THE SPAN OF A RIEMANN SURFACE

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In their book on capacity functions [2], L. Sario and K. Oikawa introduce what they call the H-span and the K-span for arbitrary Riemann surfaces. For planar surfaces, they show that if the K-span vanishes at some point for some choice of local variable at that point, then it vanishes everywhere. They then ask whether the same is true without the restriction that the surface be planar. We shall give an example of a Riemann surface of infinite genus having the property that the K-span vanishes at some point for all choices of local variable at that point, but nevertheless does not vanish everywhere. B. Rodin [1] answered the corresponding question for the H-span in the negative. In our example, it turns out that the H-span and the K-span coincide; and thus we have also obtained a Riemann surface with the property that the H-span vanishes at some point for every choice of local variable at that point, but nevertheless does not vanish identically.

1. Let R_0 be a hyperbolic Riemann surface having one ideal boundary component, and with the property that the vector lattice HD of harmonic functions with finite Dirichlet integral consists only of constants. Let $\{\gamma_n\}$ denote a sequence of analytic Jordan arcs on R_0 such that $\gamma_n \cap \gamma_m = \emptyset$ for $n \neq m$, and such that for each compact subset K of R_0, the intersection $\gamma_n \cap K$ is empty for all sufficiently large n. Let $R_0' = R_0 - \bigcup_{n=1}^{\infty} \gamma_n$, and take the sequence $\{\gamma_n\}$ so that R_0' does not belong to the class SO_{HD}, in other words, so that there exists a nonnegative, Dirichlet-finite function on R_0 that is harmonic on R_0' and vanishes quasi-everywhere on $\bigcup_{n=1}^{\infty} \gamma_n$ but does not vanish quasi-everywhere on R_0. Let R_1' and R_2' be two copies of R_0. Denote by γ_n^+ (respectively, by γ_n^-) the positive (negative) edge of γ_n. For each n, identify γ_n^+ of R_1' with γ_n^- of R_2' and γ_n^- of R_1' with γ_n^+ of R_2'. The resulting Riemann surface R' has a single ideal boundary component. Furthermore, $R' \in O_{HD}^2 - O_{HD}^1$; that is, HD has dimension 2. Let $\{\delta_n\}$ denote a sequence of analytic Jordan arcs on R' such that $\delta_n \cap \delta_m = \emptyset$ for $n \neq m$, and such that for each compact subset K of R', the set $\delta_n \cap K$ is empty for all sufficiently large n. Let $R'' = R' - \bigcup_{n=1}^{\infty} \delta_n$, and take the sequence $\{\delta_n\}$ so that R'' belongs to the class SO_{HD}. Let R_1'' and R_2'' be two copies of R''. Again, denote by δ_n^+ (by δ_n^-) the positive (negative) edge of δ_n. For each n, identify δ_n^+ of R_1'' with δ_n^- of R_2'' and δ_n^- of R_1'' with δ_n^+ of R_2''. The resulting Riemann surface R still has a single ideal boundary component and belongs to $O_{HD}^2 - O_{HD}^1$. Since R has a single ideal boundary component, KD = HD, where KD denotes the space of HD-functions u such that $*du$ has vanishing periods along all dividing cycles. Hence R possesses nonconstant KD-functions. Furthermore, since both KD(R') and KD(R) have dimension 2, every $u \in KD(R)$ can be written as $u' \circ \sigma$, where $u' \in KD(R')$ and σ denotes the projection mapping of R onto R'. Since σ has critical points at the branch points of R, it

Received December 18, 1970.
This research was supported in part by NSF GP-8260 at Arizona State University.

Michigan Math. J. 18 (1971).

follows that every $u \in KD(R)$ has critical points at the branch points of R, of which there are infinitely many. The K-span is defined to be $\left.\frac{\partial p}{\partial x}\right|_{z=\zeta}$, where dp is the reproducing kernel for the space dKD; therefore, $(du, dp) = \pi \left.\frac{\partial u}{\partial x}\right|_{z=\zeta}$ for all $du \in dKD$, and z denotes a local variable at ζ. Hence the K-span vanishes at the branch points of R, for every choice of local variable; nevertheless, there exist nonconstant KD-functions on R.

REFERENCES

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2. L. Sario and K. Oikawa, *Capacity functions*. Springer-Verlag, New York, 1969.

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