ON A PARTITION THEOREM OF SYLVESTER

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1. INTRODUCTION

G. E. Andrews [1] recently gave an analytic proof of a classical theorem (a generalization of Euler's partition theorem $\prod_{n=1}^{\infty}(1+q^n) = 1/\prod_{n=1}^{\infty}(1-q^{2n-1})$) due to Sylvester:

THEOREM. *Let* $A_k(n)$ *denote the number of partitions of* n *into odd parts (repetition allowed) with exactly* k *distinct parts. Let* $B_k(n)$ *denote the number of partitions of* n *into mutually distinct parts such that* k *maximal sequences of consecutive integers appear in each partition. Then* $A_k(n) = B_k(n)$.

In his paper, Andrews asked for a direct proof of the identity

$$(1)\quad F(a, q) = 1 + \sum_{k,n} B_k(n)a^k q^n = \sum_{r=1}^{\infty} q^{r(r-1)/2} \frac{(1+(a-1)q)\cdots(1+(a-1)q^r)}{(1-q)\cdots(1-q^{r-1})},$$

and we now give such a proof.

Andrews also gave a proof of the following identity of V. A. Lebesgue (see [3, p. 42]):

$$(2)\quad \sum_{r=0}^{\infty} q^{r(r+1)/2} \frac{(1+\beta q)(1+\beta q^2)\cdots(1+\beta q^r)}{(1-q)(1-q^2)\cdots(1-q^r)} = \prod_{r=1}^{\infty}\left(\frac{1+\beta q^{2r}}{1-q^{2r-1}}\right).$$

We derive this identity by proving the more general identity

$$(3)\quad \sum_{m=0}^{\infty} q^{m(m+1)/2} \frac{(z)_m}{(q)_m}\alpha^m = (z)_\infty(-\alpha q)_\infty \sum_{n=0}^{\infty} \frac{z^n}{(q)_n(-\alpha q)_n},$$

where

$$(a)_n \equiv (a; q)_n \equiv (1-a)(1-aq)\cdots(1-aq^{n-1}) \quad \text{and} \quad (a)_\infty \equiv (a; q)_\infty \equiv \lim_{n\to\infty} (a; q)_n .$$

This identity becomes obvious if we expand both sides of (3) in powers of z, α, and q, and compare the coefficients of similar terms. We point out that the identity (2) is a special case of the q-analogue of Kummer's theorem [5] (let $b \to \infty$ in Daum's identity). The identity (3) is a special case of a theorem of E. Heine [6, p. 106]. To see this relation, replace α with α/τ in equation (1.6) of [2], and then set $\gamma = 0$ and let $\tau \to 0$. An identity more general than (3) is also found in the Notebooks of S. Ramanujan [7, p. 194]:

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(4) $$\sum_{m=0}^{\infty} q^{m(m+1)/2} \frac{(z)_m}{(q)_m} \alpha^m = (-\alpha q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} \alpha^n z^n}{(q)_n(-\alpha q)_n} .$$

The identities (2) and (4) were also obtained anew by L. Carlitz (see [4]).

2. PROOF OF (1)

We split the series

$$\sum_{r=1}^{\infty} q^{r(r-1)/2} \frac{(-(a-1)q)_r}{(q)_{r-1}}$$

into the two series

(a) $$\sum_{r=1}^{\infty} q^{r(r-1)/2} \frac{(-(a-1)q)_{r-1}}{(q)_{r-1}}$$

and

(b) $$\sum_{r=1}^{\infty} q^{r(r+1)/2} (a-1) \frac{(-(a-1)q)_{r-1}}{(q)_{r-1}} .$$

Combining the $(r+1)$st term of (a) with the rth term of (b), we get the equation

(5) $$\sum_{r=1}^{\infty} q^{r(r-1)/2} \frac{(-(a-1)q)_r}{(q)_{r-1}} = 1 + \sum_{r=1}^{\infty} q^{r(r+1)/2} a \frac{(-(a-1)q)_{r-1}}{(q)_r} .$$

Now

$$\frac{1+(a-1)q^s}{1-q^s} = 1 + \frac{aq^s}{1-q^s} = 1 + aq^s + aq^{2s} + \cdots .$$

Hence, by virtue of (5), the identity (1) becomes

(1') $$\sum_{k,n} B_k(n) a^k q^n = \sum_{r=1}^{\infty} a(q + aq^2 + aq^3 + \cdots)(q^2 + aq^4 + aq^6 + \cdots) \cdots (q^{r-1} + aq^{2(r-1)} + aq^{3(r-1)} + \cdots)(q^r + q^{2r} + q^{3r} + \cdots) .$$

In (1'), consider the term

(6) $$a(q + aq^2 + aq^3 + \cdots)(q^2 + aq^4 + aq^6 + \cdots) \cdots (q^{m-1} + aq^{2(m-1)} + aq^{3(m-1)} + \cdots)(q^m + q^{2m} + q^{3m} + \cdots)$$

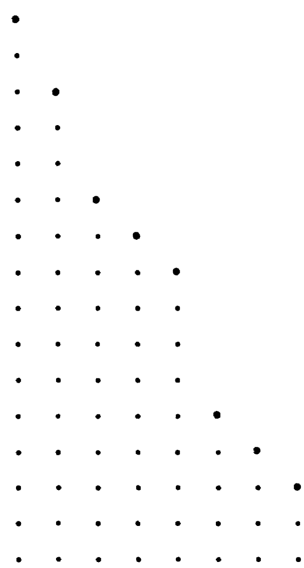
for some $m \geq k$, and expand it in powers of a and q. A term $a^k q^n$ occurs in this expansion when

$$n = 1 \cdot p_1 + 2 \cdot p_2 + \cdots + m \cdot p_m \qquad (m \geq k), \tag{7}$$

where the p_s $(s = 1, 2, \cdots, m)$ are positive integers, and where *exactly* $k - 1$ of the p_s $(s = 1, 2, \cdots, m - 1)$ are greater than 1. We interpret the right-hand side of (7) as a partition consisting of p_1 parts of length 1, p_2 parts of length 2, $\cdots$, and p_m parts of length m, with exactly $k - 1$ of the p_s $(s = 1, 2, \cdots, m - 1)$ being greater than 1. It is easy to see that if $p_{s_1}, p_{s_2}, p_{s_3}, \cdots, p_{s_{k-1}}$ $(1 \leq s_t \leq m - 1)$ are the p_s greater than 1, then a maximal sequence of consecutive parts ends at the first part whose length is s_t $(1 \leq t \leq k - 1)$, and the last maximal sequence ends at the first part whose length is m. The graph below is one such partition corresponding to $k = 4$, $m = 8$, $n = 72$. In this partition,

$$p_1 = 2, \quad p_2 = 3, \quad p_5 = 4,$$

$$p_3 = p_4 = p_6 = p_7 = 1, \quad p_8 = 3.$$



Also, each s $(1 \leq s \leq m)$ occurs at least once as a part. Therefore the parts of the conjugate partition (obtained on reading the graph by columns instead of rows) are distinct, the number of maximal sequences remaining the same. Hence the coefficient of $a^k q^n$ in (1') enumerates all partitions of n into mutually distinct parts having k maximal sequences. That is, it is the same as $B_k(n)$. This completes the proof of (1).

3. PROOF OF (2)

Let $P_r^0 \equiv P_r$; for $k \geq 1$, let P_r^k denote a partition consisting of at most r parts, no part being less than k, and let D_r^k denote a partition consisting of r different parts, none less than k.

Transferring the product $(z)_\infty$ on the right side to the left, we can write the identity (3) in the form

$$\sum_{m=0}^{\infty} \frac{\alpha^m q^{m(m+1)/2}}{(q)_m} \frac{1}{(q^m z)_\infty} = \sum_{n=0}^{\infty} \frac{z^n}{(q)_n} (-\alpha q^{n+1})_\infty . \tag{3'}$$

Let $L(N, n, m)$ and $R(N, n, m)$ denote the coefficients of $z^n \alpha^m q^N$ on the left and right sides of (3'). It is easy to see that $L(N, n, m)$ enumerate all partitions of N that are obtained as the combination

$$D_m^1 \oplus P_n^m ,$$

or equivalently, as the combination

$$D_m^1 \oplus P_n \oplus mn;$$

and R(N, n, m) enumerates all partitions of N obtained as the combination

$$P_n \oplus D_m^{n+1},$$

or equivalently, as the combination

$$P_n \oplus D_m^1 \oplus mn.$$

Therefore L(N, n, m) and R(N, n, m) enumerate partitions of the same type of N. That is, L(N, n, m) = R(N, n, m). This completes the proof of (3).

Putting $z = -\beta q$ and $\alpha = 1$ in (3), we get the relations

$$\sum_{m=0}^{\infty} q^{m(m+1)/2} \frac{(-\beta q)_m}{(q)_m} = (-\beta q)_\infty (-q)_\infty \sum_{n=0}^{\infty} \frac{(-\beta q)^n}{(q^2; q^2)_n}$$

$$= (-\beta q)_\infty (-q)_\infty / (-\beta q; q^2)_\infty = (-\beta q^2; q^2)_\infty / (q; q^2)_\infty .$$

Hence (2) is true.

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