COHOMOLOGY OF COMPACT MINIMAL SUBMANIFOLDS

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1. INTRODUCTION

Let N be a Riemannian manifold, and let $f: M \to N$ be an immersion. M is said to be *minimal* in N if the mean curvature of M in N is identically zero. In [4], J. Simons studied minimal immersions by considering elliptic differential equations involving cross-sections of various Riemannian vector bundles. In view of his results and the classical relation between harmonic forms and the cohomology of a Riemannian manifold, it is perhaps natural to ask whether there is any connection between minimality and cohomology. In this note we prove the following proposition.

THEOREM. *If* N *is a compact, connected, orientable Riemannian manifold with positive-semidefinite Ricci curvature and* $f: M \to N$ *is a minimal immersion of a compact, connected, orientable manifold* M *such that the image of* M *is not contained in a totally geodesic submanifold of* N, *then the natural map*

$$f^*: H^1(N, \mathbb{R}) \to H^1(M, \mathbb{R})$$

is one-to-one and into.

This result should be compared with the work of T. T. Frankel [1] on minimal hypersurfaces in manifolds with positive-definite Ricci curvature.

2. NOTATION

Let N be a Riemannian manifold with connection $\overline{\nabla}$, and let $f: M \to N$ be an immersion; we shall not in general differentiate between a point p in M and its image in N. There is an orthogonal decomposition $N_p = M_p \oplus M_p^\perp$ with respect to the metric on N. If U is a vector field on N, we shall denote its component tangent to M by U^T, and its component normal to M by U^N. If ∇ is the connection on M with respect to the induced metric, then for tangential vector fields X and Y,

$$\nabla_X Y = (\overline{\nabla}_X Y)^T. \tag{1}$$

If ξ is a normal vector field on M and X is a tangential vector field, define

$$A_\xi X = -(\overline{\nabla}_X \xi)^T. \tag{2}$$

It is well known (see [3, p. 14]) that $(A_\xi X)_p$ depends only on X_p and ξ_p, so that A_{ξ_p} is well-defined and is a symmetric linear operator on M_p. We recall that M is minimal in N if and only if trace $A_{\xi_p} = 0$ for all normal vector fields ξ and all $p \in M$.

Received April 29, 1971.
This research was supported by N.R.C. Grant No. A7846.

Michigan Math. J. 19 (1972).

If U is a vector field on N, it gives rise to two vector fields on M, the tangential field $V = U^T$ and the normal field $\overline{V} = U^N$.

LEMMA 1. *If* X *is a tangential field, then*

$$\nabla_X V = (\overline{\nabla}_X U)^T + A_{\overline{V}} X .$$

Proof. By (1), $\nabla_X V = (\overline{\nabla}_X V)^T$. Since the right-hand side is equal to $(\overline{\nabla}_X U)^T - (\overline{\nabla}_X \overline{V})^T$, the lemma follows from (2).

3. PROOF OF THE THEOREM

Throughout this section, we assume that N is compact, connected, and orientable with positive-semidefinite Ricci curvature, and that M is a compact, connected, and orientable manifold immersed in N.

LEMMA 2. *If* U *is a harmonic vector field on* N, *and if* M *is immersed minimally in* N, *then* V *is harmonic on* M.

Proof. It is well known that if N has positive-semidefinite Ricci curvature, then a harmonic vector field is covariant constant (see [2, p. 87]). Lemma 1 thus implies that $\nabla_X V = A_{\overline{V}} X$, for any vector field X on M.

Now div V = trace $(X \to \nabla_X V)$. Hence div V = trace $A_{\overline{V}} \equiv 0$, by the assumption of minimality.

If w is the one-form on M given by $w(X) = (V, X)$, then w is closed (since it is the pull-back of a closed form on N) and co-closed. Hence V is harmonic. ■

LEMMA 3. *If* U *is a harmonic vector field on* N *such that* $V \equiv 0$, *then* M *is contained in a totally geodesic submanifold of* N.

Proof. Since U is covariant constant on N, the distribution H on N given by $H_p = U_p^{\perp}$ is involutive. The maximal integral submanifolds of H are totally geodesic; for if c(t) is a geodesic and $T = c'(t)$, then

$$T(U, T) = (\overline{\nabla}_T U, T) + (U, \overline{\nabla}_T T) = 0 .$$

At each point of M, $M_p \subset H_p$; therefore, if r(t) is a curve in M, then $r'(t) \subset H_{r(t)}$ for all t; thus r(t) lies in a maximal integral submanifold of H. Thus, since M is path-wise connected, we see that if $p \in M$, then M must lie in the maximal integral submanifold of H through p. ■

The theorem now follows from Lemmas 2 and 3.

COROLLARY. *If an immersion of the* n*-torus* T^n *in* T^{n+p} *is minimal with respect to the flat metric, then it is the standard immersion as a subtorus.*

Proof. The corollary follows immediately from the fact that the torus is completely parallelisable.

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