THE POWERS OF AN OPERATOR OF NUMERICAL RADIUS ONE

Michael J. Crabb

Let H be a complex Hilbert space, and T a bounded linear operator on H. As usual, the numerical radius $\omega(T)$ of T is defined as

$$\sup\{|(Tx, x)|: x \in H, \|x\| = 1\}.$$

C. A. Berger and J. G. Stampfli [1] have shown that if $\omega(T) = 1$, then

$$\limsup \|T^n x\| \leq \sqrt{3}\,\|x\|$$

(see also T. Kato [3], [4]), and they ask for the best constant K such that $\limsup \|T^n x\| \leq K\|x\|$. They give an example (due to A. L. Shields) of an operator T and an element $x \in H$ with $\omega(T) = \|x\| = 1$ and $\|T^n x\| = \sqrt{2}$ $(n = 1, 2, \cdots)$. Hence $K \geq \sqrt{2}$. We show here that the best constant K is $\sqrt{2}$, and further, that if $\|x\| = \omega(T) = 1$, then $\|T^n x\| \to \ell$ for some $\ell \leq \sqrt{2}$.

Berger and Stampfli [2] show that if $\omega(T) = \|x\| = 1$ and $\|T^n x\| = 2$ for some n, then $T^{n+1}x = 0$. We give another proof of this, and we also show that $\|Tx\| = \|T^2 x\| = \cdots = \|T^{n-1}x\| = \sqrt{2}$, that x, Tx, $\cdots$, $T^n x$ are mutually orthogonal, and that their linear span forms a reducing subspace of T. This was proved in the case $n = 1$ by J. P. Williams and T. Crimmins [5].

THEOREM 1. *Suppose that* $\omega(T) = \|x\| = 1$. *Then* $\|T^n x\| \to \ell$, *where* $0 \leq \ell \leq \sqrt{2}$. *If* $\ell = \sqrt{2}$, *then* $\|T^n x\| = \sqrt{2}$ $(n = 1, 2, \cdots)$.

Proof. Let n be a positive integer, and let $a_k \in \mathbb{R}$ $(k = 0, 1, \cdots, n)$. Then, for $y = a_0 x + \cdots + a_n T^n x$, $|(Ty, y)| \leq (y, y)$, in other words,

$$\left| a_0 a_1 \|Tx\|^2 + a_1 a_2 \|T^2 x\|^2 + \cdots + \sum_{\substack{j,k=0 \\ j+1 \neq k}}^{n} a_j a_k (T^{j+1}x, T^k x) \right| \tag{1}$$

$$\leq a_0^2 + a_1^2 \|Tx\|^2 + \cdots + \sum_{\substack{j,k=0 \\ j \neq k}}^{n} a_j a_k (T^j x, T^k x).$$

Replacing T by $e^{i\theta}T$ and integrating both sides over $[0, 2\pi]$, we obtain the inequality

$$m_1 a_0 a_1 + m_2 a_1 a_2 + \cdots + m_n a_{n-1} a_n \leq a_0^2 + m_1 a_1^2 + \cdots + m_n a_n^2,$$

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where $m_n = \|T^n x\|^2$. Let

$$D_n = \begin{vmatrix} 1 & -m_1/2 & & & & \\ -m_1/2 & m_1 & & & \mathbf{0} & \\ & & \cdot & & & \\ & & & \cdot & & \\ & \mathbf{0} & & & \cdot & -m_n/2 \\ & & & & -m_n/2 & m_n \end{vmatrix}$$

and put $D_0 = 1$. Since the associated quadratic form is nonnegative, $D_n \geq 0$. Expanding by the last row, we find that

$$D_n = m_n D_{n-1} - \frac{1}{4} m_n^2 D_{n-2}. \tag{2}$$

If D_n vanishes for some n, let k be the smallest suffix for which $D_k = 0$. Then $D_{k+1} = -\frac{1}{4} m_{k+1}^2 D_{k-1}$, so that $m_{k+1} = 0$. In this case, $\|T^n x\| \to 0$.

If $D_n > 0$ for all n, then

$$\frac{D_n}{D_{n-1}} = \frac{m_n D_{n-1} - \frac{1}{4} m_n^2 D_{n-2}}{D_{n-1}} \leq \frac{D_{n-1}}{D_{n-2}},$$

since $D_{n-1}^2 - m_n D_{n-1} D_{n-2} + \frac{1}{4} m_n^2 D_{n-2}^2 \geq 0$. Therefore $\frac{D_n}{D_{n-1}} \to \ell$ as $n \to \infty$, for some $\ell \geq 0$. By (2),

$$m_n = 2\left\{\frac{D_{n-1}}{D_{n-2}} \pm \left[\left(\frac{D_{n-1}}{D_{n-2}}\right)^2 - \frac{D_n}{D_{n-1}} \frac{D_{n-1}}{D_{n-2}}\right]^{1/2}\right\},$$

so that $m_n \to 2\ell$. Also, $\ell \leq D_1/D_0 = m_1 - \frac{1}{4} m_1^2 \leq 1$.

Clearly, $\ell = 1$ only if $m_1 = m_2 = \cdots = 2$.

THEOREM 2. *Suppose that* $\omega(T) = \|x\| = 1$ *and that* $\|T^n x\| = 2$ *for some integer* n. *Then* $T^{n+1} x = 0$, $\|T^k x\| = \sqrt{2}$ $(k = 1, 2, \cdots, n-1)$, *the elements* $x, Tx, \cdots, T^n x$ *are mutually orthogonal, and their linear span is a reducing subspace of* T.

Proof. With the notation of Theorem 1, $m_n = 4$. Therefore

$$D_n = 4(D_{n-1} - D_{n-2}) \geq 0,$$

so that

$$1 \leq \frac{D_{n-1}}{D_{n-2}} \leq \frac{D_{n-2}}{D_{n-3}} \leq \cdots \leq \frac{D_1}{D_0} \leq 1.$$

Hence $D_1 = D_2 = \cdots = D_{n-1} = 1$, and $m_1 = m_2 = \cdots = m_{n-1} = 2$. Also, $D_n = 0$, so that $m_{n+1} = 0$. Let $(T^j x, T^k x) = R_{jk} e^{i\alpha_{jk}}$. Replacing T by $e^{i\theta} T$ in (1), and taking real parts, we obtain the inequality

$$(3)\quad \begin{aligned} &2a_0a_1 + \cdots + 2a_{n-2}a_{n-1} + 4a_{n-1}a_n + \sum_{\substack{j,k=0\\ j+1\neq k}}^{n} a_ja_kR_{j+1,k}\cos((j+1-k)\theta + \alpha_{j+1,k}) \\ &\leq a_0^2 + 2a_1^2 + \cdots + 2a_{n-1}^2 + 4a_n^2 + \sum_{\substack{j,k=0\\ j\neq k}}^{n} a_ja_kR_{jk}\cos((j-k)\theta + \alpha_{jk}) \,. \end{aligned}$$

If we put $a_0 = a_1 = \cdots = a_{n-1} = 1$ and $a_n = 1/2$, then the terms independent of θ on each side of (3) are equal. Hence the integrals of each side over $[0, 2\pi]$ are equal. If we had strict inequality in (3) for some θ, the integral on the left-hand side would be less than that on the right. Therefore, we have equality in (3) for these $a_0, \cdots, a_n$. Hence, the partial derivatives of the two sides of (3) with respect to $a_0, \cdots, a_n$ must be equal at this point. For a_n, this gives the relation

$$\begin{aligned} &R_{1,n}\cos((n-1)\theta - \alpha_{1,n}) + \cdots + R_{n-1,n}\cos(\theta - \alpha_{n-1,n}) \\ &\qquad = 2R_{0,n}\cos(n\theta - \alpha_{0,n}) + \cdots + 2R_{n-1,n}\cos(\theta - \alpha_{n-1,n}) \,. \end{aligned}$$

Since this holds for $0 \leq \theta \leq 2\pi$, we conclude that $R_{0,n} = \cdots = R_{n-1,n} = 0$. Similarly, differentiating with respect to $a_{n-1}, a_{n-2}, \cdots$ successively, we find that $R_{ij} = 0$ for $i \neq j$.

Let $L = \mathrm{lin}(x, Tx, \cdots, T^n x)$. Suppose $a \perp L$. Putting

$$y = x + Tx + \cdots + T^{n-1}x + \frac{1}{2}T^n x + ta \,,$$

where $t > 0$, we obtain the inequality $\Re(Ty, y) \leq (y, y)$, which implies that

$$t\,\Re\left(x + Tx + \cdots + \frac{1}{2}T^n x,\ Ta\right) + t^2\,\Re(a, Ta) \leq t^2\,\|a\|^2 \qquad (t > 0)\,.$$

Hence $\Re\left(x + \cdots + \frac{1}{2}T^n x,\ Ta\right) \leq 0$. Replacing a by $e^{i\theta}a$, we obtain the equation $\left(x + \cdots + \frac{1}{2}T^n x,\ Ta\right) = 0$, and replacing T by $e^{i\theta}T$, we deduce that $(x, Ta) = \cdots = (T^n x, Ta) = 0$, or $Ta \perp L$. Hence L is a reducing subspace of T.

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King's College
Aberdeen, Scotland