PRODUCT MANIFOLDS THAT ARE NOT NEGATIVE SPACE FORMS

Patrick Eberlein

INTRODUCTION

In [2], it was shown that for every integer $k \geq 1$ there exist compact Riemannian manifolds M of dimension 4k and sectional curvature K ($-4 \leq K \leq -1$) that do not admit a Riemannian metric with $K \equiv -1$. In this paper, we show by a different method that there exist many (noncompact) product manifolds M that admit a complete metric with sectional curvature $K \leq -1$, but admit no complete metric with $K \equiv -1$. For example, if B is a compact Riemannian manifold of dimension $n \geq 2$ and sectional curvature $K < 0$, then $M = R \times B \times S^1$, where S^1 and R denote the unit circle and real line, is such a manifold.

If M is a complete Riemannian manifold with sectional curvature $K \leq 0$, then M is a quotient H/D, where H is a complete, simply connected manifold with $K \leq 0$, and D is a properly discontinuous group of isometries of H. As in [3], one may construct the set $H(\infty)$ of points at infinity for H and define a limit set $L(D) \subset H(\infty)$ for the group D. If $M = H/D$ has curvature $K \leq -1$, then the limit set must be one of three types described in Theorem 1. If $\pi_1(M)$ is a nontrivial direct product, then the discussion of monic groups shows that L(D) must be of the first type. If M admits a complete metric with $K \equiv -1$, and if L(D) is of the first type, then by Proposition 4, $\pi_1(M)$ is isomorphic to $\pi_1(M')$, where M' is a complete flat manifold. Using warped products, we construct manifolds M with sectional curvature $K \leq -1$ such that $\pi_1(M)$ is a nontrivial direct product not isomorphic to $\pi_1(M')$ for any flat manifold M'. Our result has the defect that the curvature of M is not bounded below; but this defect is inherent in our method.

The necessary background results are developed in [1] and [3], and more detailed discussions of points at infinity, limit sets, monic groups, and warped products may be found there with proofs of our unproved assertions.

MANIFOLDS WITH SECTIONAL CURVATURE $K \leq 0$

Let H denote a complete, simply connected Riemannian manifold of dimension $n \geq 2$ and sectional curvature $K \leq 0$. Unit-speed geodesics γ and σ in H are *asymptotic* if there exists a number $c > 0$ such that $d(\gamma t, \sigma t) \leq c$ for all $t \geq 0$, where d denotes the Riemannian metric of H. The relation of being asymptotic is an equivalence relation on the geodesics of H (which shall always be assumed to have unit speed), and the equivalence classes are *points at infinity* for H. If $H(\infty)$ denotes the set of points at infinity, then the space $\overline{H} = H \cup H(\infty)$ with a natural topology is homeomorphic to the closed unit ball in R^n, and $H(\infty)$ is homeomorphic to the bounding sphere S^{n-1}.

Received June 4, 1971.
This research was supported in part by NSF Grants GP-11476 and GP-20096.

If γ is a geodesic of H, let $\gamma(\infty)$ denote the equivalence class of γ, and let $\gamma(-\infty)$ denote the equivalence class of the reverse curve $t \to \gamma(-t)$. If ϕ is an isometry of H and x is a point in $H(\infty)$, let $\phi(x) = (\phi \circ \gamma)(\infty)$, where γ denotes any geodesic in the equivalence class x. The mapping ϕ is well-defined and becomes a homeomorphism of $\overline{H}$. Let D denote a nonempty, properly discontinuous group of isometries of H, together with its extension to a group of homeomorphisms of $\overline{H}$. The *limit set* of D, denoted by L(D), is defined as the set of accumulation points in $H(\infty)$ of an orbit D(p); here p denotes any point of H (its choice is immaterial). The set L(D) is nonempty and closed, and it is invariant under D.

A geodesic γ in H is said to *join* points $x \neq y$ in $H(\infty)$ if $\gamma(-\infty) = x$ and $\gamma(\infty) = y$. A manifold H satisfies *Axiom* 1 if for each pair of points $x \neq y$ in $H(\infty)$ there exists a geodesic γ joining x to y. Axiom 1 is satisfied, for example, if the sectional curvature satisfies an inequality of the form $K \leq c < 0$ [3].

If ϕ is an isometry of H, then ϕ has at least one fixed point in $\overline{H}$, since $\overline{H}$ is a cell. An isometry ϕ is *elliptic* if it has a fixed point in H. If ϕ is not elliptic, and if H satisfies Axiom 1, then ϕ has either exactly one fixed point in $H(\infty)$ (ϕ is *parabolic*), or exactly two fixed points in $H(\infty)$ (ϕ is *axial*).

The following result classifies the sets L(D).

THEOREM 1. *Let* H *satisfy Axiom* 1, *and let* D *be a properly discontinuous group of isometries of* H. *Then either*

(1) L(D) *is a singleton* $\{x\}$, *and every element of* D *is parabolic with fixed point* x; *or*

(2) L(D) *consists of two points* x *and* y, D *is infinite cyclic, and every element of* D *is axial with fixed points* x *and* y; *or*

(3) L(D) *is an infinite set, and the elements of* D *have no common fixed point.*

The group D and the quotient manifold H/D are called *parabolic, axial,* or *Fuchsian* according as L(D) is of type (1), (2), or (3). If M is a compact Riemannian manifold with $K < 0$, then M is Fuchsian. In particular, if M is a compact, orientable surface of genus $n \geq 2$, then M admits a Riemannian metric of Gaussian curvature $K \equiv -1$, and thus M is Fuchsian. There exist many noncompact Fuchsian manifolds [3].

MONIC GROUPS

A *disjoint decomposition* of a group G is an indexed collection $\{G_i\}$ of subgroups such that

(1) $G = \bigcup_i G_i$, and for each pair i, j, either $G_i = G_j$ or $G_i \cap G_j = \{1\}$,

(2) each G_i has strictly disjoint conjugates; that is, if $xG_ix^{-1} \cap G_i \neq \{1\}$, then $x \in G_i$.

A group G is *monic* if the only disjoint decomposition it possesses is the trivial one such that $G_i = G$ for all i; otherwise, G is *multic*. Let H satisfy Axiom 1, and let D be a properly discontinuous group of isometries of H. For each $x \in H(\infty)$, let $D_x = \{\phi \in D: \phi x = x\}$. By Proposition 9.2 of [3], the stability groups D_x form a disjoint decomposition of $D \simeq \pi_1(M)$, where $M = H/D$. If D is monic, then D has a fixed point in $H(\infty)$, and by Theorem 1, D cannot be Fuchsian. The following result is Proposition 9.4 of [3].

PROPOSITION 1. *A group* G *is monic if*

(1) G *has nontrivial center; or*

(2) G *is a nontrivial direct product; or*

(3) G *has a monic normal subgroup* $N \neq \{1\}$.

WARPED PRODUCTS

For the construction of product manifolds with sectional curvature $K \leq -1$, we shall need the concept of warped products, developed by R. L. Bishop and B. O'Neill in Section 7 of [1].

Let B and F be Riemannian manifolds, and let f be a positive differentiable function on B. Consider the differentiable product manifold $B \times F$ with projections $\pi: B \times F \to B$ and $\eta: B \times F \to F$. The *warped product* $M = B \times_f F$ is the manifold $B \times F$ furnished with the Riemannian structure such that

$$\|x\|^2 = \|\pi_*(x)\|^2 + f^2(\pi m)\,\|\eta_*(x)\|^2$$

for every tangent vector $x \in M_m$. If B and F are complete Riemannian manifolds, then M is a complete Riemannian manifold.

We shall be interested in the case where B is the real line R. If π is a 2-plane tangent to $R \times_f F$ at (t, p), let π have an orthonormal basis consisting of $x + v$ and w, where x is tangent to R at t, while v and w are tangent to F at p. If v and w span a plane σ, let L(v, w) be the sectional curvature in F of σ; otherwise, let L(v, w) be zero. In $R \times_f F$, the sectional curvature of π is given by the formula

$$K(\pi) = \frac{-f''(t)}{f(t)}\,\|x\|^2 + \frac{L(v, w) - f'(t)^2}{f(t)^2}\,\|v\|^2 ,$$

where the warped-product norms $\|x\|^2$ and $\|v\|^2$ satisfy the condition $\|x\|^2 + \|v\|^2 = 1$ (for a derivation of the formula, see [1, pp. 23-27]).

From the formula for $K(\pi)$ we easily obtain the following result.

PROPOSITION 2. *Let* $M = R \times_f F$; *let* L *and* K *denote the sectional curvature of* F *and* M, *respectively.*

(1) *If* $L \leq 0$ *and* $f(t) = e^t$, *then* $K \leq -1$.

(2) *If* $L \equiv 0$ *and* $f(t) = e^{ct}$, *where* c *denotes a positive constant, then* $K \equiv -c^2$.

(3) *If there exist positive constants* c *and* d *such that* $-c^2 \leq L \leq -d^2 < 0$ *and if* $f(t) = \cosh t$, *then* $-\alpha^2 \leq K \leq -\beta^2$, *where* $\alpha = \max\{1, c\}$ *and* $\beta = \min\{1, d\}$.

The sectional curvature K of a Riemannian manifold X is *negatively pinched* if there exist positive constants c and d such that $-c^2 \leq K(\pi) \leq -d^2 < 0$, for all 2-planes π tangent to X. We show that cases (2) and (3) in the preceding result provide essentially the only possibilities for the construction of warped-product manifolds of the form $R \times F$ with negatively pinched sectional curvature.

PROPOSITION 3. *Let* F *be a complete Riemannian manifold, and let* f *denote a positive* C^∞*-function on* R. *Suppose that the sectional curvature of* $M = R \times_f F$ *is negatively pinched. Then either*

(1) f *is a strictly convex function on* R *with a minimum, and the sectional curvature* L *of* F *is negatively pinched; or*

(2) f *is a strictly convex function on* R, *with* $\inf f = 0$, *and* F *is a flat manifold* $(L \equiv 0)$.

Proof. By Lemma 7.6 of [1], F has sectional curvature $L \leq 0$, and f is strictly convex ($f''(t) > 0$ for all t). This can also be seen directly from the curvature formula. Let c and d be positive numbers such that $-c^2 \leq K \leq -d^2$. We divide the proof into two cases.

Case 1: f has a minimum at t_0. Let σ be any 2-plane tangent to F at p, and let σ_{t_0} be the induced 2-plane tangent to M at (t_0, p). By the curvature formula, $K(\sigma_{t_0}) = \frac{L(\sigma)}{f(t_0)^2}$. Hence

$$-f(t_0)^2 c^2 \leq L(\sigma) \leq -f(t_0)^2 d^2 ,$$

and L is negatively pinched.

Case 2: f has no minimum. Replacing f by the function $g(t) = f(-t)$, if necessary, we may assume that f(t) decreases to $a \geq 0$ as $t \to +\infty$. The manifold $M' = R \times_g F$ is isometric to $M = R \times_f F$, under the map $(t, p) \to (-t, p)$.

We show that $a = 0$. Since f(t) is decreasing, there exists a sequence $\{t_n\}$ of positive real numbers such that $t_n \to \infty$ and $f''(t_n) \to 0$ as $n \to \infty$. For each integer n, let π_n be a 2-plane containing a vector tangent to R at t_n. Then $K(\pi_n) = \frac{-f''(t_n)}{f(t_n)}$. If $a > 0$, then $K(\pi_n) \to 0$ as $n \to \infty$; this contradicts the assumption that K is negatively pinched.

We now show that F is flat. Suppose that $L(\sigma) < 0$ for some 2-plane σ tangent to F at p. For any $t \in R$, let σ_t denote the induced 2-plane tangent to M at (t, p). Then

$$K(\sigma_t) = \frac{L(\sigma) - f'(t)^2}{f(t)^2} \leq \frac{L(\sigma)}{f(t)^2} \to -\infty$$

as $t \to +\infty$; this contradicts the fact that K is negatively pinched.

PARABOLIC MANIFOLDS WITH $K \equiv -1$

If a manifold M admits a complete Riemannian metric with $K \equiv -1$, then M must be parabolic, axial, or Fuchsian, by Theorem 1. The following result and the discussion after it classify the parabolic manifolds with $K \equiv -1$.

PROPOSITION 4. *Let* M *be a parabolic manifold with sectional curvature* $K \equiv -1$. *If* M *has dimension* 2, *then* $\pi_1(M)$ *is infinite cyclic. If* M *has dimension* $n \geq 3$, *then there exists a flat manifold* M' *of dimension* $n - 1$ *such that* $\pi_1(M)$ *is isomorphic to* $\pi_1(M')$.

Proof. We shall consider only the case where M has dimension $n \geq 3$. The case $n = 2$ is treated similarly.

The n-dimensional hyperbolic space H^n is the unique complete, simply connected, Riemannian manifold with sectional curvature $K \equiv -1$. It follows that if M is an n-dimensional parabolic manifold with sectional curvature $K \equiv -1$, then M is a quotient manifold H^n/D, where the isometry group D has a unique fixed point $x \in H^n(\infty)$. We recall that

$$H^n = \{(x_1, \cdots, x_n): x_i \in R \text{ and } x_n > 0\},$$

and that the inner product at $x = (x_1, \cdots, x_n)$ is the usual dot product multiplied by by the factor $1/x_n^2$. Let y denote the asymptotic equivalence class of the geodesic $\gamma(t) = (0, 0, \cdots, 0, e^t)$, and let T be an isometry of H^n such that $T(y) = x$. If $D^* = T^{-1} \circ D \circ T$, then D^* is a properly discontinuous group of parabolic isometries of H^n, and it has the unique fixed point y. Since $\pi_1(M)$ is isomorphic to D^*, it suffices to show that D^* is isomorphic to $\pi_1(M')$ for some complete, flat manifold M'.

The set $L = \{(x_1, \cdots, x_n) \in H^n: x_n = 1\}$ is a complete hypersurface of sectional curvature $K \equiv 0$. This is the *horosphere* determined by y that passes through the point $(0, 0, \cdots, 0, 1)$. We show that L is invariant under D^*.

Given $p = (x_1, \cdots, x_{n-1}, 1) \in L$ and $\phi^* \in D^*$, let $\phi^*(p) = (y_1, \cdots, y_n)$. If $\gamma(t) = (x_1, \cdots, x_{n-1}, e^t)$, then $\phi^*\gamma(t) = (y_1, \cdots, y_{n-1}, y_n e^t)$, since ϕ^* fixes $y \in H^n(\infty)$. It is easy to show that if d is the Riemannian metric in H^n, then $d(\gamma t, \phi^*\gamma t) \to |\log y_n|$ as $t \to \infty$. On the other hand, $d(\gamma t, \phi^*\gamma t) \to 0$ as $t \to \infty$, by Propositions 10.8 and 10.9 of [1]. Hence $y_n = 1$ and D^* preserves L.

For each $\phi^* \in D^*$, we define a map $\phi': R^{n-1} \to R^{n-1}$ such that

$$(\phi'(x_1, \cdots, x_{n-1}), 1) = \phi^*(x_1, \cdots, x_{n-1}, 1)$$

for each point $x = (x_1, \cdots, x_{n-1}) \in R^{n-1}$. It is not difficult to show that $D' = \{\phi': \phi^* \in D^*\}$ is a properly discontinuous group of isometries of R^{n-1}. If $M' = R^{n-1}/D'$, then $\pi_1(M) \simeq D \simeq D^* \simeq D' \simeq \pi_1(M')$.

Remark 1. (1) For any number $a > 0$, we could have considered the horosphere $L_a = \{(x_1, \cdots, x_n) \in H^n: x_n = a\}$. D^* preserves L_a and induces the same group D'.

(2) If ϕ' is an isometry of R^n, then we may define an isometry ϕ^* of H^{n+1} by setting $\phi^*(x_1, \cdots, x_{n+1}) = (\phi'(x_1, \cdots, x_n), x_{n+1})$. If D' is a properly discontinuous group of isometries of R^n, then $D^* = \{\phi^*: \phi' \in D'\}$ is a properly discontinuous group of parabolic isometries of H^{n+1} with unique fixed point $\gamma(\infty)$, where $\gamma(t) = (0, \cdots, 0, e^t)$. The flat manifold R^n/D' induces a parabolic manifold H^{n+1}/D^* with $K \equiv -1$.

(3) $R^+ = (0, \infty)$ is complete in the metric determined by the formula $\|d/du(t)\|^2 = 1/t^2$. If $f(t) = 1/t$, then $R^+ \times_f R^n$ is isometric to H^{n+1} under the map $(t, (x_1, \cdots, x_n)) \to (x_1, \cdots, x_n, t)$. If D' and D^* are groups related to each other as above, then the manifolds $R^+ \times_f (R^n/D')$ and H^{n+1}/D^* are isometric. Hence, for $n \geq 3$, the parabolic manifolds M of dimension n and curvature $K \equiv -1$ are precisely the warped products $M = R^+ \times_f M'$, where M' is a complete manifold of dimension n - 1 and curvature $K \equiv 0$. The warped product $R^+ \times_f S^1$ is the only parabolic manifold of dimension 2 and curvature $K \equiv -1$.

PRODUCT MANIFOLDS THAT ARE NOT NEGATIVE SPACE FORMS

THEOREM 2. *Let* B *and* F *be complete, multiply connected, Riemannian manifolds of arbitrary dimension greater than* 1 *and with sectional curvature* $K \leq 0$, *and let* B *be Fuchsian. Let* S^1 *and* R *denote the unit circle and the real line. Then the differentiable product manifolds* $R \times B \times F$ *and* $R \times B \times S^1$ *admit a complete metric with sectional curvature* $K \leq -1$, *but they admit no complete metric with* $K \equiv -1$.

Proof. The Riemannian products (with warping function $f \equiv 1$) $B \times F$ and $B \times S^1$ are complete manifolds with curvature $K \leq 0$. By Proposition 2, the warped products $M = R \times_{e^t} (B \times F)$ and $N = R \times_{e^t} (B \times S^1)$ are complete and have curvature $K \leq -1$. We show that neither manifold admits a complete metric with $K \equiv -1$.

Suppose that one of these manifolds, say M, admits a complete metric g with $K \equiv -1$. By Proposition 1 and the discussion preceding it, M cannot be Fuchsian with respect to g, since $\pi_1(M)$ is a nontrivial direct product. Since $\pi_1(M)$ is not infinite cyclic, M must be parabolic with respect to g, by Theorem 1. By Proposition 4, $\pi_1(M) \simeq \pi_1(B) \times \pi_1(F) \simeq \pi_1(M')$, where M' is a complete, flat manifold. According to [5, p. 106], there exists a compact, flat manifold M'' such that $\pi_1(M'') \simeq \pi_1(M')$. If we regard $\pi_1(M'')$ as a properly discontinuous group of isometries of a Euclidean space, then a theorem of Bieberbach [5, p. 100] states that the translations in $\pi_1(M'')$ form a normal abelian subgroup of finite index. Projecting from $\pi_1(M)$ onto $\pi_1(B)$, we see that $\pi_1(B)$ contains a normal abelian subgroup G of finite index in $\pi_1(B)$. Since $\pi_1(B)$ is an infinite group, $G \neq \{1\}$. By Proposition 1, G is a monic normal subgroup of $\pi_1(B)$, and therefore $\pi_1(B)$ is monic. This is impossible, since B is Fuchsian by hypothesis. The contradiction shows that neither M nor N is a negative space form.

Remark 2. (1) The argument above shows that neither $B \times F$ nor $B \times S^1$ is a negative space form. If either manifold admits a complete metric with $K \leq c < 0$, then we need not form the product with R to obtain our desired example. It is unclear under what conditions such a metric may exist. For example, if B and F are compact, then neither $B \times F$ nor $B \times S^1$ admits a metric with $K < 0$ [4].

(2) If B is the double torus, then $R \times B \times S^1$ is the simplest example (with respect to our method) of a C^∞-manifold that admits a complete metric with $K \leq -1$, but admits no complete Riemannian metric with $K \equiv -1$. We do not know whether such examples exist in dimensions 2 and 3. Our method has the defect that the sectional curvature of the product manifold is not negatively pinched. We shall now show that we can not correct the defect by choosing a different warping function. The sectional curvature of a Fuchsian manifold B is negative on some 2-planes, since a Euclidean space does not satisfy Axiom 1. Hence, the Riemannian products $B \times F$ and $B \times S^1$ have sectional curvature that is zero on some 2-planes and negative on others. By Proposition 3, there exists no positive C^∞-function f on R such that $R \times_f (B \times F)$ or $R \times_f (B \times S^1)$ has negatively pinched sectional curvature.

REFERENCES

1. R. L. Bishop and B. O'Neill, *Manifolds of negative curvature.* Trans. Amer. Math. Soc. 145 (1969), 1-49.

2. P. Eberlein, *Manifolds admitting no metric of constant negative curvature.* J. Differential Geometry 5 (1971), 59-60.

3. P. Eberlein and B. O'Neill, *Visibility manifolds* (to appear).

4. A. Preissmann, *Quelques propriétés globales des espaces de Riemann.* Comment. Math. Helv. 15 (1943), 172-216.

5. J. A. Wolf, *Spaces of constant curvature.* McGraw-Hill, New York, 1967.

University of California
Los Angeles, California
and
University of California
Berkeley, California 94720