HYPERINVARIANT SUBSPACES AND TRANSITIVE ALGEBRAS

R. G. Douglas and Carl Pearcy

0. INTRODUCTION

All Hilbert spaces to be discussed in this paper will be *complex*, and all operators to be considered will be *linear*. Moreover, unless we specifically state the contrary, an operator under discussion will be assumed to be *everywhere defined* and bounded. If $\mathscr{H}$ is a Hilbert space, we denote by $\mathscr{L}(\mathscr{H})$ the algebra of all bounded operators on $\mathscr{H}$. All subalgebras $\mathscr{A}$ of $\mathscr{L}(\mathscr{H})$ will be assumed to contain the identity operator $1_{\mathscr{H}}$; the commutant of $\mathscr{A}$ will be denoted by $\mathscr{A}'$. If T is an operator in $\mathscr{L}(\mathscr{H})$, then the weakly closed subalgebra generated by T (that is, the weak closure of the set of all polynomials in T) will be denoted by $\mathscr{A}_T$. Clearly, the algebra consisting of all operators that commute with T is exactly $(\mathscr{A}_T)'$. We say that a subspace $\mathscr{M}$ of $\mathscr{H}$ is a *hyperinvariant subspace* for T if $\mathscr{M}$ is invariant under each operator in $(\mathscr{A}_T)'$, and if in addition $\mathscr{M}$ is different from (0) and $\mathscr{H}$.

Thus far, most attempts to obtain structure theorems for classes of operators on Hilbert space have depended on the exhibition of a sufficiently large supply of invariant subspaces for the operator. However, a careful scrutiny of the situation in finite-dimensional spaces shows that the determination of the hyperinvariant subspaces of an operator is likely to be more worthwhile. Thus, the problem of determining whether every operator on Hilbert space has a hyperinvariant subspace may be of more importance than the corresponding problem for invariant subspaces.

The study of hyperinvariant subspaces for certain classes of operators was begun in [7] and [20], and continued in [5], [8], [9], and [16]. In particular, complete information concerning hyperinvariant subspaces has been obtained for normal operators [7], finite-rank operators [8], and isometries [5]. Furthermore, hyperinvariant subspaces have been shown to exist for operators that are quasi-similar to normal operators [20], and also for operators that generate a finite von Neumann algebra of type I [9]. In Section 2 of this paper we introduce the concept of a disjoint ordered pair of operators, and we exhibit an interesting connection between disjoint pairs of operators and hyperinvariant subspaces.

There is another problem, closely related to that of the existence of hyperinvariant subspaces, to which this paper makes a contribution. A subalgebra $\mathscr{A}$ of $\mathscr{L}(\mathscr{H})$ is said to be *transitive* if the only subspaces invariant for all operators in $\mathscr{A}$ are (0) and $\mathscr{H}$. Hence, an operator T fails to have a hyperinvariant subspace if and only if $(\mathscr{A}_T)'$ is transitive. If $\mathscr{H}$ is finite-dimensional, then it follows from a well-known theorem of Burnside [10, p. 276] that the only transitive subalgebra of $\mathscr{L}(\mathscr{H})$ is $\mathscr{L}(\mathscr{H})$ itself. The difficult and still open corresponding question for an infinite-dimensional Hilbert space $\mathscr{H}$ is whether every transitive subalgebra $\mathscr{A}$ of $\mathscr{L}(\mathscr{H})$ is strongly dense (equivalently, weakly dense) in $\mathscr{L}(\mathscr{H})$. In [1], W. B. Arveson introduced a technique for studying this question, and he succeeded in giving an affirmative answer in case $\mathscr{A}$ contains a maximal abelian von Neumann algebra or a pure isometry of multiplicity one. Further results along these lines have been

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given in [11], [14], and [15]. In Section 1, we discuss the relation between the question whether every transitive algebra is strongly dense and the question whether every operator has invariant and hyperinvariant subspaces. We also introduce an elementary localization principle that yields immediate generalizations of previous results in these directions. Finally, in Section 3 we generalize Arveson's fundamental result by proving that if a transitive algebra $\mathcal{A}$ acts on a separable Hilbert space $\mathcal{H}$ and contains a Hermitian operator that is not of uniform infinite multiplicity, then $\mathcal{A}$ is strongly dense in $\mathcal{L}(\mathcal{H})$.

1. OPEN QUESTIONS AND LOCALIZATION

A subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ is said to be n-*transitive* if for every linearly independent set $\{x_1, \cdots, x_n\}$ in $\mathcal{H}$ and for every set $\{y_1, \cdots, y_n\}$ in $\mathcal{H}$, there exists a sequence $\{A_k\}_{k=1}^{\infty}$ in $\mathcal{A}$ such that $\lim_k \|A_k x_i - y_i\| = 0$ for $1 \leq i \leq n$. It is easy to see that an algebra $\mathcal{A}$ is transitive if and only if it is 1-transitive. Consider now the following sequence of propositions, each of which represents an important open question in operator theory.

(P_1) Every transitive subalgebra of $\mathcal{L}(\mathcal{H})$ is strongly dense in $\mathcal{L}(\mathcal{H})$.

Arveson observed in [1] that an algebra is strongly dense in $\mathcal{L}(\mathcal{H})$ if and only if it is n-transitive for every positive integer n. Thus, this proposition implies the following proposition.

(P_2) Every transitive subalgebra of $\mathcal{L}(\mathcal{H})$ is 2-transitive.

Another observation made in [1] is that a transitive algebra $\mathcal{A}$ is 2-transitive if and only if every closed, densely defined linear transformation that commutes with $\mathcal{A}$ is a scalar multiple of the identity operator. Hence, (P_2) implies the following.

(P_3) Every transitive subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ satisfies the relation $\mathcal{A}' = \{\lambda 1\}$.

Observe that if there exists a transitive algebra whose commutant contains a nonscalar operator T, then T cannot have a hyperinvariant subspace. Furthermore, if there exists a nonscalar operator T with no hyperinvariant subspace, then $(\mathcal{A}_T)'$ is a transitive algebra such that $(\mathcal{A}_T)'' \neq \{\lambda 1\}$. Hence this last proposition is equivalent to the following.

(P_4) Every nonscalar operator on Hilbert space has a hyperinvariant subspace.

Since every operator T satisfies the condition $T \in (\mathcal{A}_T)'$, it is obvious that (P_4) implies the following proposition.

(P_5) Every operator on Hilbert space has a nontrivial invariant subspace.

The following theorem summarizes the relations between these propositions.

THEOREM 1.1. *The propositions above are related in the following way:*

$$(P_1) \Rightarrow (P_2), \quad (P_2) \Rightarrow (P_3), \quad (P_3) \Leftrightarrow (P_4), \quad (P_4) \Rightarrow (P_5).$$

Two additional remarks concerning these propositions are worth making. First, while (P_5) is of interest only for separable, infinite-dimensional Hilbert spaces, the other four propositions are of interest for all infinite-dimensional Hilbert spaces. The second remark concerns (P_3). It is not hard to construct nontransitive algebras $\mathcal{A}$ (even on finite-dimensional spaces) such that $\mathcal{A}' = \{\lambda 1\}$.

We now turn to the question of localization. If $\mathscr{A}$ is a subalgebra of $\mathscr{L}(\mathscr{H})$ and E is a projection in $\mathscr{A}$, then $E\mathscr{A}E = \{ETE: T \in \mathscr{A}\}$ is a subalgebra of $\mathscr{A}$, and $E\mathscr{A}E$ may also be viewed as a subalgebra of $\mathscr{L}(\mathscr{H}_E)$, where $\mathscr{H}_E$ denotes the range of E. We show that questions concerning $\mathscr{A}$ can sometimes be localized so as to become questions about algebras of the form $E\mathscr{A}E$.

LEMMA 1.2. *If $\mathscr{A}$ is an n-transitive subalgebra of $\mathscr{L}(\mathscr{H})$ and E is a nonzero projection in $\mathscr{A}$, then $E\mathscr{A}E$ is an n-transitive subalgebra of $\mathscr{L}(\mathscr{H}_E)$.*

Proof. If $\dim \mathscr{H}_E < n$, the result is automatic. Therefore, let $\{x_1, \cdots, x_n\}$ be any linearly independent set in $\mathscr{H}_E$, and let $\{y_1, \cdots, y_n\}$ be a set in $\mathscr{H}_E$. Since $\mathscr{A}$ is n-transitive, there exists a sequence $\{A_k\}$ in $\mathscr{A}$ such that $\lim_k \|A_k x_i - y_i\| = 0$ for $1 \leq i \leq n$. Thus

$$\|EA_kEx_i - y_i\| = \|E(A_kx_i - y_i)\| \leq \|A_kx_i - y_i\| ,$$

and it follows easily that $E\mathscr{A}E$ is n-transitive.

The next two theorems constitute our principal localization results.

THEOREM 1.3. *If $\mathscr{A}$ is a transitive subalgebra of $\mathscr{L}(\mathscr{H})$ and E is a projection in $\mathscr{A}$, then $\mathscr{A}$ is strongly dense in $\mathscr{L}(\mathscr{H})$ if and only if $E\mathscr{A}E$ is strongly dense in $\mathscr{L}(\mathscr{H}_E)$.*

Proof. If $\mathscr{A}$ is strongly dense in $\mathscr{L}(\mathscr{H})$, it is obvious that $E\mathscr{A}E$ is strongly dense in $\mathscr{L}(\mathscr{H}_E)$. To argue the other way, suppose that $E\mathscr{A}E$ is strongly dense in $\mathscr{L}(\mathscr{H}_E)$, and suppose, without loss of generality, that $\mathscr{A}$ is strongly closed. Then $E\mathscr{A}E = \mathscr{L}(\mathscr{H}_E)$, and it follows that $\mathscr{A}$ contains finite-rank operators. Thus from Corollary 2 of [11] we may conclude that $\mathscr{A} = \mathscr{L}(\mathscr{H})$, as desired.

THEOREM 1.4. *Suppose that T is an operator in $\mathscr{L}(\mathscr{H})$, and suppose that $\mathscr{M}$ is a nontrivial reducing subspace for T such that $T \mid \mathscr{M}$ has a hyperinvariant subspace. Then T has a hyperinvariant subspace.*

Proof. If E denotes the projection in $\mathscr{L}(\mathscr{H})$ with range $\mathscr{M}$, then $E \in (\mathscr{A}_T)'$. Furthermore, an easy computation shows that $E(\mathscr{A}_T)'E = (\mathscr{A}_{ETE})'$. Since ETE has a hyperinvariant subspace, $E(\mathscr{A}_T)'E$ is not transitive, and thus by Lemma 1.2, $(\mathscr{A}_T)'$ is not transitive. Thus T has a hyperinvariant subspace.

Theorems 1.3 and 1.4 furnish immediate generalizations of previous results concerning the strong density of transitive algebras and the existence of hyperinvariant subspaces. Two corollaries of particular interest are the following.

COROLLARY 1.5. *Suppose that $A \in \mathscr{L}(\mathscr{H})$, that A is not a scalar, and that the von Neumann algebra generated by A has a direct summand that is finite and of type* I. *Then A has a hyperinvariant subspace.*

Proof. This is an immediate consequence of Theorem 1.4 and the result of Hoover [9] mentioned above.

COROLLARY 1.6. *If $\mathscr{A}$ is a transitive subalgebra of $\mathscr{L}(\mathscr{H})$ and $\mathscr{A}$ contains a von Neumann algebra that has a maximal abelian direct summand, then $\mathscr{A}$ is strongly dense in $\mathscr{L}(\mathscr{H})$.*

Proof. This is an immediate consequence of Theorem 1.3 and Arveson's theorem [1].

To continue our study of hyperinvariant subspaces, we introduce the concept of a disjoint ordered pair of operators.

2. DISJOINT ORDERED PAIRS OF OPERATORS

Let A and B be operators on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively. We shall say that the ordered pair (A, B) of operators is *disjoint* if the only bounded operator X mapping $\mathscr{K}$ into $\mathscr{H}$ and satisfying the equation $AX = XB$ is $X = 0$. [To see that a pair (A, B) may be disjoint while (B, A) is not disjoint, consider the pair $(0, U^*)$, where U is any non-unitary isometry in $\mathscr{L}(\mathscr{H})$. In view of this, it is necessary to consider ordered pairs.] We begin our discussion with some elementary propositions concerning disjoint pairs of operators.

PROPOSITION 2.1. *A pair* (A, B) *of operators is disjoint if and only if the pair* (B^*, A^*) *is also disjoint.*

PROPOSITION 2.2. *If* A *and* B *are operators whose spectra are disjoint, then both pairs* (A, B) *and* (B, A) *are disjoint.*

Proof. This follows immediately from the fact that under the hypothesis, the operator $X \to AX - XB$ is invertible [18].

PROPOSITION 2.3. *If* A *and* B *are normal operators, then the pair* (A, B) *is disjoint if and only if the pair* (B, A) *is disjoint.*

Proof. This is an immediate consequence of the Fuglede-Putnam theorem [13].

The next result requires some additional terminology. Suppose that M and N are normal operators on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively, and let E_M and E_N be their respective spectral measures. We say that the normal operators M and N are *mutually singular* if for every vector x in $\mathscr{H}$ and for every vector y in $\mathscr{K}$, the scalar measures $\mu_x(\cdot) = (E_M(\cdot)x, x)$ and $\nu_y(\cdot) = (E_N(\cdot)y, y)$ are singular.

PROPOSITION 2.4. *If* M *and* N *are normal operators, then the pair* (M, N) [*and hence the pair* (N, M)] *is disjoint if and only if* M *and* N *are mutually singular.*

Proof. This is a restatement of Lemma 4.1 of [6].

This theorem about normal operators represents a considerable improvement over Proposition 2.2, which is valid for an arbitrary pair of operators. To see this, note first that if M and N are normal operators such that the spectrum of M is disjoint from the spectrum of N, then clearly M and N are mutually singular. On the other hand, let $\mathscr{H}$ be the Hilbert space $L_2[0, 1]$, where the measure on the interval [0, 1] is understood to be Lebesgue measure, and let M denote multiplication by the coordinate function $f(x) = x$. Furthermore, let N be a diagonal matrix relative to some orthonormal basis for $\mathscr{H}$ whose diagonal entries are exactly the rational numbers between 0 and 1. Easy spectral theory shows that the operators M and N are mutually singular, but that M and N have exactly the same spectrum—namely, the interval [0, 1]. This example shows that Proposition 2.4 is a refinement for normal operators of Proposition 2.2.

We consider now the connection between disjoint pairs of operators and hyperinvariant subspaces, which motivated this discussion of disjoint pairs. If $\mathscr{M}$ is any subspace of $\mathscr{H}$, we denote by $P_{\mathscr{M}}$ the projection in $\mathscr{L}(\mathscr{H})$ whose range is $\mathscr{M}$. The basic new idea in the following theorem is due to H. Radjavi and P. Rosenthal [16].

THEOREM 2.5. *Let* A *be an operator on* $\mathscr{H}$, *and suppose that there exist nonzero subspaces* $\mathscr{M}$ *and* $\mathscr{N}$ *such that* $\mathscr{M}$ *is invariant for* A, $\mathscr{N}$ *is invariant for* A^*, *and the pair* $(P_{\mathscr{N}} A \mid \mathscr{N}, A \mid \mathscr{M})$ *is disjoint. Then* A *has a hyperinvariant subspace.*

Proof. We show first that $P_{\mathscr{N}} B P_{\mathscr{M}} = 0$ for every operator B in $(\mathscr{A}_A)'$. This follows immediately from the equation

$$(P_{\mathscr{N}} A \mid \mathscr{N})(P_{\mathscr{N}} B P_{\mathscr{M}})x = (P_{\mathscr{N}} A P_{\mathscr{N}})(B P_{\mathscr{M}})x = (P_{\mathscr{N}} A)(B P_{\mathscr{M}})x$$
$$= (P_{\mathscr{N}} B)(A P_{\mathscr{M}})x = (P_{\mathscr{N}} B P_{\mathscr{M}})(A \mid \mathscr{M})x \qquad (x \in \mathscr{M})$$

and the fact that the pair $(P_{\mathscr{N}} A \mid \mathscr{N}, A \mid \mathscr{M})$ is disjoint. (Notice, in particular, that since $1_{\mathscr{H}}$ belongs to $(\mathscr{A}_A)'$, the subspaces $\mathscr{M}$ and $\mathscr{N}$ are orthogonal.) Thus, if x and y are nonzero vectors in $\mathscr{M}$ and $\mathscr{N}$, respectively, and B is any operator in $(\mathscr{A}_A)'$, then

$$\|Bx - y\| = \|B P_{\mathscr{M}} x - P_{\mathscr{N}} y\| \geq \|P_{\mathscr{N}} B P_{\mathscr{M}} x - P_{\mathscr{N}} y\| = \|y\| .$$

It follows that $(\mathscr{A}_A)'$ is not 1-transitive, and thus that A has a hyperinvariant subspace.

Theorem 2.5 may be useful in establishing the existence of hyperinvariant subspaces for operators that are known to have invariant subspaces (for example, compact operators). At present, the difficulty that one encounters when trying to apply the theorem is the lack of useful criteria for the disjointness of a pair of operators. Thus the search for theorems better than Propositions 2.1 to 2.4 would be a worthwhile endeavor.

The following corollary shows how Theorem 2.5 can be applied to operators that are 2-by-2 matrices.

COROLLARY 2.6. *Suppose that* A *and* B *are operators on Hilbert spaces* $\mathscr{H}$ *and* $\mathscr{K}$, *respectively. Suppose also that* $\mathscr{M} \subset \mathscr{H}$ *and* $\mathscr{N} \subset \mathscr{K}$ *are nonzero invariant subspaces for* A *and* B^*, *respectively, such that the pair* $(P_{\mathscr{N}} B \mid \mathscr{N}, A \mid \mathscr{M})$ *is disjoint. Then every operator of the form*

$$\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$$

acting on $\mathscr{H} \oplus \mathscr{K}$ *has a hyperinvariant subspace.*

Some additional results along these lines are the following.

THEOREM 2.7. *Suppose that* A *and* B *are operators on Hilbert spaces* $\mathscr{H}$ *and* $\mathscr{K}$, *respectively. Suppose also that either* A *or* B *has a hyperinvariant subspace, and that there exists a quasi-invertible operator* $J: \mathscr{K} \to \mathscr{H}$ *such that* $AJ = JB$. *Then every operator of the form*

$$\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$$

acting on $\mathscr{H} \oplus \mathscr{K}$ *has a hyperinvariant subspace.*

Proof. Let T be the operator

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

where C is an arbitrary operator mapping $\mathscr{K}$ into $\mathscr{H}$. We give the proof in the case where A has a hyperinvariant subspace $\mathscr{M}$; the other argument is similar. It suffices to prove that the algebra $(\mathscr{A}_T)'$ is not transitive. Thus, let

$$T' = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

be an arbitrary element of $(\mathscr{A}_T)'$. An easy matricial calculation shows that $BY = YA$, and thus that $AJY = JBY = JYA$. Hence JY commutes with A, and thus $JY(\mathscr{M}) \subset \mathscr{M}$. Since J is quasi-invertible (that is, since J has dense range and no null space), it follows that the subspace

$$\mathscr{N} = \bigvee_{T' \in (\mathscr{A}_T)'} Y(\mathscr{M})$$

cannot be all of $\mathscr{H}$. Thus, if $x \in \mathscr{M}$, we see that for every $T' \in (\mathscr{A}_T)'$, $T'(x \oplus 0) = y \oplus z$, where $z \in \mathscr{N}$, and it follows immediately that the algebra $(\mathscr{A}_T)'$ is not 1-transitive.

COROLLARY 2.8. *If* A *is an operator on* $\mathscr{H}$ *that has a hyperinvariant subspace, then every operator on* $\mathscr{H} \oplus \mathscr{H}$ *of the form*

$$\begin{pmatrix} A & * \\ 0 & A \end{pmatrix}$$

has a hyperinvariant subspace.

COROLLARY 2.9. *If* V *is any operator on* $\mathscr{H}$ *whose lattice of invariant subspaces is linearly ordered (for example, if* V *is the Volterra operator), then every operator of the form*

$$\begin{pmatrix} V & * \\ 0 & V \end{pmatrix}$$

has a hyperinvariant subspace.

3. A GENERALIZATION OF ARVESON'S THEOREM

The main purpose of this section is to generalize the fundamental theorem of Arveson [1, Theorem 3.3] concerning the strong density of transitive operator algebras. We shall eventually prove the following theorem.

THEOREM 3.1. *Suppose that* $\mathscr{A}$ *is a transitive algebra acting on a separable Hilbert space* $\mathscr{H}$, *and suppose that* $\mathscr{A}$ *contains a von Neumann algebra that has an abelian direct summand of finite multiplicity. Then* $\mathscr{A}$ *is strongly dense in* $\mathscr{L}(\mathscr{H})$.

A less technical but equivalent statement of this theorem is as follows.

THEOREM 3.1'. *Suppose that* $\mathscr{A}$ *is a transitive algebra acting on a separable Hilbert space* $\mathscr{H}$, *and suppose that* $\mathscr{A}$ *contains a Hermitian operator that is not of uniform multiplicity* $\aleph_0$. *Then* $\mathscr{A}$ *is strongly dense in* $\mathscr{L}(\mathscr{H})$.

To prove the theorem, we shall need some additional notation and two fundamental lemmas.

We denote by $\mathbb{C}_n$ the n-dimensional Hilbert space of n-tuples of complex numbers, and by $\{e_1, \cdots, e_n\}$ the canonical orthonormal basis for $\mathbb{C}_n$. We remind the reader that a *Stonian space* X is an extremely disconnected, compact Hausdorff space. We shall say that a measure μ on X is *perfect* if μ is a finite, regular Borel measure on X with the property that a measurable set E has positive measure if and only if E has nonvoid interior. In particular, if μ is perfect, then every set of the first category in X has measure zero, and thus, corresponding to every function f in $L_\infty(X, \mu)$, there exists a unique continuous function in C(X) that differs from f on a set of measure zero.

If X is a Stonian space and μ is a perfect measure on X, we denote by $L_2(X, \mathbb{C}_n)$ the Hilbert space consisting of all weakly μ-measurable, square-integrable functions from X to $\mathbb{C}_n$. The inner product of two functions f and g in $L_2(X, \mathbb{C}_n)$ is given by the equation

$$(f, g) = \int_X (f(t), g(t))_{\mathbb{C}_n} \, d\mu .$$

Let M_n denote the von Neumann algebra of all n-by-n complex matrices, and observe that $M_n = \mathscr{L}(\mathbb{C}_n)$. Denote by $M_n(X)$ the C*-algebra of all continuous functions from X to M_n, and note that $M_n(X)$ may be identified with a von Neumann subalgebra of $\mathscr{L}(L_2(X, \mathbb{C}_n))$, where the operator on $L_2(X, \mathbb{C}_n)$ corresponding to an element A in $M_n(X)$ is defined by the equation

$$(Af)(t) = A(t)f(t) \qquad (f \in L_2(X, \mathbb{C}_n),\ t \in X) .$$

Henceforth, we shall assume that this identification has been made.

We shall make special use of $D_n(X)$, the abelian von Neumann subalgebra of $M_n(X)$ consisting of all operators A in $M_n(X)$ such that A(t) is a scalar matrix for each t in X. For each $\phi \in C(X)$, let M_ϕ denote the operator in $D_n(X)$ defined by the equation

$$(M_\phi f)(t) = \phi(t)f(t) \qquad (f \in L_2(X, \mathbb{C}_n),\ t \in X) .$$

Clearly, the correspondence $\phi \leftrightarrow M_\phi$ is a C*-isomorphism between C(X) and $D_n(X)$.

The terminology above will remain fixed in what follows. Furthermore, the following well-known characterization of normal operators of uniform finite multiplicity will be needed (see [12], [19]).

PROPOSITION 3.2. *Let* N *be a normal operator of uniform multiplicity* $n < \aleph_0$ *acting on a separable Hilbert space* $\mathscr{H}$. *Then there exist a Stonian space* X *and a perfect measure* μ *on* X *such that* $\mathscr{H}$ *is isomorphic to* $L_2(X, \mathbb{C}_n)$ *and such that, under this isomorphism, the von Neumann algebra* $\mathscr{V}$ *generated by* N *is carried onto the algebra* $D_n(X)$.

The first of our fundamental lemmas is the following.

LEMMA 3.3. *Suppose, with the notation as above, that* $\mathscr{D}$ *is a dense linear manifold in the Hilbert space* $\mathscr{H} = L_2(X, \mathbb{C}_n)$, *and suppose that* $\mathscr{D}$ *is invariant under all multiplications by operators in* $D_n(X)$. *Then there exists a nonvoid compact open set* $U \subset X$ *such that the vectors* $f_1, \cdots, f_n$ *in* $\mathscr{H}$ *defined by the equation*

$$f_j(t) = \begin{cases} e_j & (t \in U), \\ 0 & (t \notin U) \end{cases}$$

belong to $\mathscr{D}$.

Proof. It follows from an easy continuity argument that there exists a positive number ε with the property that if $\{h_1, \cdots, h_n\}$ is any set of vectors in $\mathbf{C}_n$ satisfying the condition $\|h_i - e_i\| < \varepsilon$ for $1 \leq i \leq n$, then the vectors $h_1, \cdots, h_n$ form a linearly independent set in $\mathbf{C}_n$. Now, for $1 \leq j \leq n$, let g_j be the constant function in $\mathscr{H}$ defined by the equation $g_j(t) \equiv e_j$. Since $\mathscr{D}$ is dense in $\mathscr{H}$, there exist vectors $x_1, \cdots, x_n$ in $\mathscr{D}$ such that

$$\int_X \left(\sum_{j=1}^{n} \|x_j(t) - g_j(t)\|^2 \right) d\mu = \sum_{j=1}^{n} \|x_j - g_j\|^2 < \delta,$$

where δ is any preassigned positive number. In particular, δ may be chosen small enough to ensure that there exists a set $U \subset X$ with positive measure such that $\|x_j(t) - g_j(t)\| < \varepsilon$ almost everywhere on U for $1 \leq j \leq n$. Moreover, since (X, μ) is regular, we may take U to be compact, and since in a perfect measure space a measurable set has positive measure if and only if it has nonvoid interior, we may also suppose that U is compact and open. Furthermore, by multiplying each x_j by M_{χ_U}, we may assume that the x_j vanish on $X \setminus U$. By modifying the x_j on a set of the first category (which necessarily has measure zero), we may suppose that the x_j are continuous on U. Thus for $1 \leq j \leq n$, we have the inequality $\|x_j(t) - e_j\| < \varepsilon$ on $U \setminus W$, where $\mu(W) = 0$. Since $U \setminus W$ is dense in U, the continuity of the x_j on U implies that $\|x_j(t) - e_j\| < \varepsilon$ for $1 \leq j \leq n$ and for all $t \in U$. Define the operator A in $M_n(X)$ to be that matrix-valued function that has the vector x_j for its jth column $(1 \leq j \leq n)$. By our choice of ε, it follows that the matrix $A(t)$ is nonsingular for each $t \in U$. Thus, there exists an element B in $M_n(X)$ such that

$$B(t) = \begin{cases} A(t)^{-1} & (t \in U), \\ 0 & (t \notin U). \end{cases}$$

It follows that the columns of the product AB, being sums of continuous multiplies of the columns of A, also lie in $\mathscr{D}$. Since $AB(t) \equiv 1$ on U, the proof is complete.

The second fundamental lemma is a slight generalization of the theorem in Section 2 of [11].

LEMMA 3.4. *Suppose that* $\mathscr{A}$ *is a transitive subalgebra of* $\mathscr{L}(\mathscr{H})$ *and that for every pair* $(\mathscr{D}, T)$ *with the three properties*

1) $\mathscr{D}$ *is a dense linear manifold in* $\mathscr{H}$,
2) T *is a linear transformation whose domain contains* $\mathscr{D}$, *and*
3) T *commutes with* $\mathscr{A}$ *on* $\mathscr{D}$ (*that is,* $A\mathscr{D} \subset \mathscr{D}$ *and* $AT = TA$ *on* $\mathscr{D}$, *for each* $A \in \mathscr{A}$),

there exist a nonzero vector $x = x(\mathscr{D}, T)$ *in* $\mathscr{D}$ *and a scalar* $\lambda = \lambda(\mathscr{D}, T)$ *such that* $Tx = \lambda x$. *Then* $\mathscr{A}$ *is strongly dense in* $\mathscr{L}(\mathscr{H})$.

Proof. Let T be a closed linear transformation with dense domain $\mathscr{E} \subset \mathscr{H}$ such that T commutes with $\mathscr{A}$ on $\mathscr{E}$. According to Corollary 2.5 of [1], to prove that $\mathscr{A}$ is 2-transitive it suffices to prove that T is a scalar. By hypothesis, there exist a nonzero vector x in $\mathscr{E}$ and a scalar λ such that $Tx = \lambda x$. It follows that every nonzero vector in the dense linear manifold $\mathscr{A}x \subset \mathscr{E}$ is also an eigenvector for T corresponding to λ. Since T is closed, a simple calculation shows that T is everywhere defined and that $T = \lambda$. Thus $\mathscr{A}$ is 2-transitive, and by induction, we may assume that $\mathscr{A}$ is n-transitive. To show that $\mathscr{A}$ is $(n+1)$-transitive, let $T_1, \cdots, T_n$ be linear transformations with common dense domain $\mathscr{E} \subset \mathscr{H}$, each commuting with $\mathscr{A}$ on $\mathscr{E}$, and suppose that $\{(x, T_1x, \cdots, T_nx): x \in \mathscr{E}\}$ is closed in the direct sum of $n + 1$ copies of $\mathscr{H}$. By Corollary 2.5 of [1], to prove that $\mathscr{A}$ is $(n+1)$-transitive it suffices to show that some T_i is closable. By hypothesis, there exist a vector x_1 in $\mathscr{E}$ and a scalar λ_1 such that $T_1 x_1 = \lambda_1 x_1$. Let

$$\mathscr{E}_1 = \{x \in \mathscr{E}: T_1 x = \lambda_1 x\}.$$

Then $\mathscr{E}_1$ is a dense linear manifold containing $\mathscr{A}x_1$, and $\mathscr{E}_1$ is invariant under $\mathscr{A}$. Furthermore, T_2 commutes with $\mathscr{A}$ on $\mathscr{E}_1$. Therefore, by hypothesis, there exist a nonzero vector x_2 in $\mathscr{E}_1$ and a scalar λ_2 such that $T_2 x_2 = \lambda_2 x_2$. Let

$$\mathscr{E}_2 = \{x \in \mathscr{E}_1: T_2 x = \lambda_2 x\}.$$

By the same reasoning as above, $\mathscr{E}_2$ contains $\mathscr{A}x_2$ and is thus a dense linear manifold in $\mathscr{H}$, and in addition, $\mathscr{E}_2$ is invariant under $\mathscr{A}$. Thus we may apply the hypothesis to the pair $(T_3, \mathscr{E}_2)$. By an obvious finite-induction argument, we obtain a linear manifold $\mathscr{E}_n$ that is dense in $\mathscr{H}$ and invariant under $\mathscr{A}$, and that satisfies the condition $T_i x = \lambda_i x$ for each $x \in \mathscr{E}_n$. From the fact that

$$\{(x, T_1 x, \cdots, T_n x): x \in \mathscr{E}\}$$

is closed it now follows easily that $\mathscr{E} = \mathscr{H}$ and that $T_i \equiv \lambda_i$ $(1 \leq i \leq n)$. This shows that each T_i is closed, and thus that $\mathscr{A}$ is $(n+1)$-transitive. By induction, $\mathscr{A}$ is k-transitive for every positive integer k, and hence $\mathscr{A}$ is strongly dense in $\mathscr{L}(\mathscr{H})$.

Our program to prove Theorem 3.1' is now fairly transparent. We shall show that every transitive algebra satisfying the hypotheses of Theorem 3.1' also satisfies the hypothesis of Lemma 3.4, and hence must be strongly dense.

Proof of Theorem 3.1'. Let H be a Hermitian operator in $\mathscr{A}$ such that H is not of uniform multiplicity $\aleph_0$. We may suppose that $\mathscr{A}$ is strongly closed, and therefore that $\mathscr{A}$ contains the von Neumann algebra $\mathscr{V}$ generated by H. By elementary multiplicity theory, $\mathscr{V}$ contains a central (in $\mathscr{V}$) projection E such that EHE is of uniform multiplicity n for some positive integer n. By virtue of Theorem 1.3, it suffices to prove that $E\mathscr{A}E$ is strongly dense in $\mathscr{L}(\mathscr{H}_E)$. To say the same thing slightly differently: by a change of notation, we may assume that $\mathscr{A}$ contains the abelian von Neumann algebra $\mathscr{V}$ generated by a Hermitian operator H of uniform multiplicity n, and also that the identity of $\mathscr{V}$ is $1_{\mathscr{H}}$.

By Proposition 3.2, there exist a Stonian space X and a perfect measure μ on X such that $\mathscr{H}$ is isomorphic to $L_2(X, \mathbb{C}_n)$ and such that $\mathscr{V}$ is unitarily equivalent under this isomorphism to $D_n(X)$. We henceforth assume that the identifications $\mathscr{H} = L_2(X, \mathbb{C}_n)$ and $\mathscr{V} = D_n(X)$ have been made, and thus that $\mathscr{A} \supset D_n(X)$. We wish to apply Lemma 3.4; therefore, let T be a linear transformation on $\mathscr{H}$ whose domain contains a dense linear manifold $\mathscr{D}$, and suppose that T commutes with $\mathscr{A}$ on

$\mathscr{D}$. We proceed to show that there exists a nonzero vector x_0 in $\mathscr{D}$ that is an eigenvector for T. This will complete the proof, by virtue of Lemma 3.4.

By hypothesis, the domain $\mathscr{D}$ is invariant under the algebra $\mathscr{A}$. In particular, since $\mathscr{A}$ is strongly closed and $\mathscr{V} = D_n(X) \subset \mathscr{A}$, the domain $\mathscr{D}$ is invariant under multiplication by every operator in $D_n(X)$. Thus, by Lemma 3.3, there exists a nonvoid compact open subset U_1 of X such that the functions $f_1, \cdots, f_n$ in $\mathscr{H}$ defined by the equations

$$f_j(t) = \begin{cases} e_j & (t \in U_1), \\ 0 & (t \notin U_1) \end{cases}$$

belong to $\mathscr{D}$. By virtue of the definition of the vectors f_j, we may write, for each $t \in U_1$,

$$(Tf_j)(t) = \alpha_{1j}(t) f_1(t) + \cdots + \alpha_{nj}(t) f_n(t) \qquad (1 \leq j \leq n).$$

It follows easily from the definition of the inner product on $\mathscr{H}$ that each of the scalar-valued functions α_{ij} is integrable $[\mu]$ over the set U_1. Since μ is a perfect measure, one sees without difficulty that there exists a nonvoid compact open set $U \subset U_1$ such that on U each of the functions α_{ij} $(1 \leq i, j \leq n)$ is bounded, and hence may be assumed to be in C(U). For $1 \leq i \leq n$, let g_i be the vector in $\mathscr{D}$ defined by the equation $g_i = M_{\chi_U} f_i$. Since T commutes with M_{χ_U} on $\mathscr{D}$, we see that $(Tg_i)(t) = 0$ for $t \in X \setminus U$ and

$$\begin{aligned}(Tg_j)(t) &= [M_{\chi_U}(Tf_j)](t) = \chi_U(t)(Tf_j)(t) = (Tf_j)(t) = \alpha_{1j}(t) f_1(t) + \cdots + \alpha_{nj}(t) f_n(t) \\ &= \alpha_{1j}(t) g_1(t) + \cdots + \alpha_{nj}(t) g_n(t) \qquad (1 \leq j \leq n),\end{aligned}$$

for $t \in U$.

We write F for the projection $F = M_{\chi_U}$ in $D_n(X)$, and $\mathscr{F}$ for the range of F, so that each vector g_j belongs to $\mathscr{D} \cap \mathscr{F}$. Furthermore, since $F \in \mathscr{A}$, we have the relations $F\mathscr{D} \subset \mathscr{D}$ and $FT = TF$ on $\mathscr{D}$. It follows easily that $\mathscr{D} \cap \mathscr{F}$ is dense in $\mathscr{F}$, that $T(\mathscr{D} \cap \mathscr{F}) \subset \mathscr{D} \cap \mathscr{F}$, and that $T = FT$ on $\mathscr{D} \cap \mathscr{F}$. In particular, observe that T commutes with the algebra $F\mathscr{A}F$ on $\mathscr{D} \cap \mathscr{F}$.

We shall complete the proof by showing that the linear transformation T has an eigenvector in $\mathscr{D} \cap \mathscr{F}$. To accomplish this, let $\tilde{T}$ be the element of the von Neumann algebra $\mathscr{V}' = M_n(X)$ defined by $\tilde{T}(t) = 0$ for $t \in X \setminus U$ and

$$\tilde{T}(t) = \begin{pmatrix} \alpha_{11}(t) & \cdots & \alpha_{1n}(t) \\ \cdots & \cdots & \cdots \\ \alpha_{n1}(t) & \cdots & \alpha_{nn}(t) \end{pmatrix}$$

for $t \in U$. We wish to show that $Tw = \tilde{T}w$ for every vector w in $\mathscr{D} \cap \mathscr{F}$. Suppose first that w has the form

$$w = M_{\phi_1} g_1 + \cdots + M_{\phi_n} g_1, \tag{1}$$

where $\phi_1, \cdots, \phi_n$ are functions in C(U). Then, since both T and $\tilde{T}$ commute with the multiplications M_{ϕ_i} on $\mathscr{D}$, and since by construction T agrees with $\tilde{T}$ on the vectors g_i, we see that $Tw = \tilde{T}w$ for all vectors w in $\mathscr{D} \cap \mathscr{F}$ of the form (1). Now suppose that w is an arbitrary nonzero vector in $\mathscr{D} \cap \mathscr{F}$. Then the best that can be said about the function $t \to \|w(t)\|_{\mathbb{C}_n}$ is that it is square-integrable. However, the function ρ defined by the equation

$$\rho(t) = \begin{cases} 1/(1 + \|w(t)\|) & (t \in U), \\ 0 & (t \notin U) \end{cases}$$

belongs to $L_\infty(X, \mu)$, and the vector $M_\rho w$ belongs to $\mathscr{D} \cap \mathscr{F}$ and is of the form (1). Hence

$$M_\rho \tilde{T} w = \tilde{T} M_\rho w = T M_\rho w = M_\rho T w .$$

Since $M_\rho \mid \mathscr{F}$ has no null space (by inspection), we obtain the desired conclusion that $\tilde{T}w = Tw$ for all $w \in \mathscr{D} \cap \mathscr{F}$.

Now where are we? We have found a nonzero projection F in $D_n(X)$ and a bounded operator $\tilde{T}$ in $M_n(X)$ such that on the set $\mathscr{D} \cap \mathscr{F}$, the operator T agrees with $\tilde{T}$. But we observed earlier that T commutes with the algebra $F\mathscr{A}F$ on $\mathscr{D} \cap \mathscr{F}$, and since $\mathscr{D} \cap \mathscr{F}$ is invariant under $F\mathscr{A}F$, it follows that $\tilde{T}$ commutes with the algebra $F\mathscr{A}F$ on $\mathscr{D} \cap \mathscr{F}$. Since $\tilde{T}$ is bounded, $\tilde{T}$ commutes with the algebra $F\mathscr{A}F$ on all of $\mathscr{F}$. In particular, $\tilde{T}\mathscr{F} \subset \mathscr{F}$, and inspection shows that $\tilde{T} \mid \mathscr{F}$ is an n-normal operator. Now suppose that $\tilde{T} \mid \mathscr{F}$ is not a scalar multiple of $1_{\mathscr{F}}$. Then $\tilde{T} \mid \mathscr{F}$ has a nontrivial hyperinvariant subspace $\mathscr{P}$, by Hoover's theorem [9] (see also [16]). This means that $\mathscr{P}$ is invariant for the commutant of $\tilde{T} \mid \mathscr{F}$, and in particular, $\mathscr{P}$ is invariant for the transitive algebra $F\mathscr{A}F$. This is a contradiction, which establishes that $\tilde{T} \mid \mathscr{F}$ is a scalar, and hence that T is a scalar on $\mathscr{D} \cap \mathscr{F}$. This completes the proof via Lemma 3.4.

Note. The reader interested in the circle of ideas concerning transitive algebras should consult the recent article [17], which was written concurrently with this paper.

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State University of New York
Stony Brook, New York 11790
and
University of Michigan
Ann Arbor, Michigan 48104