RINGS OF TYPE II

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1. INTRODUCTION

We define a *ring of type* II to be an integral domain R with the following properties:

(1) R is a complete, Noetherian local ring of Krull dimension one.

(2) Every ideal of R can be generated by two elements.

Of course, a complete discrete valuation ring is a ring of type II. But not all rings of type II are valuation ring (example: the ring of all formal power series in one variable over a field, with the linear term missing). It is the purpose of this paper to characterize rings of type II in terms of a Hausdorff condition and the structure of certain modules.

Definition. An integral domain R is said to have *property* D if every torsion-free R-module of finite rank is a direct sum of R-modules of rank 1.

Definition. The *Krull dimension* of an integral domain is the maximal number of terms in a chain of nonzero prime ideals.

In [4] we proved the following theorem.

THEOREM 1 [4, Theorem 4]. *If* R *is an integral domain, the following statements are equivalent.*

(1) R *is a ring of type* II.

(2) R *is a Noetherian integral domain with property* D.

The aim of this paper is to replace the Noetherian assumption with the weaker Hausdorff assumption that $\bigcap I^n = 0$ for every proper ideal I of R. We shall prove the following theorem (see Section 4):

THEOREM 11. *If* R *is an integral domain, the following statements are equivalent.*

(1) R *is a ring of type* II.

(2) R *has property* D, *and* $\bigcap I^n = 0$ *for every proper ideal* I *of* R.

2. REVIEW

Definition. An integral domain R is said to have a *remote quotient field* Q if there exists an R-module S such that $R \subset S \subsetneq Q$ and $S^{-1} = 0$, where $S^{-1} = \{x \in Q \mid xS \subset R\}$.

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Definition. A ring R is called a *local ring* if it has a single maximal ideal (no Noetherian conditions are assumed).

Definition. An integral domain R is said to be an h-*local ring* if every nonzero ideal of R is contained in only a finite number of maximal ideals of R, and if every nonzero prime ideal of R is contained in only one maximal ideal of R.

Definition. An integral domain R is said to be a *ring of type* I if it satisfies the following conditions:

(1) R has exactly two maximal ideals M_1 and M_2.

(2) $M_1 \cap M_2$ does not contain a nonzero prime ideal of R.

(3) R_{M_1} and R_{M_2} are maximal valuation rings.

In earlier papers, we proved the following theorems:

THEOREM 2 [6, Theorem B]. *Let* R *be an integral domain. Then the following statements are equivalent:*

(1) R *is a ring of type* I.

(2) R *has property* D *and a remote quotient field.*

THEOREM 3 [6, Theorem B']. *Let* R *be an integral domain. Then the following statements are equivalent:*

(1) R *is a ring of type* I.

(2) R *has property* D *and is not complete* (*in the* R-*topology*).

THEOREM 4 [4, Theorem 2]. *A valuation ring has property* D *if and only if it is a maximal valuation ring.*

3. TECHNICAL LEMMAS

The following lemma is due to H. Bass [1, Proposition 7.5].

LEMMA 5. *Let* R *be a local integral domain with property* D. *Then every finitely generated, torsion-free* R-*module of rank* 1 *can be generated by two elements.*

Proof. Let M be the maximal ideal of R, and let I be a finitely generated, torsion-free R-module of rank 1. Then I has a minimal generating set $\{a_1, \cdots, a_n\}$. We can assume that $n \geq 2$. Since I is isomorphic to an ideal of R, we can assume without loss of generality that I is an ideal of R.

Let F be a direct sum of n copies of R; then $x = (a_1, \cdots, a_n) \in F$. Let B be the pure submodule of rank 1 of F that is generated by x. Then $F/B = C$ is a torsion-free R-module of rank $n - 1$. Because R has property D, $C = C_1 \oplus \cdots \oplus C_{n-1}$, where each C_i is a torsion-free R-module of rank 1. By the theory of projective covers of finitely generated modules over local rings, there exist decompositions $F = F_1 \oplus \cdots \oplus F_{n-1}$ and $B = B_1 \oplus \cdots \oplus B_{n-1}$, where $B_i \subset F_i$. Since B is indecomposable, we can assume that $B = B_1$.

Now the coordinates of x relative to any free basis of F form a minimal generating set for I. Hence x is not contained in any proper direct summand of F. Thus $F = F_1$, and therefore $n - 1 = 1$. Hence $n = 2$.

LEMMA 6. *Let* R *be an integral domain such that every finitely generated ideal of* R *can be generated by two elements. Then the integral closure of* R *is a Prüfer ring.*

Proof. Clearly, every finitely generated torsion-free R-module of rank 1 is isomorphic to an ideal of R, and can thus be generated by two elements. This property is obviously inherited by every ring between R and its quotient field. Thus we may assume, without loss of generality, that R is an integrally closed local ring, and we must prove that R is a valuation ring.

Let M be the maximal ideal of R, and let Q be the quotient field of R. Let V be a valuation ring in Q that dominates R; that is, suppose $R \subset V$ and $m(V) \cap R = M$, where $m(V)$ is the maximal ideal of V. We shall prove that $R = V$.

Suppose that $R \neq V$. If every unit of V is contained in R, then $V = R$. Hence there exists a unit x of V that is not in R. Let A be the R-module generated by 1, x, and x^2. By assumption, A can be generated by two elements. Since R is a local ring, two of the elements 1, x, x^2 generate A. However, since R is integrally closed, x is not integral over R, and thus 1 and x cannot generate A.

In fact, 1 and x^2 generate A. For if x and x^2 generate A, then there exist elements a and b in R such that $1 = ax + bx^2$. If both a and b are in M, then $1 \in VM \subset m(V)$, and this is a contradiction. Hence either $a \notin M$ or $b \notin M$. If b is not in M, then x is integral over R, which is impossible. Thus a is not in M, and hence we see that 1 and x^2 generate A.

Thus we have shown that there exist elements c and d in R such that $x = c + dx^2$. We see that $d \in M$, since x is not integral over R. However, $c \notin M$, since x is a unit in V. But then $1/x$ is integral over R, and hence $1/x \in M$. Therefore $1 = x \cdot 1/x \in VM \subset m(V)$. This contradiction shows that $R = V$.

COROLLARY 7. *Let* R *be an integral domain with property* D. *Then the integral closure of* R *is a Prüfer ring.*

Proof. This is an immediate consequence of Lemmas 5 and 6.

LEMMA 8. *Let* R *be an integral domain whose quotient field* Q $(Q \neq R)$ *is not remote, and suppose that* $\bigcap I^n = 0$ *for every proper principal ideal* I *of* R. *Then* R *is a local ring of Krull dimension* 1.

Proof. Suppose that R has two distinct nonzero prime ideals P_1 and P_2. We can assume that $P_1 \not\subset P_2$. Choose an element $a \in P_1$ such that $a \notin P_2$, and let $S = \{a^n\}$ be the multiplicatively closed set generated by a. Now $R_S^{-1} = \bigcap Ra^n$, and therefore $R_S^{-1} = 0$, by assumption. Since Q is not remote from R, we conclude that $R_S = Q$. However, $P_2 \cap S$ is empty, and thus $R_S P_2$ is a nonzero, proper, prime ideal of R_S. Therefore, R_S cannot be a field. This contradiction shows that R has only one nonzero prime ideal.

LEMMA 9. *Let* R *be an integral domain with property* D *whose quotient field* Q $(Q \neq R)$ *is not remote. Suppose that* $\bigcap I^n = 0$ *for every proper principal ideal* I *of* R. *Then the integral closure of* R *is a maximal valuation ring of Krull dimension* 1.

Proof. Let F be the integral closure of R. By Lemma 8, R is a local ring of Krull dimension 1. Thus F also has Krull dimension 1. By Corollary 7, F is a Prüfer ring. Suppose that F has two distinct maximal ideals N_1 and N_2. Then F_{N_1} and F_{N_2} are valuation rings. F_{N_1} and F_{N_2} have property D, by [3, Lemma

6.2], and thus F_{N_1} and F_{N_2} are maximal valuation rings, by Theorem 4. Let $S = F_{N_1} \cap F_{N_2}$; then S has Krull dimension 1, and hence S is a ring of type I. Hence, by Theorem 2, S has a remote quotient field. But since Q is not remote from R, it is certainly not remote from S. This contradiction shows that F is a local ring. A local Prüfer ring is a valuation ring. Therefore F is a maximal valuation ring, by Theorem 4.

4. THE MAIN THEOREMS

THEOREM 10. *Let* R *be an integrally closed domain. Then the following statements are equivalent:*

(1) R *has property* D, *and* $\bigcap I^n = 0$ *for every proper principal ideal* I *of* R.

(2) R *has Krull dimension* 1, *and* R *is either a maximal valuation ring or a ring of type* I.

Proof. (1) ⇒ (2): Suppose that R has property D, and $\bigcap I^n = 0$ for every proper principal ideal I of R. If R has a remote quotient field, then R is a ring of type I, by Theorem 2. If R does not have a remote quotient field, then R is a maximal valuation ring, by Lemma 9. Thus R is either a maximal valuation ring or a ring of type I. We must prove that R has Krull dimension 1.

Suppose that R does not have Krull dimension 1. Then some nonzero prime ideal P of R is not a maximal ideal. Suppose that R is a valuation ring. Then there exists a nonunit $a \in R$ such that $a \notin P$. It follows that $P \subset Ra^n$ for all n. Thus $P \subset \bigcap Ra^n = 0$. This contradiction shows that R is a ring of type I.

Let M_1 and M_2 be the two maximal ideals of R. Since R is an h-local ring, we can assume that $P \subsetneqq M_1$ and $P \not\subset M_2$. If $(M_1 - P) \subset M_2$, then $P \subset M_2$. Hence there exists an element $b \in M_1$ such that $b \notin P$ and $b \notin M_2$.

Now, if I is any ideal of R that is contained in M_1, but not in M_2, then $R_{M_1} I \cap R = I$. For if $x \in R_{M_1} I \cap R$, then $x = c/s$, where $c \in I$ and $s \in R - M_1$. Since neither M_1 nor M_2 contains $I + Rs$, there exist elements $d \in I$ and $t \in R$ such that $1 = d + ts$. Hence, $x = dx + tsx = dx + tc \in I$.

Thus, for any integer $n > 0$, we have the relations $R_{M_1} b^n \cap R = Rb^n$ and $R_{M_1} P \cap R = P$. Since $b^n \notin P$ and R_{M_1} is a valuation ring, $R_{M_1} P \subset R_{M_1} b^n$. Thus

$$P = (R_{M_1} P \cap R) \subset (R_{M_1} b^n \cap R) = Rb^n$$

for all $n > 0$. Therefore $P \subset \bigcap Rb^n = 0$. This contradiction shows that R has Krull dimension 1.

(2) ⇒ (1): Assume that R is either a maximal valuation ring or a ring of type I. By Theorem 4 or by Theorem 2, R has property D. We now assume that R also has Krull dimension 1, and we must show that $\bigcap I^n = 0$ for every proper principal ideal I of R.

Let r be any nonzero element of the Jacobson radical of R, and let S denote the multiplicatively closed subset $\{r^n\}$ generated by r. Since every nonzero prime ideal of R meets S, we see that R_S is the quotient field Q of R. Let $J = \bigcap Rr^n$, and suppose $J \neq 0$. If a is a nonzero element of J, then, because $R_S = Q$, there exist $b \in R$ and an integer $n > 0$ such that $\frac{1}{a} = \frac{b}{r^n}$. But $a = cr^{n+1}$ for some $c \in R$, and hence $cbr = 1$. Thus r is a unit in R, and this is a contradiction. Therefore $\bigcap Rr^n = 0$.

This disposes of the case where R is a valuation ring, and hence we may assume that R is a ring of type I with two maximal ideals M_1 and M_2. In the light of the preceding paragraph, it will be sufficient to show that if $b \in M_1$ and $b \notin M_2$, then $\bigcap Rb^n = 0$. Now R_{M_1} is a local ring of Krull dimension 1, and hence, by the preceding paragraph, $\bigcap R_{M_1} b^n = 0$. As in the proof that (1) $\Rightarrow$ (2), we see that $R_{M_1} b^n \cap R = Rb^n$. Thus $\bigcap Rb^n = 0$.

We are now ready to prove the main theorem of this paper.

THEOREM 11. *Let* R *be an integral domain, but not a field. Then the following statements are equivalent:*

(1) R *is a ring of type* II.

(2) R *has property* D, *and* $\bigcap I^n = 0$ *for every ideal* I *of* R.

Proof. If R is a ring of type II, then R has property D, by Theorem 1. Since R is a Noetherian ring, $\bigcap I^n = 0$ for every ideal I of R. Conversely, assume that R has property D and that $\bigcap I^n = 0$ for every ideal I of R. We shall prove that R is a ring of type II.

First we show that R does not have a remote quotient field. Suppose that R has a remote quotient field. Then, by Theorem 2, R is a ring of type I. Let M_1 and M_2 be the maximal ideals of R. Then

$$\bigcap_n (R_{M_1} M_1)^n = \bigcap_n (R_{M_1} M_1^n) = R_{M_1}\left(\bigcap M_1^n\right) = 0.$$

Since R_{M_1} is a maximal valuation ring, this implies that R_{M_1} is a complete discrete valuation ring. Similarly, R_{M_2} is a complete discrete valuation ring. Since R is an h-local ring, we deduce from [5, Lemma, p. 258] that R is a Noetherian ring. But Noetherian rings of type I do not exist, as was proved by F. K. Schmidt [7]. This contradiction shows that R does not have a remote quotient field.

We now see by Lemma 8 that R is a local ring of Krull dimension 1 and with maximal ideal M. Let F be the integral closure of R. Then, by Lemma 9, F is a maximal valuation ring of Krull dimension 1, with maximal ideal N. We assert that N is a principal ideal of F.

Suppose that N is not a principal ideal of F. By Lemma 5, $\dim_{R/M} F/FM \leq 2$. If $FM \neq N$, then there exists an element $x \in N - FM$, and therefore $FM \subsetneq Fx \subset N$. But $\dim_{R/M} N/FM = 1$, in this case, and thus $N = Fx$ is a principal ideal of F. Thus we can assume that $FM = N$. Suppose $I = F^{-1}$. Since R does not have a remote quotient field, $I \neq 0$. Since I is an ideal of F, we see that

$$IN^k = I(FM)^k = I(FM^k) = IM^k \subset M^k$$

for every integer $k > 0$. Therefore,

$$I\left(\bigcap_k N^k\right) \subset \bigcap_k IN^k \subset \bigcap_k M^k = 0 .$$

Hence, $\bigcap N^k = 0$. But since F is a valuation ring, this implies that N is a principal ideal of F. Thus N is a principal ideal of F, in all cases, and since F has Krull dimension 1, we see that $\bigcap N^k = 0$. From this it follows immediately that F is a complete discrete valuation ring.

Let x be an element of F such that $N = Fx$. Every ideal of F is a power of N. Now $\dim_{R/M} F/N \leq \dim_{R/M} F/FM \leq 2$. Since $Fx^i/Fx^{i+1} \cong F/Fx = F/N$, it follows that if J is any nonzero ideal of F, then F/J is an R-module of finite length. Let $I = F^{-1}$; since R does not have a remote quotient field, I is a nonzero ideal of F that is contained in R. We have just seen that F/I is an R-module of finite length. Since F/I maps onto F/R, we see that F/R is an R-module of finite length. Thus F is a finitely generated R-module.

Since every ideal of F is isomorphic to F, every ideal of F is a finitely generated R-module. Thus I is a finitely generated ideal of R. Now M/I is an R-submodule of F/I. Thus M/I is an R-module of finite length. Therefore, M is a finitely generated ideal of R. Since M is the only nonzero prime ideal of R, it follows from a theorem of I. S. Cohen [2, Chapter I, Theorem 3.4] that R is a Noetherian ring.

By Lemma 5, every ideal of R can be generated by two elements. We have already seen that R is a local ring of Krull dimension 1. By Theorem 3, R is complete in the R-topology. But for a Noetherian local domain of Krull dimension 1, the R-topology and the M-adic topology are the same. Hence R is a ring of type II. This completes the proof of the theorem.

Remark. It is interesting to compare Theorem 11 with the main theorem of [5], where we proved that every ideal of an integral domain R can be generated by two elements if and only if R is a Noetherian ring such that for every maximal ideal M, R_M has property D for finitely generated torsion-free modules.

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