LOCALLY COMPACT LATTICES WITH SMALL LATTICES

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In [4], L. Anderson asked whether each locally compact, connected topological lattice has a base consisting of open sublattices. We shall show that this question has a negative answer even in a compact, connected, metrizable distributive lattice. However, we shall see that if a lattice has finite dimension (either codimension of H. Cohen [9] or inductive dimension of Urysohn and Menger), then it has such a base. The following natural question arises: What is a necessary and sufficient condition for a lattice to have such a base? In the first section, we shall answer this question. We shall then prove that no locally compact, connected, complemented lattice has a base consisting of open sublattices. This implies that each locally compact, relatively complemented lattice that is either finite-dimensional or has a base of open sublattices is totally disconnected. J. Lawson [11] studied the parallel problem for a semilattice. He proved that locally compact, locally connected, finite-dimensional semilattices have small semilattices.

The following theorem was conjectured by A. D. Wallace [14], and it was proved in [2] and [7]: If L is a compact, connected lattice of codimension at most n, then the number of elements in its center, denoted by Card(Cen(L)), is at most 2^n. In the second section, we shall see that this theorem also holds in a locally compact, connected lattice with 0 and 1. Furthermore, if the lattice is not compact, then Card(Cen(L)) $\leq 2^{n-1}$.

For a pair of subsets A and B of a topological lattice L, we use $A \wedge B$ and $A \vee B$ to denote the sets

$$\{a \wedge b \mid a \in A \text{ and } b \in B\} \qquad \text{and} \qquad \{a \vee b \mid a \in A \text{ and } b \in B\},$$

respectively. For a subset A of L, we let A^*, A°, and $F(A) = A^* \setminus A^\circ$ denote the closure, the interior, and the boundary of A, respectively. All other terms and definitions used in this paper are the same as in [3] and [7]. It is known ([1], [3], and [5]) that every locally compact, connected lattice is chain-wise connected, locally convex, and locally connected.

1. LATTICES WITH SMALL LATTICES

A topological lattice that has a base consisting of open sublattices is called a lattice with small lattices. Recently, J. Lawson [12] gave an example of a compact, connected, metrizable, distributive lattice L that admits no nontrivial lattice-homomorphism into the unit interval I with the usual order, that is, every lattice-homomorphism of L into I is a constant mapping. We show that this lattice has no base consisting of open sublattices. Suppose that the lattice L has such a base. Then, by [13, Theorem 5], the topology of L must be the interval topology of L. By [13, Theorem 6], L admits enough lattice-homomorphisms to separate points of L.

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It is known that every finite-dimensional, compact, connected, distributive lattice admits enough lattice-homomorphisms to separate points. Therefore, the lattice in the example is infinite-dimensional.

For a finite-dimensional lattice, we have the following result.

THEOREM 1. *If* L *is a finite-codimensional, locally compact, connected lattice, then* L *has small lattices.*

Proof. It is known that the breadth is at most equal to the codimension in every locally compact, connected lattice. For a neighborhood U of a point x in L, there exist two elements y and z in L and a neighborhood V of x such that $V \subset [y, z] \subset U$ [8, Theorem 1]. Since L is locally compact, we may assume that $M = [y, z]$ is compact. By [8, Corollary 1], the relative topology M in L must be the interval topology of M. Again by [13, Theorem 5], M has small lattices. Hence there exists an open sublattice W in M such that $x \in W \subset V$. Clearly, W is an open sublattice in L. The proof is now complete.

Finite dimensionality is clearly not a necessary condition for L to have small lattices. Therefore it may not be easy to obtain a necessary and sufficient condition in terms of dimension for L to have small lattices.

E. B. Davies [10] has given several necessary and sufficient conditions for a compact lattice to have small lattices.

In the following theorem, we give analogous conditions for locally compact, connected lattices.

THEOREM 2. *Let* L *be a locally compact, connected lattice. Then the following conditions are equivalent.*

(i) L *has small lattices.*

(ii) *If* $x \not\geq y$ *in* L, *then there exists an element* z *in* L *such that* $x \in (z \wedge L)^{\circ}$ *and* $z \not\geq y$, *and dually.*

(iii) L *has a base consisting of closed intervals of* L.

Proof. (i) → (ii). Let $x \not\geq y$, and let U be a neighborhood of x such that $u \not\geq y$ for all $u \in U$. Choose a neighborhood V of x such that V is an open sublattice of L, V^* is compact (and hence a sublattice), and $V^* \subset U$. Let z denote the maximal element of V^*. Then $x \in (z \wedge L)^{\circ}$ and $z \not\geq y$, and the dual statement also holds.

(ii) → (iii). Let W be a neighborhood of x. Choose open neighborhoods U_1 and U_2 of x such that U_1 is convex, U_2^* is compact, and $U_1 \subset U_2 \subset U_2^* \subset W$. Let

$$A = U_2^* \cap (L \setminus (L \wedge U_1)).$$

Then A is a compact subset of L. Let P be the set of all $y \in L$ such that $x \in (y \wedge L)^{\circ}$. By (ii), P is clearly not empty. We show first that there exists $b \in P$ such that $(b \wedge L) \cap A$ is empty. Suppose that $(y \wedge L) \cap A$ is nonempty for each $y \in P$. Then it is easy to see that the family of sets of the form $(y \wedge L) \cap A$, where $y \in P$, has the finite-intersection property. Since A is compact, there exists an element

$$u \in \bigcap_{y \in P} (y \wedge L) \cap A,$$

and $y \geq u$ for all $y \in P$. On the other hand, we see that $x \not\geq u$, because $A \cap (x \wedge L) = \square$ and $u \in A$. By (ii), it follows that there exists $z \in L$ such that

$x \in (z \wedge L)^{\circ}$ and $z \not\geq u$, and hence $z \in P$. This is a contradiction. Now we show that $b \in U_1$. Suppose that $b \notin U_1$. Then either $b \in A$ or $b \in L \setminus U_2^*$, because $b \notin L \wedge U_1$. The first case is clearly impossible. Thus $b \in L \setminus U_2^*$. Let C be a connected chain from x to b. Clearly, $C \cap F(U_2) \neq \square$. Every element p of $C \cap F(U_2)$ belongs to $(b \wedge L) \cap U_2^*$. But $p \notin L \wedge U_1$, because $x \leq p$, $U_1 \subset U_2$, and U_1 is convex. It follows that $C \cap F(U_2) \subset (b \wedge L) \cap A$. Hence the other case is also impossible. Dually, there exists an element a in U_1 such that $x \in (a \vee L)^{\circ}$. Hence $x \in (a \vee L)^{\circ} \cap (b \wedge L)^{\circ} = [a, b]^{\circ} \subset [a, b] \subset U_1 \subset W$.

(iii) → (i). Let W be a neighborhood of x. Choose a closed interval $M = [a, b]$ and an open subset V in L such that $x \in V \subset M \subset W$ (note that M is compact). The relative topology of M in L has a base of closed intervals, since L has such a base. Thus, by a result of Davies [10], M has a small lattice. Therefore, V contains a sublattice of L that is open in L.

LEMMA 1. *No nondegenerate, locally compact, connected, complemented lattice has small lattices.*

Proof. Suppose that such a lattice L has a small lattice. Take an open sublattice U containing the zero 0 of L, such that U^* is compact and $1 \notin U$, where 1 is the unit of L. Now choose an open convex sublattice V (such a V always exists if L has a small lattice and is locally convex) such that $0 \in V \subset V^* \subset U$. Then $V^* \subset b \wedge L$ for the maximal element b of V^*. Let z be a complement of b. Then $z \not\leq b$, because $b \neq 1$. By [1, Lemma 6], we have the inclusion

$$b \wedge [L \setminus (b \wedge L)] \subset F(b \wedge L).$$

Hence $0 \in F(b \wedge L)$. This is a contradiction.

The following corollary follows immediately from Theorem 1.

COROLLARY 1. *Every nondegenerate, locally compact, connected, complemented lattice has infinite codimension.*

THEOREM 3. *Every locally compact, relatively complemented lattice that is either finite-dimensional or has small lattices is totally disconnected.*

Proof. Suppose that such a lattice L is not totally disconnected. Then there exists some x in L whose connected component C contains an element other than x. Therefore C is a nondegenerate, locally compact, connected sublattice under its relative topology. Since L has finite dimension or small lattices, the lattice C itself has small lattices. By Theorem 2, there exists a closed interval [a, b] of C that constitutes a neighborhood of x in C and is therefore nondegenerate. Since C is convex in L, [a, b] is also a closed interval in L, which is complemented. By Lemma 1, this is a contradiction, because [a, b] itself has small lattices.

COROLLARY 2. *Every locally compact, orthomodular lattice that has finite codimension or small lattices is totally disconnected.*

2. DIMENSION AND CENTERS

An element c of a lattice L is said to be *neutral* if each triple c, x, y of elements in L generates a distributive sublattice. An element of a lattice with 0 and 1 is called a *center element* if it is a neutral element and is complemented. It is well known that the set of all center elements in a lattice with 0 and 1 forms a Boolean algebra with the same 0 and 1.

THEOREM 4. *Let* L *be a locally compact, connected lattice with* 0 *and* 1. *If the codimension of* L *is* n, *then* $\mathrm{Card}(\mathrm{Cen}(L)) \leq 2^n$.

Proof. Let C be the set of all center elements of L. We show first that Card(Cen(L)) must be finite. Suppose that it is infinite. It is known [6, Theorem 4, p. 59] that a Boolean algebra of finite length is finite. Therefore we can choose a chain $0 = c_0 < c_1 < c_2 < \cdots < c_{n+1} = 1$ of $(n+2)$ elements in C. Let $x_{k-1} = c_{k-1} \vee c_k'$, where c_k' is the complement of c_k in C $(k = 1, 2, \cdots, n+1)$. Clearly, $x_k \vee x_j = 1$ $(k \neq j)$. Furthermore, the meet $x_0 \wedge \cdots \wedge x_n$ is not a meet of a subset of n of the x_i, because if $x_0 \wedge \cdots \wedge x_n = x_0 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_n$, then by distributivity $x_i = 1$, from which it follows that $c_{i-1} = c_i$. This implies that C has breadth greater than n. It is known [2] that in such a lattice the breadth is less than or equal to the codimension. This is a contradiction. Hence C is a finite Boolean algebra.

Let Card (C) $= 2^m$, where m is a positive integer. Suppose that $m > n$. Let the atoms of C be $a_1, \cdots, a_m$. Consider a compact, connected chain X_i in L from 0 to a_i $(i = 1, 2, \cdots, m)$. Let f be the mapping from $X_1 \times \cdots \times X_m$ into L defined by the relation

$$f(y_1, \cdots, y_m) = y_1 \vee y_2 \vee \cdots \vee y_m.$$

Let g be the mapping from $f(X_1 \times \cdots \times X_m)$ into $X_1 \times \cdots \times X_m$ defined by the relation $g(x) = (x \wedge a_1, x \wedge a_2, \cdots, x \wedge a_m)$. Then f and g are both continuous, and furthermore, $g = f^{-1}$, because $a_1, \cdots, a_m$ are neutral elements in L. Since X_i is nondegenerate, the codimension of $X_1 \times \cdots \times X_m$ is m [7]. This is a contradiction. Hence the proof is now complete.

LEMMA 2. *Let* L *be a locally compact, connected lattice with* 0 *and* 1, *and let* Ca(L) *denote the number of all the atoms of the center of* L. *Then* L *is iseomorphic with a cartesian product of* n *nondegenerate, compact, connected chains if and only if* $\mathrm{Ca}(L) = \mathrm{cd}(L) = n$, *where* cd(L) *denotes the codimension of* L.

Proof. Suppose L is iseomorphic with a cartesian product $J_1 \times \cdots \times J_n$ of n nondegenerate, compact, connected chains. Let 0_i and 1_i be the zero and the unit of J_i $(i = 1, 2, \cdots, n)$, respectively. Clearly,

$$(1_1, 0_2, \cdots, 0_n), \quad (0_1, 1_2, 0_3, \cdots, 0_n), \quad \cdots, \quad (0_1, 0_2, \cdots, 1_n)$$

are all atoms of the center of $J_1 \times \cdots \times J_n$. Conversely, let $c_1, \cdots, c_n$ denote all the atoms of the center of L. Consider the mapping

$$f: L \to (c_1 \wedge L) \times \cdots \times (c_n \wedge L) \qquad \text{defined by} \qquad f(x) = (c_1 \wedge x, \cdots, c_n \wedge x)$$

and the mapping

$$g: (c_1 \wedge L) \times \cdots \times (c_n \wedge L) \to L \qquad \text{defined by} \qquad g(x_1, \cdots, x_n) = x_1 \vee \cdots \vee x_n.$$

Then f and g are both continuous, and furthermore $g = f^{-1}$. By [7, Lemma 2.7], $\mathrm{cd}(c_i \wedge L) = 1$ $(i = 1, 2, \cdots, n)$. Therefore $c_i \wedge L$ is a locally compact, connected chain, and hence it is compact.

COROLLARY 2. *If* L *satisfies the conditions in Theorem* 4 *and is not compact, then* $\mathrm{Card}(\mathrm{Cen}(L)) \leq 2^{n-1}$.

The following example shows that in some respect the result above is the best possible.

Example. In Euclidean 3-space, let

$$L = \{(x, y, z) \mid 0 < x, y, z < 1\} \cup \{(0, 0, z) \mid 0 \leq z \leq 1\} \cup \{(1, 1, z) \mid 0 \leq z \leq 1\} .$$

Then L is a locally compact, connected lattice with 0 and 1 under the order of cardinal product and the usual topology of Euclidean 3-space, and it is not compact. The center of L is $\{(0, 0, 0), (0, 0, 1), (1, 1, 0), (1, 1, 1)\}$.

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