NORMAL ANALYTIC FUNCTIONS AND LINDELÖF'S THEOREM

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1. INTRODUCTION

This paper deals with various weakenings of the hypotheses in Lindelöf's classical limit theorem for bounded analytic functions.

Let D and Γ denote the unit disk $|z| < 1$ and the unit circle $|z| = 1$, respectively. The open subarc of Γ with endpoints $z = 1$ and $\zeta = e^{i\theta}$ $(0 < \theta < 2\pi)$ is denoted by $A(0, \theta)$, while the domain bounded by the arc $A(0, \theta)$ and the closed chord subtending $A(0, \theta)$ is denoted by $G(0, \theta)$. For points $\zeta = e^{i\theta}$, we write $\zeta \to 1^+$ if $\theta \to 0^+$.

Lindelöf's theorem [1, p. 42] is the following proposition.

THEOREM L. *Suppose that* f *is a bounded analytic function in* D, *that*

$$\lim_{\zeta \to 1^+} |f(\zeta)| = \lim_{\zeta \to 1^+} (\limsup_{z \to \zeta} |f(z)|) = 0, \tag{1}$$

and that $0 < \theta < 2\pi$. *Then* $f(z) \to 0$ *as* $z \to 1$ *in* $G(0, \theta)$.

It is known [2, Theorem 5.6] that the condition (1) can be replaced by the condition

$$\lim_{\zeta \to 1^+, \zeta \in \Gamma - E} |f(\zeta)| = 0, \tag{2}$$

where $\mu E = 0$ (μ denotes Lebesgue measure on Γ). Moreover, it follows from a theorem of C. Carathéodory (see [1, p. 207] or [2, Theorem 5.5]) that if the radial limits of f satisfy the condition

$$\left|\lim_{r \to 1} f(r\zeta)\right| < \varepsilon$$

for almost every point ζ in some arc $A(0, \theta)$, then

$$|f(\zeta)| \leq \varepsilon \quad (\zeta \in A(0, \theta)).$$

(The latter inequality can also be deduced from the representation of f by its Poisson integral.) Thus we can replace the condition (1) in Theorem L by the condition

$$\lim_{\zeta \to 1^+, \zeta \in \Gamma - E} (\lim_{r \to 1} f(r\zeta)) = 0, \tag{3}$$

where $\mu E = 0$ and f has a radial limit at each $\zeta \in \Gamma - E$.

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In [3, Theorem 3], J. L. Doob uses a condition weaker than (3) to obtain a weaker form of Theorem L for bounded analytic functions. *Let* δ_ε *be the lower metric density from the north at* $z = 1$ *of the set*

$$R_\varepsilon = \{\zeta : |\lim_{r \to 1} f(r\zeta)| < \varepsilon\};$$

in other words, let

$$\delta_\varepsilon = \liminf_{\theta \to 0^+} \frac{\mu[A(0, \theta) \cap R_\varepsilon]}{\theta}.$$

If $\lim_{\varepsilon \to 0} (\varepsilon)^{\delta_\varepsilon} = 0$, *then* f *has angular limit zero at* $z = 1$.

O. Lehto and K. I. Virtanen [5, Theorem 2] have shown that Theorem L remains true under the weaker assumption that f is a normal meromorphic function in D. (A meromorphic function f is *normal* in D if and only if the family $\{f(S(z))\}$ is normal in D in the sense of Montel, where S ranges over all conformal maps of D onto itself.) Theorem 2 shows that the conclusion of Lindelöf's theorem need not hold for a normal analytic function satisfying the condition (2). However, the conclusion of Lindelöf's theorem does hold for any normal analytic function f satisfying a condition similar to (3), if in addition f has ∞ as a radial limit at no point of some arc $A(0, \theta)$ (Theorem 1). In this result, "analytic" cannot be replaced by "meromorphic". Finally, Theorem 3 shows that Doob's result need not hold for a normal analytic function f even if ∞ is a radial limit of f at no point of some arc $A(0, \theta)$ and $\delta_\varepsilon = 1$ for all $\varepsilon > 0$.

2. NOSHIRO'S PRINCIPLE

Several results of K. Noshiro [6] imply the following generalization of the aforementioned result of Carathéodory.

THEOREM N. *Suppose* f *is analytic in* D, *and for almost every point* $\zeta \in A(0, \theta)$,

$$|f_{\gamma(\zeta)}(\zeta)| \equiv \limsup_{z \to \zeta,\, z \in \gamma(\zeta)} |f(z)| \leq \varepsilon,$$

where $\gamma(\zeta)$ *is some arc in* D *ending at* ζ. *If* f *has* ∞ *as an asymptotic value at no point of* $A(0, \theta)$, *then*

$$|f(\zeta)| \leq \varepsilon \qquad (\zeta \in A(0, \theta)).$$

Proof. If $|f(\zeta)| = +\infty$ for some point $\zeta \in A(0, \theta)$, then ∞ is an asymptotic value of f at some point of $A(0, \theta)$ (see [6, Theorem 2]). Thus $|f(\zeta)|$ is finite for each $\zeta \in A(0, \theta)$. By [6, Theorem 1], $|f(\zeta)| \leq \varepsilon$ for each $\zeta \in A(0, \theta)$.

We now easily prove our first result.

THEOREM 1. *Let* f *be a normal analytic function in* D. *Suppose that*

$$\lim_{\zeta \to 1^+,\, \zeta \in \Gamma - E} |f_{\gamma(\zeta)}(\zeta)| = 0, \tag{4}$$

where $\mu E = 0$ *and* $\gamma(\zeta)$ *is some arc in* D *at* $\zeta \in \Gamma - E$, *and that* $0 < \theta < 2\pi$. *Then* $f(z) \to 0$ *as* $z \to 1$ *in* $G(0, \theta)$ *if (and only if)* f *has* ∞ *as a radial limit at no point of some arc* $A(0, \theta')$ $(\theta' \leq \theta)$.

Proof. If f is normal and has ∞ as a radial limit at no point of $A(0, \theta')$, then f has ∞ as an asymptotic value at no point of $A(0, \theta')$ (see [5, Theorem 2]). The condition (1) now follows from (4) by Theorem N. Finally, Lehto and Virtanen's generalization of Lindelöf's theorem implies the desired conclusion.

3. EXAMPLE 1

The following example shows that the condition (4) in Theorem 1 can hold when ∞ is an asymptotic value of f in each arc $A(0, \theta)$.

THEOREM 2. *There exists a function* f, *normal and analytic in* D, *such that*

$$\lim_{\zeta \to 1^+, \zeta \in \Gamma - E} |f(\zeta)| = 0, \tag{5}$$

where $\mu E = 0$; *but, for each* θ $(0 < \theta < 2\pi)$, $\lim f(z)$ *does not exist as* $z \to 1$ *in* $G(0, \theta)$. *It is possible to construct the function* f *so that it has a nonzero angular limit at* $z = 1$.

Proof. Let $w = \lambda(\tau)$ denote the elliptic modular function defined in the disk D_τ: $|\tau| < 1$, where the fundamental non-Euclidean triangle has vertices $\tau = 1$, $e^{2\pi i/3}$, $e^{4\pi i/3}$ that are mapped to $w = 0, 1, \infty$, respectively. Then λ has radial limit zero at $\tau_1 = 1$ as well as at all points $\tau_n = e^{i\phi_n}$ $(n \geq 2)$ that are obtained from $\tau_1 = 1$ by the reflections used to extend λ from the fundamental triangle to all of D_τ. The unit circle $|\tau| = 1$ is denoted by Γ_τ.

For each positive integer ν, the set

$$H_\nu = \left\{ \tau : |\lambda(\tau)| < \frac{1}{\nu + 1} \right\}$$

consists of an enumerable collection of domains $\Delta_{\nu,n}$ $(n = 1, 2, \cdots)$ such that each closure $\overline{\Delta}_{\nu,n}$ is a closed Jordan region satisfying the relation

$$\overline{\Delta}_{\nu,n} \cap \Gamma_\tau = \{\tau_n\}.$$

For each integer ν, we shall delete from D_τ a certain number n_ν of the closed regions $\overline{\Delta}_{\nu,n}$ so that if G is the resulting simply connected domain and $\tau = \Psi(z)$ maps D conformally onto G, then $\mu E = 0$, where E is the subset of Γ that corresponds to Γ_τ under Ψ. If $e^{i\phi}$ is a point of Γ_τ such that $\phi \neq \phi_n$ for $n \geq 1$, then either λ has angular limit 1 or ∞ at $e^{i\phi}$, or λ has no angular limit at $e^{i\phi}$. Set $f(z) = \lambda(\Psi(z))$, where we assume that $\Psi(1) = e^{i\phi}$. By [5, p. 57], f is normal in D. For each integer ν, at most finitely many of the closed regions $\overline{\Delta}_{\nu,n}$ are deleted from D_τ; therefore $\lambda(\tau) \to 0$ as $\tau \to e^{i\phi}$ with the restriction that τ be a boundary point of G in D_τ. Thus (5) holds. For each θ $(0 < \theta < 2\pi)$, $\lim f(z)$ does not exist as $z \to 1$ in $G(0, \theta)$, because (5) holds and f does not have angular limit zero at $z = 1$.

It only remains to exhibit a sequence $\{n_\nu\}$ such that $\mu E = 0$. First, we note that the circle Γ_τ has harmonic measure $\omega \equiv 0$ with respect to each region $D_\tau - H_\nu$. For if $\tau = \Psi_\nu(z)$ maps D conformally onto $D_\tau - H_\nu$, then

$|\lambda(\Psi_\nu(z))| > 1/(\nu+1)$. Thus $\lambda(\Psi_\nu(z))$ has a radial limit at almost every point of Γ. Hence Ψ_ν maps almost every point of Γ onto a boundary point of $D_\tau - H_\nu$ lying in D_τ, and therefore $\omega(z, \Gamma_\tau, D_\tau - H_\nu) \equiv 0$. Now, we can choose an integer n_ν such that $\omega(0, \Gamma_\tau, G_\nu) < 1/\nu$, where

$$G_\nu = D_\tau - \bigcup_{n=1}^{n_\nu} \overline{\Delta}_{\nu,n}.$$

We set

$$G = D_\tau - \bigcup_{\nu=1}^{\infty} \bigcup_{n=1}^{n_\nu} \overline{\Delta}_{\nu,n} = \bigcap_{\nu=1}^{\infty} G_\nu .$$

By the principle of monotoneity,

$$\omega(0, \Gamma_\tau, G) < \omega(0, \Gamma_\tau, G_\nu) < \frac{1}{\nu} \qquad (\nu = 1, 2, \cdots),$$

and therefore $\omega(z, \Gamma_\tau, G) \equiv 0$. Thus, if $\tau = \Psi(z)$ maps D conformally onto G and $E \subset \Gamma$ corresponds to Γ_τ under Ψ, then $\mu E = 0$.

Remark 1. Theorem 1 (and also Theorem N) no longer holds if f is assumed to be a normal meromorphic function. We first apply the technique of Theorem 2 to a Schwarz triangle function $w = \lambda(\tau)$ for which the fundamental non-Euclidean triangle has exactly one vertex on Γ_τ and that vertex corresponds to $w = 0$. Then λ, and hence f, does not have ∞ as an asymptotic value. Also, (4) holds, but it is easily seen that

$$\limsup_{\zeta \to 1^+} |f(\zeta)| = +\infty .$$

4. EXAMPLE 2

We now show that Theorem 1 is no longer true if we replace the condition that E be of measure zero by the condition that E have metric density zero from the north at $z = 1$.

THEOREM 3. *There exists a function* F, *normal and analytic in* D_τ, *for which the following conditions hold:*

(i) $$\lim_{\tau \to 1^+,\ \tau \in \Gamma_\tau - E_\tau} |F(\tau)| = 0,$$

where $E_\tau \subset \Gamma_\tau$ *has metric density zero from the north at* $\tau = 1$;

(ii) F *has* ∞ *as an asymptotic value at no point of some arc* $A(0, \phi)$ $(e^{i\phi} \in \Gamma_\tau)$;

(iii) F *does not have angular limit zero at* $\tau = 1$.

Proof. The set E in Theorem 2 is a perfect, nowhere dense subset of Γ of measure zero. Hence, the part of E lying in $A(0, \pi/2)$ can be covered by a sequence of open arcs

$$(e^{i\alpha(n)}, e^{i\beta(n)}) \qquad (0 < \alpha(n) < \beta(n) < \pi/2,\ n = 1, 2, \cdots)$$

such that if A_n is the closed arc $[e^{i\alpha(n)}, e^{i\beta(n)}]$, then $\bigcup A_n$ has metric density zero from the north at $z = 1$. It can be assumed that the sequence $\{A_n\}$ converges monotonically (in the obvious sense) to $z = 1$. If

$$B = \Gamma - \bigcup_{n=1}^{\infty} A_n,$$

then

(6) $$\lim_{\zeta \to 1^+, \zeta \in B} |f(\zeta)| = 0,$$

where f is the function in Theorem 2. Also, f has ∞ as an asymptotic value at no point of $B \cap A(0, \pi/2)$.

For each integer n, let $C_{\alpha(n)}$ and $C_{\beta(n)}$ be circles of radius 1/2 internally tangent to Γ at $e^{i\alpha(n)}$ and $e^{i\beta(n)}$, respectively, and let C_n be the circle of radius 1 that intersects A_n and is tangent to each of $C_{\alpha(n)}$ and $C_{\beta(n)}$. Let Λ_n be the Jordan arc traversed as follows: Beginning at $e^{i\alpha(n)}$, follow $C_{\alpha(n)}$ counterclockwise to its intersection with C_n, follow C_n clockwise to its intersection with $C_{\beta(n)}$, and follow $C_{\beta(n)}$ counterclockwise to $e^{i\beta(n)}$. If we set $\delta_n = \frac{1}{2}[\beta(n) - \alpha(n)]$, then

(7) $$\mu\Lambda_n = \delta_n + 3 \arcsin \frac{\sin \delta_n}{3} < 3\delta_n = \frac{3}{2}\mu A_n.$$

Also, because all radii concerned are at least 1/2, the angle of inclination $\theta(s)$ of the tangent to Λ_n as a function of arclength s on Λ_n satisfies the inequality

(8) $$|s' - s| \geq \frac{1}{2}|\theta(s') - \theta(s)|.$$

If

$$\Lambda = B \cup \bigcup_{n=1}^{\infty} \Lambda_n,$$

then Λ is a smooth Jordan curve for which (8) holds, where s now denotes arclength on Λ. Using (7), one can easily show that B (considered as a subset of Λ) has metric density 1 from the north at $z = 1$.

Let $z = \Psi(\tau)$ $(\Psi(1) = 1)$ map D_τ conformally onto the interior of Λ. By Kellogg's theorem [4, p. 374], $\Psi'(\tau)$ is continuous and nonzero on the closed disk $|\tau| \leq 1$. Let B_τ denote the set on Γ_τ that corresponds to B under $\Psi(\tau)$. We claim that B_τ has metric density 1 from the north at $\tau = 1$. Given an arc $A(0, \phi)$ on Γ_τ, let Λ_ϕ denote the subarc of Λ that corresponds to $A(0, \phi)$ under Ψ. If we set $B_{\tau,\phi} = B_\tau \cap A(0, \phi)$ and $M_\phi = \sup |\Psi'(\tau)|$ on $A(0, \phi)$, then

$$\mu[B \cap \Lambda_\phi] = \int_{B_{\tau,\phi}} |\Psi'(e^{i\phi})|\, d\phi \leq (\mu[B_{\tau,\phi}])M_\phi.$$

Thus, as we asserted,

$$\lim_{\phi \to 0^+} \frac{\mu[B_{\tau,\phi}]}{\phi} \geq \lim_{\phi \to 0^+} \frac{\mu[B \cap \Lambda_\phi]}{\mu[\Lambda_\phi]} \cdot \frac{\int_0^\phi |\Psi'(e^{i\phi})|\, d\phi}{\phi} \cdot \frac{1}{M_\phi}$$

$$= 1 \cdot |\Psi'(1)| \cdot \frac{1}{|\Psi'(1)|} = 1 .$$

The function $F(\tau) = f(\Psi(\tau))$ is normal in D_τ and has ∞ as an asymptotic value at no point of the arc $A(0, \phi)$, where $e^{i\phi} = \Psi(e^{i\alpha(1)})$. Since f does not have angular limit zero at $z = 1$, the same is true for F at $\tau = 1$. If $E_\tau = \Gamma_\tau - B_\tau$, then (6) implies that the condition (i) holds for F.

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