}$ is compact, then by Lemma 5, $G_0/B(G_0)$ is compact. Since G_0 is compactly generated, $B(G_0)$ is also compactly generated. As a compactly generated, locally compact, abelian group, $B(G_0) = V \times Z^n \times K$, where V is a vector group, Z is the set of integers, and K is a compact abelian group. It is not hard to see that K is a characteristic subgroup of $B(G_0)$ and is therefore a normal subgroup of G_0. It follows that K is trivial and $G_0/V \times Z^n$ is compact. Choose a free abelian group $Z^m \subseteq V$, where m is the dimension of the vector group V. Then $Z^m \times Z^n = Z^{m+n}$ is a uniform subgroup of G_0; that is, the space G_0/Z^{m+n} is compact. The structure of a connected, locally compact group with a uniform, finitely generated, free abelian subgroup has been studied in a paper by the first author [11]. Here we record the main result in that paper for reference.

PROPOSITION. *Let* G *be a connected locally compact group, and let* Z^r *be a finitely generated, free abelian, uniform subgroup of* G. *Then* G *contains a compact normal subgroup* K *of* G *such that* $H = G/K$ *contains no compact nondiscrete normal subgroups. Let* R *be the radical of* H, *and let* N *be the nilradical of* H. *Then* H/R *is compact and* R *splits over* N.

THEOREM 7. *Let* G *be a locally compact group such that* $G/\overline{B(G)}$ *is compact. Then* G *is an extension of a compact group by a locally compact group* $\tilde{G}$ *whose identity component* $\tilde{C}$ *is an analytic group without compact, normal subgroups other than the identity subgroup, and there exist subgroups* $\tilde{K}$ *and* $\tilde{Q}$ *of* $\tilde{G}$ *such that*

(1) $\tilde{K}$ *is compact and totally disconnected, and* $\tilde{Q} \subseteq B(\tilde{G})$,

(2) $\tilde{K}\tilde{C}$ *is isomorphic to* $\tilde{K} \times \tilde{C}$ *and is open,*

(3) $\tilde{G}/\tilde{K}\tilde{C}\tilde{Q}$ *is a finite discrete space,*

(4) *the elements of* $\overline{\tilde{Q}}$ *commute with the elements in* $\tilde{C}$, *and* $\overline{\tilde{Q}} \cap \tilde{C}$ *is the center of* $\tilde{C}$,

(5) $\tilde{C}/B(\tilde{C})$ *is compact.*

Proof. By Lemma 3, G is an extension of a compact group by a locally compact group $\tilde{G}$ whose identity component $\tilde{C}$ is an analytic group without compact, normal subgroups other than the identity subgroup. By Lemma 4, there exists a compact subgroup $\tilde{K}$ of $\tilde{G}$ such that $\tilde{K}\tilde{C}$ is open and is isomorphic to $\tilde{K} \times \tilde{C}$. Clearly, $\tilde{K}$ is isomorphic to $\tilde{K} \times \tilde{C}/\tilde{C}$ and is totally disconnected. Let

$$\tilde{Q} = \{x \in B(\tilde{G}) \mid xy = yx \text{ for all } y \in \tilde{C}\}.$$

It is easy to see that $\tilde{Q}$ is a characteristic subgroup of $\tilde{G}$. Clearly, $\tilde{Q} \subseteq B(\tilde{G})$ and the elements of $\overline{\tilde{Q}}$ commute with the elements of $\tilde{C}$.

Next we show that $\tilde{C}\tilde{Q} = \tilde{C}B(\tilde{G})$. Let $x \in B(\tilde{G})$; then the inner automorphism I_x induced by x is a bounded automorphism. By Theorem B, there exists an element $y \in B(\tilde{C})$ such that $I_x(g) = I_y(g)$ for all $g \in \tilde{C}$. This implies that $y^{-1}xgx^{-1}y = g$ for all $g \in \tilde{C}$. Since $B(\tilde{C}) \subseteq B(\tilde{G})$ by Lemma 8, $y^{-1}x$ is in $B(\tilde{G})$ and hence in $\tilde{Q}$. Therefore $x \in y\tilde{Q} \subseteq \tilde{C}\tilde{Q}$. It follows that $\tilde{C}\tilde{Q} = \tilde{C}B(\tilde{G})$.

Since $G/\overline{B(G)}$ is compact, $\tilde{G}/\overline{B(\tilde{G})}$ is also compact. This implies that $\tilde{G}/\tilde{K}\tilde{C}\overline{B(\tilde{G})}$ is compact. Since $\tilde{K}\tilde{C}$ is open,

$$\tilde{K}\tilde{C}\overline{B(\tilde{G})} = \tilde{K}\tilde{C}B(\tilde{G}) = \tilde{K}\tilde{C}\tilde{Q}.$$

Hence $\tilde{G}/\tilde{K}\tilde{C}\tilde{Q}$ is a finite discrete space.

Now we denote the center of $\tilde{C}$ by $Z(\tilde{C})$ and show that $\overline{\tilde{Q}} \cap \tilde{C} = Z(\tilde{C})$. Clearly, $\overline{\tilde{Q}} \cap \tilde{C} \subseteq Z(\tilde{C})$. Let x be an element of $Z(\tilde{C})$. Then x is in $B(\tilde{C})$ and hence in $B(\tilde{G})$. This implies that $x \in \tilde{Q} \cap \tilde{C}$, by the definition of $\tilde{Q}$. Consequently, $\overline{\tilde{Q}} \cap \tilde{C} = Z(\tilde{C})$.

Finally, the compactness of $\tilde{C}/B(\tilde{C})$ follows from Lemma 9. The proof is complete.

6. EXAMPLES

In this section we give some examples. First, we prove two propositions that are helpful in constructing examples of locally compact, totally disconnected groups with dense bounded parts.

PROPOSITION 5. *Let* G *be a locally compact, totally disconnected group; let* N *be a closed normal subgroup of* G *such that* $B(N) = N$, *and let* K *be a compact open subgroup of* G. *If* $G = NK$, *then* G *is an extension of a compact group by a semidirect product of a discrete* FC-*group and a compact group.*

Proof. Consider the subgroup $N \cap K$, which is compact and open in N. By Corollary 4, $N \cap K$ is contained in a compact, open, normal subgroup L of N. Since $N \cap K$ is open in N, we can assume that L is the smallest closed normal subgroup of N containing $N \cap K$; that is, L is the subgroup generated by the set $\{nxn^{-1} \mid x \in N \cap K, n \in N\}$. Now we show that LK is actually a subgroup. It suffices to show that $LK = KL$. Let $n \in N$, $x \in N \cap K$, and $k \in K$. Because N is a normal subgroup of G, $kn = n'k$ for some $n' \in N$; also,

$$k(nxn^{-1})k^{-1} = n'kxk^{-1}n'^{-1} = n'x'n'^{-1},$$

where $x' = kxk^{-1}$. Since $N \cap K$ is normal in K, x' is in $N \cap K$ and $n'x'n'^{-1}$ is in L. It follows that for each $r \in L$, there exists $r' \in L$ such that $krk^{-1} = r'$. This implies that $LK = KL$ and that L is normal in $NK = G$. Clearly, N/L is a discrete FC-group and LK/L is a compact group.

It remains to prove that G/L is a semidirect product of N/L and LK/L. Clearly, $N \cap LK = L$. Since LK/L is compact, the restriction to LK/L of the natural mapping $G/L \to (G/L)/(N/L)$ is an isomorphism. Consequently, $G/L = (N/L)$ Ⓢ (LK/L).

PROPOSITION 6. *Let* G *be a topological group,* N *a closed normal subgroup of* G, *and* K *a compact subgroup of* G. *Suppose* $G = NK$, $\overline{B(N)} = N$, *and the set* K' *of elements* k *in* K *for which the restriction of* I_k *to* N *is a bounded automorphism is dense in* K. *Then* $\overline{B(G)} = G$.

Proof. Let k_0n_0 be an element of G, where $k_0 \in K$ and $n_0 \in N$. Since $\overline{K'} = K$ and $\overline{B(N)} = N$, there exist nets $\{k_\lambda\}$ and $\{n_\mu\}$ in K' and B(N), respectively, such that $k_\lambda \to k_0$ and $n_\mu \to n_0$. This implies that $k_\lambda n_\mu \to k_0n_0$. Now let kn be an element in G such that $k \in K$ and $n \in N$. We see that

$$(kn)(k_\lambda n_\mu)(kn)^{-1} = knk_\lambda n^{-1}k_\lambda^{-1}k_\lambda n n_\mu n^{-1}k^{-1} = k(nI_{k_\lambda}(n^{-1}))k_\lambda (n n_\mu n^{-1})k^{-1}$$

is contained in a compact subset of G. It follows that $k_\lambda n_\mu$ is contained in B(G). Hence $\overline{B(G)} = G$.

Example 1. Let A be a finite abelian group, and let B be the group of all automorphisms on A. Let I be an infinite index set, and let $A_i \cong A$, $B_i \cong B$ for all $i \in I$. Suppose N is the weak direct product of A_i $(i \in I)$ with discrete topology and K is the complete direct product of B_i $(i \in I)$ with product topology. Then we can form the semidirect product $G = N \circledS K$ in the natural fashion. Let K' be the set of elements k in K such that the inner automorphism on N induced by k is bounded. It is not hard to see that $\overline{K'} = K$. Hence $\overline{B(G)} = G$.

Example 2. Let $G = Z \times Z_2$ with natural actions on Z, where Z is the set of integers and Z_2 is a group of order 2. Then we see that $B(G) = Z \times 0$ and P(G) is the union of $\{(0, 0)\}$ and $Z \times 1$. Also, $G/\overline{B(G)}$ is compact and $\overline{B(G)} \neq G$; however, P(G) does not form a subgroup.

Example 3. Let R be the set of real numbers. Suppose R acts on R^2 by rotations: $tzt^{-1} = |z| e^{i2\pi t}$, where $z \in R^2$ and $t \in R$. Then we have a topological semidirect product $R^2 \circledS R$. Let

$$G = \frac{(R^2 \circledS R) \times Z}{H},$$

where Z is the set of integers and H is the cyclic group generated by (0, 1, 2). Then $B(G) = \dfrac{R^2 \times Z \times Z}{H}$ and $G/\overline{B(G)}$ is compact.

In Theorem 7, we proved that the elements of $\tilde{\tilde{Q}}$ commute with the elements of the identity component $\tilde{C}$ of $\tilde{G}$. One may ask the following question. Does there exist a subgroup $\tilde{Q}_1$ of $\tilde{Q}$ such that $\tilde{C}\tilde{Q} = \tilde{C} \times \tilde{Q}_1$? Our example shows that it may be impossible. In fact, using the same notation as in Theorem 7, we may take G as $\tilde{G}$ in our example. It is not hard to see that $\tilde{Q}$ is equal to $\dfrac{0 \times Z \times Z}{H}$ and is closed and isomorphic to Z. If $\tilde{C}\tilde{Q}$ is isomorphic to $C \times \tilde{Q}_1$ for some subgroup $\tilde{Q}_1$ of $\tilde{Q}$, then $\tilde{Q} = (\tilde{Q} \cap \tilde{C}) \times \tilde{Q}_1$. Since $\tilde{Q}$ is isomorphic to Z and $\tilde{Q} \cap \tilde{C}$ is nontrivial, it follows that $\tilde{Q}_1$ must be trivial. This implies that $\tilde{Q} \subseteq \tilde{C}$. However, this is not true, since the element (0, 0, 1)H is in $\tilde{Q}$, but is not in $\tilde{C}$.

Added June 12, 1972. The authors wish to call to attention that the following paper has significant results in the same area: S. Grosser and M. Moskowitz, *Compactness conditions in topological groups*. J. Reine Angew. Math. 246 (1971), 1-40.

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