ON EQUATIONS IN FREE GROUPS

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1. INTRODUCTION

It is well known that if elements $u_1, \cdots, u_n$ in a free group satisfy some nontrivial relation $w(u_1, \cdots, u_n) = 1$, then the rank of the free subgroup generated by $u_1, \cdots, u_n$ is at most $n - 1$. We are interested in conditions on w under which such a subgroup can in fact have rank $n - 1$. We obtain a necessary and sufficient condition (see Theorem 3), namely, that $w = w(x_1, \cdots, x_n)$ lie in the normal closure of some element from a free basis for the free group F freely generated by $x_1, \cdots, x_n$. Unfortunately, this is not entirely satisfactory, since no general method is known for deciding whether a word w meets the criterion. However, for special classes of w, we succeed in making this condition more explicit.

One special case of our problem has received some attention. If elements a, b, and c of a free group satisfy a relation $a^m b^n c^p = 1$, where $|m|, |n|, |p| \geq 2$, then the rank of the group generated by a, b, and c is at most 1. This was proved for $|m| = |n| = |p| = 2$ by R. C. Lyndon [6], for $|m| = |n| = |p| \geq 2$ by E. Schenkman [11], J. Stallings [13], and G. Baumslag [1], and for general $|m|, |n|, |p| \geq 2$ by M. P. Schützenberger [12] and by Schützenberger and Lyndon [7].

The last result is contained in the following theorem of Baumslag [2]. Suppose that $w = W(x_1, \cdots, x_n)$ is an element of the free group F freely generated by $x_1, \cdots, x_n$, that w is not a *primitive,* in other words, is not a member of a free basis of F, and that w is not a proper power, that is, $w \neq u^k$ if $k > 1$ and $u \in F$. If elements $y_1, \cdots, y_{n+1}$ satisfy the relation $W(y_1, \cdots, y_n) = y_{n+1}^m$ for some $m > 1$ and generate a free group, then the rank of this free group is at most $n - 1$.

We obtain a theorem that contains Baumslag's result:

THEOREM 1. *Let* $w = W(X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m))$ $(w \neq 1)$ *be an element of the free group* F *freely generated by* $x_1, \cdots, x_n, y_1, \cdots, y_m$. *Suppose that neither* X *nor* Y *is a proper power, and set* $W(X, 1) = X^h$ and $W(1, Y) = Y^k$. *In order that elements* $u_1, \cdots, u_n, v_1, \cdots, v_m$ *satisfying the relation*

$$W(X(u_1, \cdots, u_n), Y(v_1, \cdots, v_m)) = 1$$

generate a free group of rank $n + m - 1$, *it is necessary and sufficient that at least one of the following three conditions hold.*

(i) $X(x_1, \cdots, x_n)$ *and* $Y(y_1, \cdots, y_m)$ *are both primitive.*

(ii) $X(x_1, \cdots, x_n)$ *is primitive and* k *is a multiple of* h, *or* $Y(y_1, \cdots, y_m)$ *is primitive and* h *is a multiple of* k.

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(iii) *The normal closure* N *in* F *of some primitive in* F *contains either* $X = X(x_1, \cdots, x_n)$ *or* $Y = Y(y_1, \cdots, y_m)$. *If* N *contains* X, *then* $k = 0$; *if* N *contains* Y, *then* $h = 0$.

We obtain Theorem 1 as a consequence of the following more general result.

THEOREM 2. *Let* $w = W(X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m), \cdots, Z(z_1, \cdots, z_\ell))$ $(w \neq 1)$ *be an element of the free group* F *freely generated by* $x_1, \cdots, x_n, y_1, \cdots, y_m, \cdots, z_1, \cdots, z_\ell$. *Suppose that none of* X, Y, $\cdots$, Z *is a proper power in* F. *In order that elements* $u_1, \cdots, u_n, v_1, \cdots, v_m, \cdots, w_1, \cdots, w_\ell$ *satisfying the relation* $W(X(u_1, \cdots, u_n), Y(v_1, \cdots, v_m), \cdots, Z(w_1, \cdots, w_\ell)) = 1$ *generate a free group of rank* $n + m + \cdots + \ell - 1$, *it is necessary and sufficient that at least one of the following two conditions hold:*

(i) a) *Let* $\overline{F}$ *denote the free group freely generated by* x, y, $\cdots$, z. *Then there exists a primitive* $P(x, y, \cdots, z)$ *whose normal closure in* $\overline{F}$ *contains* $W(x, y, \cdots, z)$.

b) *There exists a partition of*

$$\{X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m), \cdots, Z(z_1, \cdots, z_\ell)\}$$

into a pair of mutually exclusive subsets,

$$\{X, Y, \cdots, Z\} = \{X_1, \cdots, X_t\} \cup \{X_{t+1}, \cdots, X_s\},$$

where $\{X_{t+1}, \cdots, X_s\}$ *is nonempty and is part of a free basis of* F.

c) *The set* $\{P_1, \cdots, P_t\}$ *is a set of primitives of* F, *where* P_i $(i = 1, 2, \cdots, t)$ *is obtained from* $P(X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m), \cdots, Z(z_1, \cdots, z_\ell))$ *by replacing* X_i *in* P *by* 1.

(ii) *One of the elements* $X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m), \cdots, Z(z_1, \cdots, z_\ell)$, *say* $X = X(x_1, \cdots, x_n)$, *is in the normal closure in* F *of some primitive* P, *and the normal closure of* X *in* F *contains* $W(X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m), \cdots, Z(z_1, \cdots, z_\ell))$.

Our method of proof, which is different from that used by others in earlier work on this problem, rests on the arguments by which W. Magnus [8] proved the *Freiheitssatz* and related results. Using these arguments, we obtain a statement (Theorem 4) that generalizes a theorem of A. Karrass, W. Magnus, and D. Solitar [4] on the elements of finite order in groups with a single defining relation. We shall use explicitly the Hauptform of the *Freiheitssatz* [8], Grushko's Theorem [5], and a result of Magnus [9], that if a group defined by n generators and a single relation $w = 1$ can be generated by $n - 1$ elements, then it is a free group.

2. DEFINITIONS AND PRELIMINARY RESULTS

If p and q are elements of a group F and q lies in the normal closure of p in F, we call p a *root* of q in F, and we write $p \to_F q$. If $F \subseteq \overline{F}$, then clearly $p \to_F q$ implies $p \to_{\overline{F}} q$, while the converse need not hold; however it is easy to see that the converse does hold if $\overline{F}$ is obtained by adjoining to F an element u such that u^t is a primitive of F for some $t \neq 0$.

A word $w = w(x_1, \cdots, x_n)$ $(w \neq 1)$ in a group F freely generated by $x_1, \cdots, x_n$ is called *simple* if there exists a free group of rank $n - 1$ generated by elements $u_1, \cdots, u_n$ that satisfy the relation $w(u_1, \cdots, u_n) = 1$. This is equivalent to the condition that the group $G = \langle x_1, \cdots, x_n; w(x_1, \cdots, x_n) = 1 \rangle$ (defined by generators

$x_1, \cdots, x_n$ and the single defining relation $w(x_1, \cdots, x_n) = 1$) has a quotient group that is free and of rank $n - 1$.

THEOREM 3. *An element* w *of a free group* F *is simple if and only if* w *has a primitive root.*

Proof. Let $w(x_1, \cdots, x_n)$ have a primitive root y_1, which must then be an element of a basis $y_1, \cdots, y_n$ for F. Suppose ϕ is the retract of F onto its subgroup generated by $y_2, \cdots, y_n$; then, since y_1 is in the kernel of ϕ, w is in the kernel of ϕ; therefore w is simple.

Conversely, let w be simple, and suppose $w\phi = 1$ for some homomorphism ϕ of F onto a free group H with basis $y_1, \cdots, y_{n-1}$. By Grushko's Theorem, F has a basis $x_1, \cdots, x_n$ such that

$$x_1\phi = y_1, \quad \cdots, \quad x_{n-1}\phi = y_{n-1}, \quad x_n\phi = 1.$$

Clearly, the kernel of ϕ is the normal closure of x_n. Since w is in the kernel of ϕ, it follows that x_n is a primitive root of w.

The application of this criterion would presumably require finding all roots of w and then deciding which of these roots, if any, are primitive. Using topological techniques, J. H. C. Whitehead [14] gave a solution of the second problem: to decide when an element u in F is primitive. Later, E. S. Rappaport [10], and more recently, J. P. Higgins and Lyndon [3], rederived Whitehead's solution, using algebraic methods. The problem of finding all roots of a word w appears to be very difficult, and we are unable to solve it. However, we do obtain conditions on the roots of w, under certain restrictions on w.

In looking for primitive roots p of an element r in a free group F, one must at times distinguish between the cases $r \in F'$ and $r \notin F'$. (F' denotes the derived group of F.) For example, we have the following result.

THEOREM 5. *Let* F *be a free group, and suppose* $p, q, r \in F$, $r \notin F'$. *If* p *is primitive,* $p \to r$, *and* $q \to r$, *then* $p \to q$.

COROLLARY. *Let* F *be free, and suppose* $p, q, r \in F$, $r \notin F'$. *If* p *and* q *are primitive,* $p \to r$, *and* $q \to r$, *then* p *is either conjugate to* q *or to* q^{-1}.

The corollary follows from the theorem and a result of Magnus [8] to the effect that if p and q are elements of a free group F, p is primitive, and $q \to p$, then p is conjugate either to q or to q^{-1}.

The hypothesis that $r \notin F'$ is necessary. Otherwise, if $r \in F'$ and F is already free of rank 2, every primitive p has the property that $p \to r$. Moreover, infinitely many primitives q in F are conjugate neither to p nor to p^{-1}. For each such q it follows that $q \to r$ and $p \not\to q$.

To prove Theorem 5, note that, since $q \to r$, we have the relation

$$r = \prod_i t_i^{-1} q^{\varepsilon_i} t_i$$

for some t_i in F and for some integers ε_i. Moreover, $\sum \varepsilon_i \neq 0$, since $r \notin F'$. Since p is primitive, we see that $F/N = F^*$, where F^* is free and N is the normal closure of p in F. From the relation $r = \prod_i t_i^{-1} q^{\varepsilon_i} t_i$ it follows that

$$rN = \prod_i (t_i N)^{-1} (qN)^{\varepsilon_i} (t_i N).$$

But $rN = 1$, since $p \to_F r$. We want to show that $p \to_F q$, in other words, that $qN = 1$.

Suppose $qN \neq 1$. Let the first term of the lower central series of F^* in which qN does not lie be the kth term, and denote it by F_k^*. If the image of qN in F^*/F_k^* is q^*, then $q^* \neq 1$, but q^* is in the center of F^*/F_k^*. Hence, by the relation

$$1 = \prod_i (t_i N)^{-1} (qN)^{\varepsilon_i} (t_i N),$$

we see that $1 = q^{*\sum \varepsilon_i}$. This is a contradiction, since $\sum \varepsilon_i \neq 0$ and F^*/F_k^* is a free nilpotent group without elements of finite order.

3. SUFFICIENCY

THEOREM 2S. *Let* $w = W(X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m), \cdots, Z(z_1, \cdots, z_\ell))$ $(w \neq 1)$ *be an element of the free group* F *freely generated by*

$$x_1, \cdots, x_n, y_1, \cdots, y_m, \cdots, z_1, \cdots, z_\ell .$$

Suppose none of the elements $X, Y, \cdots, Z$ *is a proper power in* F. *Let* $\overline{F}$ *be the free group on the free generators* $x, y, \cdots, z$. *Then* w *is simple if one of the following two conditions holds:*

(i) a) *There exists a primitive root* $P(x, y, \cdots, z)$ *of* $W(x, y, \cdots, z)$ *in* $\overline{F}$.

b) *There exists a partition of* $\{X, Y, \cdots, Z\}$ *into a pair of mutually exclusive subsets,* $\{X, Y, \cdots, Z\} = \{X_1, \cdots, X_t\} \cup \{X_{t+1}, \cdots, X_s\}$, *where* $\{X_{t+1}, \cdots, X_s\}$ *is not empty and is part of a free basis of* F. (It may be that $t = 0$ and $\{X_1, \cdots, X_t\}$ is empty.)

c) *The set* $\{P_1, P_2, \cdots, P_t\}$ *is a set of primitives of* F, *where* P_i $(i = 1, 2, \cdots, t)$ *is obtained from* $P(X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m), \cdots, Z(z_1, \cdots, z_\ell))$ *by replacing* X_i *in* P *by* 1. (If $t = 0$, this condition is vacuous.)

(ii) *There exists a primitive* p *such that* $p \to_F U$ *and* $U \to_F w$, *where* U *is one of the elements* $X, Y, \cdots, Z$.

LEMMA 1. *If condition* (i) *of Theorem* 2S *holds, then*

$$P(X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m), \cdots, Z(z_1, \cdots, z_\ell))$$

is primitive in F.

Proof. For the sake of definiteness, assume that the elements $X, Y, \cdots, Z$ have been ordered so that $X = X_1$, $Y = X_2$, $\cdots$, $Z = X_s$, and that $x, y, \cdots, z$ have been relabeled so that $u_1 = x$, $u_2 = y$, $\cdots$, $u_s = z$. If $t = 0$, the statement is obvious. Otherwise, we show successively that

$R_1 = P(X_1, u_2, \cdots, u_s)$ is primitive,

$R_2 = P(X_1, X_2, u_3, \cdots, u_s)$ is primitive, $\cdots$,

$R_i = P(X_1, X_2, \cdots, X_i, u_{i+1}, \cdots, u_{t+1}, \cdots, u_s)$ is primitive, $\cdots$,

$R_t = P(X_1, X_2, \cdots, X_t, u_{t+1}, \cdots, u_s)$ is primitive.

Observe that R_1 is primitive if and only if $G = \langle x_1, \cdots, x_n, u_2, \cdots, u_s ; R_1 = 1 \rangle$ is free. But G is a free product of a pair of free groups with an amalgamated subgroup; in particular, G has the presentation

$$\langle u_1, \cdots, u_s ; P(u_1, \cdots, u_s) = 1 \rangle \underset{u_1 = X_1}{*} \langle x_1, \cdots, x_n \rangle .$$

Clearly, G is free if u_1 is a primitive in the free group

$$\langle u_1, \cdots, u_s ; P(u_1, \cdots, u_s) = 1 \rangle ,$$

or, equivalently, if $\langle u_1, \cdots, u_s ; P(u_1, \cdots, u_s) = 1, u_1 = 1 \rangle$ is free and of rank $s - 2$. But

$$\langle u_1, u_2, \cdots, u_s; P(u_1, u_2, \cdots, u_s) = 1, u_1 = 1 \rangle = \langle u_2, \cdots, u_s; P(1, u_2, \cdots, u_s) = 1 \rangle$$

is free and of rank $s - 2$, since, by (i), $P_1 = P(1, Y, \cdots, Z)$ is primitive in F, and hence is primitive in the free subgroup generated by $Y, \cdots, Z$. Similarly, the fact that R_i is primitive implies that R_{i+1} is primitive; hence,

$$R_t = P(X_1, \cdots, X_t, u_{t+1}, \cdots, u_s)$$

is primitive. Since $\{X_{t+1}, \cdots, X_s\}$ is part of a basis of F, the conclusion now follows.

The proof of Theorem 2S is now clear. For if condition (i) holds, then Lemma 1 implies that $P(X, Y, \cdots, Z)$ is primitive in F. Moreover, by (i), $P(x, y, \cdots, z)$ is a root of $W(x, y, \cdots, z)$; therefore, $P(X, Y, \cdots, Z)$ is a root of $w = W(X, Y, \cdots, Z)$, and w is simple by Theorem 3. If (ii) holds, then X is a root of $w = W(X, Y, \cdots, Z)$ and P is a primitive root of X. This implies P is a primitive root of w, and again w is simple.

THEOREM 1S. *Let* $w = W(X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m))$ $(w \neq 1)$ *be an element of the free group* F *freely generated by* $x_1, \cdots, x_n, y_1, \cdots, y_m$. *Suppose that neither* $X = X(x_1, \cdots, x_n)$ *nor* $Y = Y(y_1, \cdots, y_m)$ *is a proper power in* F, *and let* $W(X, 1) = X^h$, $W(1, Y) = Y^k$. *Then* w *is simple if one of the following conditions holds:*

(i) X *and* Y *are both primitive;*

(ii) X *is primitive and* k *is a multiple of* h, *or* Y *is primitive and* h *is a multiple of* k;

(iii) X *has a primitive root and* $k = 0$, *or* Y *has a primitive root and* $h = 0$.

Proof. We apply Theorem 2S. Note first that condition (i) a), which asserts that there exists a primitive root $P(x, y)$ of $W(x, y)$, is always satisfied. Moreover, if $P(x, 1) = x^a$ and $P(1, y) = y^b$, then $(a, b) = 1$, and $P(x, y)$ can be chosen so that $ka = hb$. (If $W(x, y)$ is not an element of the commutator subgroup $\overline{F}'$ of the free group on x and y, then $|a|$ and $|b|$ are fixed by the requirement that $P(x, y)$ be a primitive root of $W(x, y)$. If $W(x, y) \in \overline{F}'$, then each primitive in $\overline{F}$ is a root of $W(x, y)$.) It now follows that the conditions of Theorem 1S imply the conditions of Theorem 2S; in each case, w is simple.

4. ON FINDING ROOTS

LEMMA 2. *Let* F *be the free group on the free generators* $x_1, \cdots, x_n$, *let* H_1 *be the subgroup with basis* $x_1^{-t} x_i x_1^t$ for $2 \leq i \leq n$ *and all integers* t, *and let* H_2 *be the subgroup with basis* x_1^k *(for some fixed* $k > 0$*) together with* $x_1^{-t} x_i x_1^t$ $(2 \leq i \leq n,\ t = 0, 1, \cdots, k-1)$. (It is well known that these are bases, that H_1 is the normal subgroup of F generated by $x_2, \cdots, x_n$, and that H_2 is the normal subgroup of F generated by $x_1^k, x_2, \cdots, x_n$.) *Suppose* $q \in H_j$, $r \in H_j$ $(j = 1$ *or* $2)$, *and* $q \to_F r$, *where* q *is cyclically reduced. If* r *contains exactly one conjugate of* x_2, *say* $x_1^{-u} x_2 x_1^u$, *and* q *contains* x_2, *then* q *has some conjugate* $q' = x_1^{-a} q x_1^a$ *such that* $q' \to_{H_j} r$.

Proof. Since q is a cyclically reduced root of r in F, we have the relation $r = \prod_i T_i^{-1} q^{\varepsilon_i} T_i$ for some elements T_i in F and certain integers ε_i. Hence,

$$r = \prod_i (x_1^{-a_i} T_i)^{-1} (x_1^{-a_i} q x_1^{a_i})^{\varepsilon_i} (x_1^{-a_i} T_i), \tag{*}$$

where, if $H_j = H_1$, a_i is the exponent sum on x_1 in T_i, and if $H_j = H_2$, a_i is the exponent sum on x_1 in T_i, reduced modulo k, so that $0 \leq a_i \leq k-1$. In each case, r is in the normal subgroup of H_j generated by the conjugates $x_1^{-a_i} q x_1^{a_i}$. But r contains exactly one conjugate of x_2. It follows, by the *Hauptform* of the *Freiheitssatz* [8], that if q contains x_2, then exactly one $a_i = a$ enters into the expression (*) above, and $x_1^{-a} q x_1^a \to_{H_j} r$.

THEOREM 4. *Let* $w = W(X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m), \cdots, Z(z_1, \cdots, z_\ell))$ $(w \neq 1)$ *be an element of a free group* F *freely generated by*

$$x_1, \cdots, x_n, y_1, \cdots, y_m, \cdots, z_1, \cdots, z_\ell.$$

Suppose none of the elements

$$X = X(x_1, \cdots, x_n), \quad Y = Y(y_1, \cdots, y_m), \quad \cdots, \quad Z = Z(z_1, \cdots, z_\ell)$$

is a proper power in F, *and let* q *denote a root of* w *in* F. *Then either*

(i) $q \to_F U$ *and* $U \to_F w$, *where* U *is one of the elements* $X, Y, \cdots, Z$, *or*

(ii) q *is conjugate to some element* $p = P(X, Y, \cdots, Z)$, *where* $P(X, Y, \cdots, Z)$ *is a word in* $X, Y, \cdots, Z$.

Moreover, if $\overline{F}$ *is the free group on the free generators* $x, y, \cdots, z$, *then* $P(x, y, \cdots, z) \to_{\overline{F}} W(x, y, \cdots, z)$.

Proof. Note first that if only one set of generators, say $x_1, \cdots, x_n$, appears in W, then $W = X^r$ for some integer r. In this case, the theorem asserts that each root q of X^r is either a root of X or is conjugate to X^t for some integer t. Equivalently, if the group $\langle x_1, \cdots, x_n; U(x_1, \cdots, x_n) = 1 \rangle$ has an element X $(X \neq 1)$ of finite order dividing r, then $U(x_1, \cdots, x_n)$ is conjugate to X^t. (This is a theorem of Karrass, Magnus, and Solitar [4].)

We may therefore assume that at least two sets of generators, say $x_1, \cdots, x_n$ and $y_1, \cdots, y_m$, appear in W. Let $q = Q(x_1, \cdots, z_\ell)$ be a cyclically reduced root of w in F. The proof proceeds by induction on the sum σ of the lengths of X, Y, $\cdots$, Z. The initial case is trivial. For the induction, we can assume that the length of X, say, exceeds 1, and, since X is not a proper power, that X contains at least two generators, say x_1 and x_2.

Suppose first that neither x_1 nor x_2 has exponent sum 0 in X, and write these two exponent sums as $d \cdot s$ and $d \cdot t$, where $(s, t) = 1$. Embed F in a free group F^* by adjoining to F an element $\bar{x}_1$ such that $\bar{x}_1^t = x_1$. Then F^* has a basis $\bar{x}_1$, $\bar{x}_2 = x_2 \bar{x}_1^s$, $\bar{x}_3 = x_3$, $\cdots$, $\bar{z}_\ell = z_\ell$.

Let $\overline{X}$ and $\overline{Q}$ denote the revisions of X and Q, respectively, written as words relative to this basis. The exponent sums on $\bar{x}_1$ in $\overline{X}$ and in $\overline{Q}$ are 0 and k, say. If $k = 0$, let F^j denote the subgroup of F^* with basis $\bar{x}_1^{-i} \bar{x}_2 \bar{x}_1^i$, $\cdots$, $\bar{x}_1^{-i} \bar{z}_\ell \bar{x}_1^i$, for all integers i; if $k \neq 0$, let F^j denote the subgroup of F^* whose basis is $\bar{x}_1^{|k|}$ together with $\bar{x}_1^{-i} \bar{x}_2 \bar{x}_1^i$, $\cdots$, $\bar{x}_1^{-i} \bar{z}_\ell \bar{x}_1^i$ $(i = 0, 1, \cdots, |k| - 1)$. Then $\overline{X}$ and $\overline{Q}$ are in F^j, and, written as words X^j and Q^j relative to the basis above, X^j has shorter length than X. Since the corresponding words Y^j, $\cdots$, Z^j have not changed, the corresponding sum σ^j is smaller than σ. Moreover, if X, Y, $\cdots$, Z are not proper powers, then X^j, Y^j, $\cdots$, Z^j are not proper powers. Also, the relation $q \rightarrow_F w$ implies that $q \rightarrow_{F^*} w$.

If x_1 or x_2, say x_1, has exponent sum 0 in X, we proceed similarly, taking $\bar{x}_1 = x_1$, $\bar{x}_2 = x_2$, $\cdots$, $\bar{z}_\ell = z_\ell$. In either case, the induction hypothesis applies to q and w as elements of F^j.

(i) Suppose now that Q is a word in $x_1, \cdots, x_n$ alone. Then, if we set $x_i = 1$ for all i, the relation $Q \rightarrow W(X, Y, \cdots, Z)$ implies that $1 \rightarrow W(1, Y, \cdots, Z)$; hence, $W(1, Y, \cdots, Z) = 1$ and $X \rightarrow W$. Also, if N is the normal closure of Q in F, then

$$F/N \simeq \langle x_1, \cdots, x_n; Q = 1 \rangle * \langle y_1, \cdots, z_\ell \rangle .$$

Now, the fact that $WN = W(XN, YN, \cdots, ZN) = 1$ in this free product implies $XN = 1$; hence $Q \rightarrow X$, and condition (i) is fulfilled.

(ii) In the remaining case, q contains y_1, say. If W^j is the revision of W in F^j, then W^j contains $\bar{x}_1^{-i} y_1 \bar{x}_1^i$ only for $i = 0$. Therefore, Lemma 2 is applicable, and q is conjugate in F^j to some element q' such that $q' \rightarrow_{F^j} W^j$. By the induction hypothesis (we have disposed of case (i)), we conclude that q' is conjugate in F^j to some element $p = P(X^j, Y^j, \cdots, Z^j)$; hence, q and $p = P(X, Y, \cdots, Z)$ are conjugate in F^j and thus are conjugate in F^*. Since p and q lie in F, it follows by an earlier remark that p and q are conjugate in F. Moreover,

$$P(x, y, \cdots, z) \rightarrow_F W(x, y, \cdots, z) .$$

5. NECESSITY

THEOREM 2N. *Let* $w = W(X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m), \cdots, Z(z_1, \cdots, z_\ell))$ $(w \neq 1)$ *be an element of the free group* F *freely generated by* $x_1, \cdots, x_n$, $y_1, \cdots, y_m, \cdots, z_1, \cdots, z_\ell$. *Suppose that none of the elements* X, Y, $\cdots$, Z *is a* *proper power in* F. *Let* $\overline{F}$ *be the free group on the free generators* $x, y, \cdots, z$. *If* w *is simple, then one of the following two conditions holds:*

(i) a) *There exists a primitive root* $P(x, y, \cdots, z)$ *of* $W(x, y, \cdots, z)$ *in* $\overline{F}$.

b) *There is a partition of* $\{X, Y, \cdots, Z\}$ *into a pair of mutually exclusive subsets*, $\{X, Y, \cdots, Z\} = \{X_1, X_2, \cdots, X_t\} \cup \{X_{t+1}, \cdots, X_s\}$, *where* $\{X_{t+1}, \cdots, X_s\}$ *is not empty and is part of a free basis of* F. (It may be that $t = 0$ and $\{X_1, \cdots, X_t\}$ is empty.)

c) *The set* $\{P_1, P_2, \cdots, P_t\}$ *is a set of primitives of* F, *where* P_i $(i = 1, 2, \cdots, t)$ *is obtained from* $P(X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m), \cdots, Z(z_1, \cdots, z_\ell))$ *by replacing* X_i *in* P *by* 1. (If $t = 0$, this condition is vacuous.)

(ii) *There exists a primitive* p *such that* $p \rightarrow_F U$ *and* $U \rightarrow_F W$, *where* U *is one of the elements* $X, Y, \cdots, Z$.

Proof. If w is simple, it has a primitive root p, by Theorem 3. By Theorem 4, either condition (ii) above holds, or we may assume that $p = P(X, Y, \cdots, Z)$ is a word in $X, Y, \cdots, Z$. Also, $P(x, y, \cdots, z)$ is a root of $W(x, y, \cdots, z)$ in $\overline{F}$, and since $P(X, Y, \cdots, Z)$ is primitive, it follows that $P(x, y, \cdots, z)$ is primitive in $\overline{F}$, and (i) a) holds. Let $\{X_{t+1}, \cdots, X_s\}$ be the maximal subset of $\{X, Y, \cdots, Z\}$ consisting of primitives. We show that $\{X_{t+1}, \cdots, X_s\}$ is not empty; clearly, it is part of a free basis of F.

Let N be the normal closure of p in F. Then F/N, free of rank $n + m + \cdots + \ell - 1$, has as a homomorphic image the free product

$$\langle x_1, \cdots, x_n; X = 1\rangle * \langle y_1, \cdots, y_m, \cdots, z_1, \cdots, z_\ell; P(1, Y, \cdots, Z) = 1\rangle .$$

By Grushko's Theorem [5], F/N has a basis consisting of $n + m + \cdots + \ell - 1$ elements whose images are wholly contained in (and therefore generate) each free factor. Therefore, either the first factor is generated by fewer than n generators, or the second factor is generated by fewer than $m + \cdots + \ell$ generators. The theorem of Magnus [9] then implies that one of the two factors is free and either X or $P(1, Y, \cdots, Z)$ is primitive. If $P(1, Y, \cdots, Z)$ is primitive, then repetitions of the argument above eventually show that at least one of the elements $X, Y, \cdots, Z$ is primitive; hence $\{X_{t+1}, \cdots, X_s\}$ is not empty. Moreover, if X_i is not primitive, then P_i is primitive.

THEOREM 1N. *Let* $w = W(X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m))$ $(w \neq 1)$ *be an element of the free group* F *freely generated by* $x_1, \cdots, x_n, y_1, \cdots, y_m$. *Suppose neither* X *nor* Y *is a proper power in* F, *and let* $W(X, 1) = X^h$, $W(1, Y) = Y^k$. *If* w *is simple, then one of the following three conditions holds:*

(i) X *and* Y *are both primitive;*

(ii) X *is primitive and* k *is a multiple of* h, *or* Y *is primitive and* h *is a multiple of* k;

(iii) X *has a primitive root and* $k = 0$, *or* Y *has a primitive root and* $h = 0$.

Proof. Theorem 1N follows immediately from Theorem 2N, for condition (ii) of Theorem 2N is equivalent to condition (iii) above. Otherwise, conditions (i) a), b), c) hold and (i) or (ii) above follow, since the fact that $W(X, 1) = X^h$ is primitive implies that $|h| = 1$ and k is a multiple of h. Similarly, the fact that $W(1, Y) = Y^k$ is primitive implies that h is a multiple of k.

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