ON COMMUTATORS IN IDEALS OF COMPACT OPERATORS

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1. INTRODUCTION

Let $\mathscr{H}$ be an infinite-dimensional, separable Hilbert space, and denote by $\mathscr{L}(\mathscr{H})$ the algebra of all bounded, linear operators on $\mathscr{H}$. All operators to which we refer will be bounded and linear. We denote by $\mathscr{K}$ the (two-sided) ideal in $\mathscr{L}(\mathscr{H})$ consisting of all compact operators, and we recall that $\mathscr{K}$ is a C*-algebra (without identity). The ideal $\mathscr{K}$ is the only ideal in $\mathscr{L}(\mathscr{H})$ that is closed in the operator norm, but $\mathscr{K}$ contains a whole chain of other ideals—the Schatten p-ideals. For our purposes, it will be convenient to employ Dixmier's method [3, p. 296] of defining the *Schatten* p-*classes* $\mathscr{C}_p$, where p is any positive number. To do this, one first defines the *trace class* $\mathscr{C}_1$ as follows. Suppose $K \in \mathscr{K}$, and let $K = UP$ be the canonical polar decomposition of K. Then P is a positive compact operator, and thus is unitarily equivalent to a diagonal matrix—say (δ_i). One says that $K \in \mathscr{C}_1$ if the sequence $\{\delta_i\}$ belongs to the Banach space (ℓ_1). The trace *norm* of an operator $K \in \mathscr{C}_1$ is defined by $\|K\|_1 = \|\{\delta_i\}\|_1$, where the norm on the right is the norm on the Banach space (ℓ_1). The set of all operators $A^{1/p}$, where A runs over all positive operators in $\mathscr{C}_1$, is the positive part of an ideal in $\mathscr{K}$ [3, Proposition 1, p. 296], and we denote this ideal by $\mathscr{C}_p$. It is easy to see that if $K \in \mathscr{C}_p$ and K has a canonical polar decomposition $K = UP$, then $P^p \in \mathscr{C}_1$, and we define the *Schatten* p-*norm* of K by $\|K\|_p = \|P^p\|_1^{1/p}$. It is known that $\mathscr{C}_p$ is a Banach *-algebra under the Schatten p-norm [6]. Furthermore, it is not hard to see that for every pair p, q of positive numbers p and q,

$$\mathscr{C}_p \cdot \mathscr{C}_q = \mathscr{C}_r, \tag{1}$$

where $r^{-1} = p^{-1} + q^{-1}$, and where the left-hand side represents the set of all finite sums of the form $\sum A_i B_i$ $(A_i \in \mathscr{C}_p,\ B_i \in \mathscr{C}_q)$. The ideals $\mathscr{C}_1$ and $\mathscr{C}_2$ are more important than the other Schatten p-ideals. The trace class $\mathscr{C}_1$ may be characterized as the set of all compact operators K with the property that the matrix of K with respect to every orthonormal basis of $\mathscr{H}$ has absolutely summable trace (whose value is independent of the orthonormal basis). We shall denote the trace on $\mathscr{C}_1$ by $\operatorname{tr}(\cdot)$, and we recall that it has all the usual properties of a trace [6]. The ideal $\mathscr{C}_2$ is called the *Hilbert-Schmidt class*, and it may be characterized as the set of operators K in $\mathscr{L}(\mathscr{H})$ such that some (and therefore every) matrix for K has square-summable entries [6]. We refer the reader to [6] for more detail concerning the Schatten p-classes.

If $\mathscr{M}$ is any ideal in $\mathscr{L}(\mathscr{H})$, we denote by $C(\mathscr{M})$ the set of all commutators of elements of $\mathscr{M}$. In other words, $C(\mathscr{M})$ consists of all operators A (necessarily in $\mathscr{M}$) such that there exist operators $B, C \in \mathscr{M}$ with $A = BC - CB$. We also denote the linear span of $C(\mathscr{M})$ by $[\mathscr{M}, \mathscr{M}]$. In other words, $[\mathscr{M}, \mathscr{M}]$ consists of all finite sums of elements from $C(\mathscr{M})$. In the case $\mathscr{M} = \mathscr{L}(\mathscr{H})$, the identification of

Received July 20, 1970.
This research was supported by the National Science Foundation.

Michigan Math. J. 18 (1971).

$C(\mathscr{L}(\mathscr{H}))$ was accomplished in [2], where it was shown that $C(\mathscr{L}(\mathscr{H}))$ consists of the union of $\mathscr{K}$ and the class of operators of type (F). Earlier, it had been shown [4] that $[\mathscr{L}(\mathscr{H}), \mathscr{L}(\mathscr{H})] = \mathscr{L}(\mathscr{H})$.

2. SOME OPEN QUESTIONS

The purpose of this paper is to raise some questions and to provide some answers concerning the identity of the classes $C(\mathscr{M})$ and $[\mathscr{M}, \mathscr{M}]$, where $\mathscr{M} = \mathscr{K}$ or $\mathscr{M} = \mathscr{C}_p$ $(p \geq 1)$. Unfortunately, many of the questions seem difficult.

PROBLEM 1. *Is* $C(\mathscr{K}) = \mathscr{K}$?

What makes this problem so difficult is that we cannot get started. We cannot show that every projection of rank 1 is the commutator of two compact operators. (We know from [1] that every compact operator is a commutator AB - BA of two bounded operators A and B; but inspection of the relevant proofs shows that, in general, neither A nor B is compact.) The question of the identity of $[\mathscr{K}, \mathscr{K}]$ is somewhat easier, and we show (Theorem 1) that $[\mathscr{K}, \mathscr{K}] = \mathscr{K}$.

Turning our attention to the Schatten p-classes $\mathscr{C}_p$ $(p > 0)$, we observe (see (1)) that if $A, B \in \mathscr{C}_p$, then $AB - BA \in \mathscr{C}_{p/2}$, so that $C(\mathscr{C}_{2p}) \subset \mathscr{C}_p$, and it is impossible that $C(\mathscr{C}_p) = \mathscr{C}_p$. Thus, the appropriate question in this case is the following:

PROBLEM 2. *Is* $C(\mathscr{C}_{2p}) = \mathscr{C}_p$ $(p > 1)$?

It is only appropriate to ask this question for $p > 1$, because the existence of a trace on $\mathscr{C}_1$ shows immediately that any operator in $C(\mathscr{C}_2)$ must be a trace-class operator having trace zero. Let us denote the class of operators in $\mathscr{C}_1$ with trace zero by $\mathscr{C}_1^0$.

PROBLEM 3. *Is* $C(\mathscr{C}_2) = \mathscr{C}_1^0$?

The techniques involved in giving an affirmative answer to this question would likely enable us to solve some stubborn problems in the theory of commutators in finite von Neumann algebras (see [5]). Problem 3 is so intractable that we cannot even answer a weaker question:

PROBLEM 3'. *Is* $[\mathscr{C}_2, \mathscr{C}_2] = \mathscr{C}_1^0$?

3. SOME PROGRESS

In this section we prove the following theorems.

THEOREM 1. *The linear span of* $C(\mathscr{K})$ *is* $\mathscr{K}$ *itself; that is,* $[\mathscr{K}, \mathscr{K}] = \mathscr{K}$.

THEOREM 2. *The relation* $[\mathscr{C}_{2p}, \mathscr{C}_{2p}] = \mathscr{C}_p$ *holds for every real* $p > 1$.

THEOREM 3. *The relation* $[\mathscr{C}_{2+\varepsilon}, \mathscr{C}_{2+\varepsilon}] = \mathscr{C}_1$ *holds for every* $\varepsilon > 0$.

In view of the preceding remarks, all of these results are best possible. The proofs of all three theorems proceed along the same lines, as follows. Let T be an arbitrary operator in $\mathscr{K}$ [respectively, in $\mathscr{C}_p$]. Let ϕ be a unitary isomorphism of $\mathscr{H}$ onto $\mathscr{H} \oplus \mathscr{H}$, and observe that this isomorphism carries T onto a 2-by-2 matrix

$$\tilde{T} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

that acts on $\mathscr{H} \oplus \mathscr{H}$ in the usual way. It is clear that each T_i belongs to $\mathscr{K}$ [respectively, each T_i belongs to $\mathscr{C}_p$], and also that it suffices to prove the theorems for $\tilde{T}$ in place of T. We write

$$\tilde{T} = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & T_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ T_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & T_4 \end{pmatrix}, \tag{2}$$

and symmetry considerations show that to prove that $\tilde{T} \in [\mathscr{K}, \mathscr{K}]$ (respectively, $\tilde{T} \in [\mathscr{C}_{2p}, \mathscr{C}_{2p}]$), it suffices to prove this for the first two summands on the right-hand side of (2). We deal first with the second summand. Let $T_2 = UP$ be the unique polar decomposition of T_2, and note that $P^{1/2} \in \mathscr{K}$ [respectively, $P^{1/2} \in \mathscr{C}_{2p}$]. Then

$$\begin{pmatrix} 0 & T_2 \\ 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} UP^{1/2} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & P^{1/2} \\ 0 & 0 \end{pmatrix} \right],$$

and each of the matrices on the right belongs to $\mathscr{K}$ [to $\mathscr{C}_{2p}$]. Thus, to complete the proofs of Theorems 1, 2, and 3, it suffices to prove the following lemma.

LEMMA. *The operator matrix*

$$\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \tag{3}$$

is the sum (AB - BA) + (CD - DC) *of two commutators. Furthermore, if* $S \in \mathscr{K}$ [*if* $S \in \mathscr{C}_1$, $\mathscr{C}_p$ $(p > 1)$], *then each of the four operators* A, B, C, D *can be taken to belong to* $\mathscr{K}$ [*to* $\mathscr{C}_{2+\varepsilon}$ $(\varepsilon > 0)$, $\mathscr{C}_{2p}$].

Proof. If the second copy of $\mathscr{H}$ in $\mathscr{H} \oplus \mathscr{H}$ is identified with countably many copies of $\mathscr{H}$, then the Hilbert space $\mathscr{H} \oplus \mathscr{H}$ becomes identified with

$$\tilde{\mathscr{H}} = \mathscr{H} \oplus (\mathscr{H} \oplus \cdots \oplus \mathscr{H} \cdots),$$

and the operator (3) becomes unitarily equivalent to the diagonal operator matrix $\mathrm{Diag}(S, 0, 0, \cdots)$ in $\mathscr{L}(\tilde{\mathscr{H}})$. We write

$$\begin{aligned} \mathrm{Diag}(S, 0, 0, \cdots, 0, \cdots) &= \mathrm{Diag}(S, -S, S/2, -S/2, \cdots, S/n, -S/n, \cdots) \\ &\quad + \mathrm{Diag}(0, S, -S/2, S/2, \cdots, -S/n, S/n, \cdots), \end{aligned} \tag{4}$$

and we observe that the proof may be completed by showing that each of the summands on the right in (4) is a commutator of the appropriate type. To accomplish this, we first introduce some convenient notation.

If $(A_{ij})_{i,j=1}^{\infty}$ is an infinite operator matrix (with entries from $\mathscr{L}(\mathscr{H})$) that represents an operator in $\mathscr{L}(\tilde{\mathscr{H}})$, and if the only nonzero entries occurring in (A_{ij}) occur on the diagonal directly above the main diagonal, we write

$$\mathrm{UDiag}(B_1, B_2, \cdots, B_n, \cdots) = \mathrm{UDiag}(\{B_n\})$$

for the matrix (A_{ij}). This implies, of course, that $A_{12} = B_1$, $A_{23} = B_2$, $\cdots$. Similarly, for a matrix (A_{ij}) all of whose nonzero entries lie on the diagonal directly below the main diagonal we write

$$\mathrm{LDiag}(C_1, C_2, \cdots, C_n, \cdots) = \mathrm{LDiag}(\{C_n\}),$$

where $C_1 = A_{21}$, $C_2 = A_{32}$, $\cdots$. It is easy to verify directly that if $\{\alpha_n\}_{n=1}^{\infty}$ is a bounded sequence of scalars and S is the operator in $\mathscr{L}(\mathscr{H})$ appearing in (4), then the commutator

$$[\mathrm{LDiag}(\{\alpha_n S\}), \mathrm{UDiag}(\{\alpha_n 1_{\mathscr{H}}\})] \tag{5}$$

is equal to the operator

$$\mathrm{Diag}(\alpha_1^2 S, (\alpha_2^2 - \alpha_1^2)S, \cdots, (\alpha_{n+1}^2 - \alpha_n^2)S, \cdots).$$

Thus the first summand on the right-hand side of (4) is the commutator (5), where the sequence $\{\alpha_n\}$ is defined by the rule $\alpha_{2n-1} = \alpha_{2n} = 1/\sqrt{n}$ $(n = 1, 2, \cdots)$. With this definition of the sequence $\{\alpha_n\}$, it follows easily that the operator $\mathrm{UDiag}(\{\alpha_n 1_{\mathscr{H}}\})$ belongs to every ideal $\mathscr{C}_p$ $(p > 2)$. Furthermore, it is easy to see that if S is compact, then the operator $\mathrm{LDiag}(\{\alpha_n S\})$ is also compact, and if $S \in \mathscr{C}_1$, then $\mathrm{LDiag}(\{\alpha_n S\})$ belongs to $\mathscr{C}_{2+\varepsilon}$ for every positive number ε. Finally, if $S \in \mathscr{C}_p$ for $p > 1$, then $\mathrm{LDiag}(\{\alpha_n S\})$ belongs to $\mathscr{C}_{2p}$. (All of these results are easy consequences of the fact that the series $\sum_{n=1}^{\infty} n^{-p}$ converges for $p > 1$.)

To complete the proof of Lemma 4, we must show that the second summand $\mathrm{Diag}(\{\beta_n S\})$ on the right-hand side of (4) is a commutator of operators of the appropriate types. This is harder, however, because the sequence $\{\beta_n\}_{n=1}^{\infty}$ of coefficients of S on the diagonal is

$$0, 1, -\frac{1}{2}, \frac{1}{2}, \cdots, -\frac{1}{n}, \frac{1}{n}, \cdots, \tag{6}$$

and the series $\sum_{n=1}^{\infty} \beta_n$ converges conditionally to 1. (This implies that if we try to make $\mathrm{Diag}(\{\beta_n S\})$ a commutator of the form (5), then the sequence $\{\alpha_n\}$ will not converge to zero, and thus the operators in (5) will fail to be compact.) Thus, we need the following auxiliary proposition.

The conditionally convergent series

$$0 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \cdots \tag{7}$$

can be rearranged into a series $\alpha_1 + \alpha_2 + \alpha_3 + \cdots$ *that converges to zero and has the further property that*

$$\left|\sum_{i=1}^{n} \alpha_i\right| \leq \frac{4}{n} \quad (1 \leq n < \infty). \tag{8}$$

That the series (7) can be rearranged into a series converging (conditionally) to zero is a well-known elementary theorem due to Riemann. The desired series

$\alpha_1 + \alpha_2 + \cdots$ is exactly the series that is produced by the standard proof of Riemann's theorem. More precisely, we define $\alpha_1 = 0$ and $\alpha_2 = 1$, and then we add negative terms $\alpha_3, \cdots, \alpha_{k_1}$ (in the order of their occurrence in the original series) until the partial sum $\alpha_1 + \cdots + \alpha_{k_1}$ becomes negative. We then define $\alpha_{k_1+1} = 1/2$, note that the partial sum $\alpha_1 + \cdots + \alpha_{k_1} + \alpha_{k_1+1}$ is positive, and add additional negative terms $\alpha_{k_1+2}, \cdots, \alpha_{k_2}$ (again in the order of their occurrence in the original series) until the partial sum $\alpha_1 + \cdots + \alpha_{k_2}$ again becomes negative. Continuing this, we obtain by induction the series $\alpha_1 + \alpha_2 + \cdots + \alpha_n + \cdots$ whose sum is obviously zero. That (8) is valid is a consequence of the following three properties of the rearranged series:

(a) The first few terms of the series are

$$0 + 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{3} - \frac{1}{8} \cdots .$$

(b) The series does not contain two consecutive positive terms.

(c) The series does not contain four consecutive negative terms.

Both (b) and (c) are easily verified by induction based on (a) and the elementary inequality

$$\frac{1}{n} \leq \frac{1}{3n - 1} + \frac{1}{3n} + \frac{1}{3n + 1}.$$

The details of these induction arguments and the verification that (8) follows from (a), (b), and (c) are left to the reader.

We return now to complete the proof of the lemma. We must show that $\mathrm{Diag}(\{\beta_n S\})$ is a commutator of the right type, where $\{\beta_n\}$ is the sequence (6). Let $\{\alpha_n\}$ be the permutation of the sequence $\{\beta_n\}$ described in the auxiliary proposition. The operator $\mathrm{Diag}(\{\beta_n S\})$ is clearly unitarily equivalent to the operator $\mathrm{Diag}(\{\alpha_n S\})$, so that it suffices to prove the result for the operator $\mathrm{Diag}(\{\alpha_n S\})$. Let the (complex) sequence $\{\gamma_n\}$ be defined so that $\gamma_n^2 = \sum_{k=1}^{n} \alpha_k$. With this definition of the sequence $\{\gamma_n\}$, it is clear that $\mathrm{Diag}(\{\alpha_n S\})$ is the commutator $[L\,\mathrm{Diag}(\{\gamma_n S\}), U\,\mathrm{Diag}(\{\gamma_n 1_{\mathscr{H}}\})]$ (see (5)). Furthermore, by virtue of (8), we have the inequality $|\gamma_n| \leq 2/\sqrt{n}$ for every positive integer n. It now follows, just as before, that $U\,\mathrm{Diag}(\{\gamma_n 1_{\mathscr{H}}\})$ belongs to every ideal $\mathscr{C}_p$ for $p > 2$. Moreover, if S belongs to $\mathscr{K}$ [respectively, to $\mathscr{C}_1$, to $\mathscr{C}_p$ $(p > 1)$], then $L\,\mathrm{Diag}(\{\alpha_n S\})$ belongs to $\mathscr{K}$ [respectively, to $\mathscr{C}_{2+\varepsilon}$, to $\mathscr{C}_{2p}$]. This completes the proof of the lemma, and therefore of Theorems 1, 2, and 3.

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