ON THE INNER GEOMETRY OF THE SECOND FUNDAMENTAL FORM

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N. Hicks [2, p. 395, Theorem 7] proved the following result.

THEOREM A. *If the connexion induced on a complete, connected surface (in Euclidean space* R^3*) by its second fundamental form is the usual Riemannian connexion, then the surface is a sphere.*

The aim of this note is to prove a local version of Theorem A. In addition, we show that the conclusion of the theorem also holds for hypersurfaces in spaces of constant curvature, under weaker conditions for the connexion.

For a hypersurface in a space of constant curvature with definite second fundamental form, let β be the connexion induced by that form. We shall define the autoparallels belonging to β as II-geodesics, and we shall prove the following local result.

THEOREM B. *Let* F *be a hypersurface in a space of constant curvature with a definite second fundamental form; each* II*-geodesic is an (ordinary) geodesic if and only if* F *is totally umbilic.*

The corollary to Theorem D will generalize Theorem B slightly.

LEMMA 1. *Every* II*-geodesic is a curve of constant normal curvature.*

The converse is not true, as can be seen from the following example. On a rotation surface in R^3, the circles of latitude are curves of constant normal curvature but generally not II-geodesics. Thus, on an arbitrary hypersurface, the class of curves with constant normal curvature generally contains the class of II-geodesics as a proper subset. The natural question, whether Theorem B can be generalized to curves of constant normal curvature, is answered in Theorem D.

Proofs of the theorems. Let M_n be an n-dimensional manifold with local parameters (u^i), and suppose that $M_n \in C^r$ $(r \geq 3)$; let N_{n+1} be a Riemannian manifold of constant curvature K_0, and let

$$x: M_n \to N_{n+1} \tag{1.1}$$

be an isometric C^r-immersion. As usual, we have on $x(M_n)$ a first and a second fundamental form; we shall assume that the second fundamental form is definite at each point of $x(M_n)$. The two forms induce symmetric connexions Γ and β on $x(M_n)$, and covariant differentiations ∇^I and ∇^{II}.

In a local coordinate system, we define g_{ij} and b_{ij} to be the tensor components of the first and second fundamental forms, Γ^k_{ij} and β^k_{ij} to be the components of the connexions Γ and β, and $g^{(ij)}$ and $b^{(ij)}$ to be the components of the inverse tensors of g_{jk} and b_{jk}.

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The Codazzi-equations are

$$\nabla^I_k b_{ij} = \nabla^I_j b_{ik} . \tag{1.2}$$

For the tensor components A^k_{ij}, defined by the equation

$$A^k_{ij} = \Gamma^k_{ij} - \beta^k_{ij} , \tag{1.3}$$

we shall prove the following result.

LEMMA 2. *For* $A_{ijk} = b_{rk} A^r_{ij}$, *we have the relation*

$$2 A_{ijk} = - \nabla^I_k b_{ij} ; \tag{1.4}$$

and A_{ijk} *is symmetric in all indices.*

Proof (compare [1], [6]). Application of the covariant derivations on b_{ij} gives the equations

$$\nabla^I_k b_{ij} = b_{ij|k} - \Gamma^s_{ik} b_{sj} - \Gamma^s_{kj} b_{si} , \tag{1.5}$$

$$0 = b_{ij|k} - \beta^s_{ik} b_{sj} - \beta^s_{kj} b_{si} , \tag{1.6}$$

where $b_{ij|k} = \dfrac{\partial b_{ij}}{\partial u^k}$; thus it follows that

$$\nabla^I_k b_{ij} = - A^s_{ki} b_{sj} - A^s_{kj} b_{si} . \tag{1.7}$$

The symmetry of $\nabla^I_k b_{ij}$ in (k, j) implies that

$$0 = A^s_{ji} b_{sk} - A^s_{ki} b_{sj} , \tag{1.8}$$

so that A_{ijk} is symmetric in all indices; the assertion (1.4) follows from (1.7) and (1.8).

Proof of Lemma 1. We consider a curve $\gamma \in C^2$ on x(M) with parameter $t \in R$. If we put $\dot{u}^i = \dfrac{du^i}{dt}$, the normal curvature of γ is $b(t) = b_{ij} \dot{u}^i \dot{u}^j$. Using covariant differentiation D^{II}/dt with respect to β along γ, we find that

$$\frac{D^{II}}{dt} b(t) = 2 b_{ij} \dot{u}^i \frac{D^{II} \dot{u}^j}{dt} .$$

If γ is II-geodesic, then $\dfrac{D^{II}}{dt} \dot{u}^j = 0$ for a suitable parameter t. Thus the function b(t) is constant.

THEOREM D. *If the second fundamental form is definite on* x(M), *then the following statements are equivalent:*

(a) *the* II-*geodesics are geodesics;*

(b) $\Gamma = \beta$;

(c) $\nabla^I_k b_{ij} = 0$ *for each point of* $x(M_n)$;

(d) *each geodesic has constant normal curvature.*

Proof. First, we prove that (a) $\Rightarrow$ (b). For two connexions with the same autoparallels, there exists a vectorfield q_i such that

$$A_{ij}^k = \Gamma_{ij}^k - \beta_{ij}^k = q_i \delta_j^k + q_j \delta_i^k \tag{1.9}$$

(compare for example [3, p. 156]). From the symmetry of A_{ijk} in (j, k) it follows that

$$0 = q_k b_{ij} - q_j b_{jk}. \tag{1.10}$$

Multiplication with $b^{(ik)}$ shows that $q_j = 0$; (1.9) and (1.3) imply (b). The implication (b) $\Rightarrow$ (a) is trivial. Lemma 2 implies that (b) $\Leftrightarrow$ (c). Lemma 1 implies that (a) $\Rightarrow$ (d). Finally, we prove that (d) $\Rightarrow$ (c). For a geodesic with natural parameter t and constant normal curvature we have the relation (see (1.3))

$$0 = b_{ij}\dot{u}^i \frac{D^{II}\dot{u}^j}{dt} = b_{ij}\dot{u}^i(\ddot{u}^j + \beta^j_{rs}\dot{u}^r\dot{u}^s)$$

$$= b_{ij}\dot{u}^i[(\ddot{u}^j + \Gamma^j_{rs}\dot{u}^r\dot{u}^s) - A^j_{rs}\dot{u}^r\dot{u}^s] = -A_{rsi}\dot{u}^r\dot{u}^s\dot{u}^i.$$

If $p \in x(M)$, then $A_{rsi}\dot{u}^r\dot{u}^s\dot{u}^i = 0$ for every tangent direction at p; hence the symmetry of A_{rsi} implies that $A_{rsi} = 0$, and therefore (c) follows from Lemma 2.

We say that a hypersurface $x(M_n) \subset N_{n+1}$ is of type (α) if each of its points is umbilic, that is, if at each of its points all the principal curvatures $k_1, k_2, \cdots, k_n$ have the same value. We say that $x(M_n)$ is of type (β) if there exists an integer p $(1 \leq p \leq n)$ such that, for a suitable numbering of the principal curvatures,

$$k_1 = k_2 = \cdots = k_p = \lambda_1 \neq 0 \quad \text{and} \quad k_{p+1} = k_{p+2} = \cdots = k_n = \lambda_2,$$

where the constants λ_1 and λ_2 satisfy the condition $\lambda_1\lambda_2 + K_0 = 0$. In [5], it was proved that a hypersurface $x(M_n)$ in N_{n+1} satisfies the condition $\nabla^I_k b_{ij} = 0$ at each point if and only if it is of type (α) or of type (β).

Together with Theorem D, this criterion gives the following generalization of Theorems A and B.

COROLLARY. *If* $x(M_n)$ *has a definite second fundamental form, then each of the conditions* (a), (b), *and* (c) *is equivalent to the condition that* $x(M_n)$ *be of type* (α) *or* (β).

Remark. If $N_{n+1} = R^{n+1}$, then $K_0 = 0$. The assumption $\det(b_{ij}) \neq 0$ excludes both surfaces of type (β) and the case $k_1 = k_2 = \cdots = k_n = 0$. Thus Theorem B is proved.

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