QUASI-p-REGULARITY OF SYMMETRIC SPACES

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INTRODUCTION

If X and Y are CW-complexes, we say that Y is p-*equivalent* to X (notation: $X \underset{p}{\simeq} Y$) if there exists a map $f: X \to Y$ such that

$$f^*: H^*(Y; Z_p) \cong H^*(X; Z_p).$$

Following [10], we say that X is p-*regular* if it is p-equivalent to a product of spheres. We call X *quasi*-p-*regular* if X is p-equivalent to a product of spheres and spaces $B_n(p)$ satisfying the condition

$$H^*(B_n(p); Z_p) \cong \Lambda(x_{2n+1}, \mathfrak{P}^1 x_{2n+1}).$$

In [7], P. G. Kumpel discussed the p-*regularity* of irreducible symmetric spaces. The purpose of this paper is to extend the study to the *quasi*-p-*regularity* of irredicible symmetric spaces.

Let G be a compact, connected, simply connected Lie group with an involution $\sigma: G \to G$. Let K be the identity component of the fixed-point set of σ, and assume that K is totally nonhomologous to zero in G with real coefficients. Then the irreducible symmetric spaces G/K satisfying the hypotheses above are

(i) $(K \times K)/K$,

(ii) $SU(2n+1)/SO(2n+1)$,

(iii) $SU(2n)/Sp(n)$,

(iv) $Spin(2n)/Spin(2n-1)$,

(v) E_6/F_4.

As is well known, $(K \times K)/K$ is isomorphic to K. The quasi-p-regularity of the Lie groups was discussed in [8]. Since

$$Spin(2n)/Spin(2n-1) = S^{2n-1},$$

the space (iv) is quasi-p-regular. Therefore it is sufficient to study the quasi-p-regularity of (ii), (iii), and (v). Our results (Theorems 4.2, 4.3, and 4.4) are as follows.

$SU(2n)/Sp(n)$ *is quasi*-p-*regular if and only if* $p \geq n$.

$SU(2n+1)/SO(2n+1)$ *is quasi*-p-*regular if and only if* $p \geq n+1$.

E_6/F_4 *is quasi*-p-*regular if and only if* $p \geq 5$.

Corollary 4.5 answers negatively a question of Kumpel [7].

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In the first section we recall the cohomology groups of G/K and study the $\mathfrak{P}^1$-operation on them. In Section 2 we give further information, due to B. Harris [4], on $H^*(G/K; Z_p)$. The space $B_n(p)$ is constructed in Section 3. In Section 4, we prove our main results.

Throughout this paper, p denotes an odd prime (unless it is otherwise stated), ${}^p\pi_i(X)$ stands for the p-primary component of $\pi_i(X)$, and $K^{(n)}$ is the n-skeleton of a complex K.

1. THE COHOMOLOGY OF G/K AND THE $\mathfrak{P}^1$-OPERATION

It is known that

(1.1) $H^*(SU(2n+1)/SO(2n+1); Z_p) \cong \Lambda(x_5, x_9, \cdots, x_{4n+1})$,

(1.2) $H^*(SU(2n)/Sp(n); Z_p) \cong \Lambda(x_5, x_9, \cdots, x_{4n-3})$,

(1.3) $H^*(E_6/F_4; Z_p) \cong \Lambda(x_9, x_{17})$.

PROPOSITION 1.1. *In* (1.1) *and* (1.2), $\mathfrak{P}^1 x_{4i+1} = 0$ *if and only if* $p \mid i$.

For the proof, see [7], for example.

PROPOSITION 1.2. *In* (1.3),

$$\mathfrak{P}^1_p x_9 = \begin{cases} x_{17} & \text{if } p = 5, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Let $q: E_6 \to E_6/F_4$ be the projection. Let y_9 and y_{17} be the elements of $H^*(E_6; Z_p)$ such that

$$q^* x_i = y_i \quad (i = 9, 17).$$

By Theorem 1.1 of [8], $\mathfrak{P}^1_5 y_9 = y_{17}$. Therefore

$$x_{17} = q^* y_{17} = q^* \mathfrak{P}^1_5 y_9 = \mathfrak{P}^1_5 q^* y_9 = \mathfrak{P}^1 x_9.$$

For dimensional reasons, $\mathfrak{P}^1_p x_9 = 0$ if $p \neq 5$.

2. THE MULTIPLICATION IN G/K

The material in this section is due to Harris [4].

Let G and K have the same meaning as in the Introduction. Let $i: K \to G$ and $\ell: G \to G/K$ be the inclusion and the projection. Let $q: G/K \to G$ be the map defined by

$$q(gK) = g \cdot \sigma(g)^{-1}.$$

PROPOSITION 2.1 (Harris). *Let* U *be the subalgebra generated by the elements* x_i *left fixed by* σ^*, *and let* V *be the subalgebra generated by the other elements* x_i. *Then*

(a) $H^*(G; Z_p) \cong U \otimes V$ *as algebras;*

(b) i^* *maps* U *isomorphically onto* $H^*(K; Z_p)$ *and is zero on the positive-dimensional elements of* V;

(c) q^* *maps* V *isomorphically onto* $H^*(G/K; Z_p)$ *and is zero on the positive-dimensional elements of* U;

(d) $\ell^* q^*: H^*(G; Z_p) \to H^*(G; Z_p)$ *is given by the formula* $\ell^* q^*(x) = x - \sigma^*(x)$ *if* x *is primitive;*

(e) $H^*(G/K; Z_p)$ *has generators* $y_1, \cdots, y_t$ *such that* $\sigma^*(y_i) = -y_i$ *and* $q^* \ell^*(y_i) = 2y_i$ $(i = 1, \cdots, t)$.

We define a mapping $w: G/K \times G/K \to G/K$ by the equation

$$w(g_1 K, g_2 K) = g_1 \cdot \sigma(g_1)^{-1} \cdot g_2 K.$$

Then $w(eK, gK) = gK$, $w(gK, eK) = \ell q(gK)$, and $w(gK, \sigma(gK)) = gK$. The product map w induces the mapping

$$w^*: H^*(G/K; Z_p) \to H^*(G/K; Z_p) \otimes H^*(G/K; Z_p)$$

given by the equation

$$w^*(x) = q^* \ell^*(x) \otimes 1 + \sum a_i \otimes b_i + 1 \otimes x.$$

In particular, if y_i is a generator (as in (e) of Proposition 2.1), then

$$w^*(y_i) = 2y_i \otimes 1 + 1 \otimes y_i + d_i, \tag{2.1}$$

where d_i involves only the generators $y_1, \cdots, y_{i-1}$. Now we define the map

$$\phi_j: \underbrace{G/K \times \cdots \times G/K}_{j} \xrightarrow{w \times 1} G/K \times \cdots \times G/K \longrightarrow \cdots \longrightarrow G/K \times G/K \xrightarrow{w} G/K.$$

It follows from (2.1) that the induced homomorphism

$$\phi_j^*: H^*(G/K; Z_p) \to H^*(G/K; Z_p) \otimes \cdots \otimes H^*(G/K; Z_p)$$

is given by

$$\begin{aligned} \phi_j^*(y_i) = {} & 2^{j-1} y_i \otimes 1 \otimes \cdots \otimes 1 + 2^{j-2} \cdot 1 \otimes y_i \otimes 1 \otimes \cdots \otimes 1 + \cdots \\ & + 1 \otimes \cdots \otimes 1 \otimes y_i + D_i, \end{aligned} \tag{2.2}$$

where D_i involves only $y_1, \cdots, y_{i-1}$.

3. THE CONSTRUCTION OF $B_n(p)$

Denote by $V_{2n+3,2}$ the Stiefel manifold $SO(2n+3)/SO(2n+1)$, which is an S^{2n+1}-bundle over S^{2n+2}. Then $B_n(p)$ is the bundle induced by the element $\frac{1}{2}\alpha_1(2n+2)$ from $V_{2n+3,2}$, where $\alpha_1(2n+2)$ is a generator of the p-component of $\pi_{2n+1+2(p-1)}(S^{2n+2})$, and we have the diagram

$$\begin{array}{ccc} S^{2n+1} & \xrightarrow{1} & S^{2n+1} \\ \downarrow & & \downarrow \\ B_n(p) & \longrightarrow & V_{2n+3,2} \\ \downarrow & & \downarrow \\ S^{2n+1+2(p-1)} & \xrightarrow[\frac{1}{2}\alpha_1(2n+2)]{} & S^{2n+2} \end{array}$$

As is well known, $H^*(B_n(p); Z_p) \cong \Lambda(x_{2n+1}, \mathfrak{P}^1 x_{2n+1})$, and $B_n(p)$ has the cell structure

$$B_n(p) \simeq S^{2n+1} \underset{\alpha_1(2n+1)}{\cup} e^{2n+1+2(p-1)} \cup e^{4n+2+2(p-1)}.$$

Let q be a prime with $(q, p) = 1$. Since the characteristic element of the bundle $B_n(p)$ is of order p, we have the isomorphism

$${}^q\pi_i(B_n(p)) \cong {}^q\pi_i(S^{2n+1} \times S^{2n+1+2(p-1)}).$$

The p-primary components of $\pi_i(B_n(p))$ are extensively calculated in [9].

4. QUASI-p-REGULARITY

In [7], Kumpel showed that

$$(4.1) \qquad S^5 \times S^9 \times \cdots \times S^{4n-3} \underset{p}{\simeq} SU(2n)/Sp(n) \qquad \text{for } n < \frac{p+1}{2}.$$

In fact, for $1 \leq i \leq \frac{p-1}{2}$ there exist maps

$$g_i\colon S^{4i+1} \to SU(p+1)/Sp\left(\frac{p+1}{2}\right)$$

such that the induced homomorphisms

$$g_i^*\colon H^*\left(SU(p+1)/Sp\left(\frac{p+1}{2}\right); Z_p\right) \to H^*(S^{4i+1}; Z_p)$$

are epimorphic. We put

$$\bar{g}_i\colon S^{4i+1} \xrightarrow[g_i]{} SU(p+1)/Sp\left(\frac{p+1}{2}\right) \longrightarrow SU(p+1+2i)/Sp\left(\frac{p+1}{2}+i\right)$$
$$(1 \leq i \leq (p-1)/2),$$

where the second map is the natural one:

$$SU(p+1)/Sp\left(\frac{p+1}{2}\right) \subset SU(p+1+2i)/Sp\left(\frac{p+1}{2}\right) \to SU(p+1+2i)/Sp\left(\frac{p+1}{2}+i\right).$$

We shall extend $\bar{g}_i$ to

$$f_i\colon B_{2i}(p) \to SU(p+1+2i)/Sp\left(\frac{p+1}{2}\right),$$

where $B_{2i}(p) \simeq S^{4i+1} \underset{\alpha_1}{\cup} e^{4i+1+2(p-1)} \cup e^{8i+2p}$. The mapping $\bar{g}_i$ is extendable to $S^{4i+1} \underset{\alpha_1}{\cup} e^{4i+1+2(p-1)}$, since

$$\pi_{4i+2(p-1)}\left(SU(p+1+2i)/Sp\left(\frac{p+1}{2}+i\right)\right) = 0.$$

By [5], ${}_p\pi_{8i+1+2(p-1)}(SU(p+1+2i)) = 0$. Hence

$${}_p\pi_{8i+1+2(p-1)}\left(SU(p+1+2i)/Sp\left(\frac{p+1}{2}+i\right)\right) = 0,$$

by (1) of [3]. Let ε_i be the attaching element of e^{8i+2p} in $B_{2i}(p)$, and let x_i be the order of $\pi_{8i+1+2(p-1)}\left(SU(p+1+2i)/Sp\left(\frac{p+1}{2}+i\right)\right)$. Then $(x_i, p) = 1$. Therefore $\bar{g}_i$ is extendable to

$$B'_{2i}(p) = S^{4i+1} \underset{\alpha_1}{\cup} e^{4i+1+2(p-1)} \underset{x_i\varepsilon_i}{\cup} e^{8i+2p}.$$

Obviously, $B_{2i}(p) \underset{p}{\simeq} B'_{2i}(p)$. Therefore the desired map is obtained as the composition

$$f_i\colon B_{2i}(p) \to B'_{2i}(p) \to SU(p+1+2i)/Sp\left(\frac{p+1}{2}+i\right).$$

With the notation $a = (p-1)/2$, consider the composition

$$A_i = \phi_a \circ (f_1 \times \cdots \times f_i \times g_{i+1} \times \cdots \times g_a);$$

it gives the mapping

$$A_i\colon \prod_{\ell=1}^{i} B_{2\ell}(p) \times \prod_{m=i+1}^{a} S^{4m+1} \longrightarrow \underbrace{G/K \times \cdots \times G/K}_{a} \xrightarrow{\phi_a} G/K,$$

where $G = SU(p+1+2i)$ and $K = Sp\left(\frac{p+1}{2}+i\right)$. Further,

$$A_i^*\colon H^*(G/K; Z_p) \to H^*\left(\prod_{\ell=1}^{i} B_{2\ell}(p) \times \prod_{m=i+1}^{a} S^{4m+1}; Z_p\right).$$

From (2.2) it follows that

$$A_i^*(x_{4j+1}) \equiv \begin{cases} f_j^*(2^{\ell-j}\, x_{4j+1}) & \text{if } j \leq i\,, \\ g_j^*(2^{\ell-j}\, x_{4j+1}) & \text{if } j \geq i+1\,, \end{cases}$$

modulo $(f_1 \times \cdots \times f_i \times g_{i+1} \times \cdots \times g_a)^*(D_j)$. By Lemma 2 of [7], the A_i^* are isomorphisms. Thus we have shown that for $1 \leq i \leq a = \frac{p-1}{2}$,

$$\text{(4.2)} \qquad \prod_{\ell=1}^{i} B_{2\ell}(p) \times \prod_{m=i+1}^{a} S^{4m+1} \underset{p}{\simeq} SU(p+1+2i)/Sp\left(\frac{p+1}{2}+i\right).$$

Together with (4.1), this implies that $SU(2n)/Sp(n)$ is quasi-p-regular if $n \leq p$.

Next we shall prove the converse: $SU(2n)/Sp(n)$ is not quasi-p-regular if $n > p$.

PROPOSITION 4.1. *Each of the following two statements implies that* G/K *is not quasi-*p*-regular:*

(a) $\pi_j(G/K)$ *and* $\pi_j\left(\prod_{\ell} B_{2\ell}(p) \times \prod_{n} S^n\right)$ *do not have isomorphic* p*-primary components.*

(b) $\mathfrak{P}^2 x \neq 0$ *for some primitive element* x *of* $H^*(G/K;\, Z_p)$.

The proof is obvious. We first show that $SU(2n)/Sp(n)$ is not quasi-2-regular for $n \geq 3$. In fact, $\pi_8(SU(2n)/Sp(n)) \simeq \pi_{10}(SO) = 0$. On the other hand, the 2-component of $\pi_8\left(\prod B_{2\ell}(p) \times \prod S^n\right)$ is Z_8. Hence $SU(2n)/Sp(n)$ is not quasi-2-regular, by (a) of Proposition 4.1.

Remark. The proof in [7] that the prime 2 is irregular for $SU(2n)/Sp(n)$ is incorrect. Actually, for $n \geq 3$,

$$\pi_5(SU(2n)/Sp(n)) \cong \pi_7(SO) \cong Z \cong \pi_5(S^5 \times \cdots \times S^{4n-3}).$$

Suppose $n > p$. Then $\mathfrak{P}^2 x_5 \neq 0$ in $H^*(SU(2n)/Sp(n);\, Z_p)$, by Proposition 1.1. Hence $SU(2n)/Sp(n)$ is not quasi-p-regular. Therefore we have proved the following result.

THEOREM 4.2. $SU(2n)/Sp(n)$ *is quasi-*p*-regular if and only if* $p \geq n$.

By a similar method one can show that

$$\text{(4.3)} \quad \prod_{\ell=1}^{i} B_{2\ell}(p) \times \prod_{m=i+1}^{a} S^{4m+1} \underset{p}{\simeq} SU(2i+p)/SO(2i+p) \qquad \text{for } 1 \leq i \leq a = \frac{p-1}{2}.$$

Hence we get the following result.

THEOREM 4.3. $SU(2n+1)/SO(2n+1)$ *is quasi-*p*-regular if and only if* $p \geq n+1$.

Next we consider the symmetric space E_6/F_4.

THEOREM 4.4. E_6/F_4 *is quasi-*p*-regular if and only if* $p \geq 5$.

Proof. Necessity. By (1.4) of [2], $\pi_{16}(E_6/F_4) = 0$. On the other hand,

$$\pi_{16}(S^9 \times S^{17}) \cong Z_{240} \quad \text{and} \quad \pi_{16}(B_4(5)) \cong Z_{48},$$

by [8]. Therefore E_6/F_4 is not quasi-p-regular for $p = 2$ and $p = 3$.

Sufficiency. First we show that

$$B_4(5) \underset{5}{\simeq} E_6/F_4. \tag{4.4}$$

Recall (see [1]) that E_6/F_4 has the cell structure

$$E_6/F_4 \simeq S^9 \underset{\alpha}{\cup} e^{17} \cup e^{26},$$

where $\alpha \in \pi_{16}(S^9)$ is the suspension of the homotopy class of the Hopf map. Let $i\colon S^9 \to E_6/F_4$ be the inclusion. By (1.4) of [2], $\pi_{16}(E_6/F_4) = 0$, and hence i can be extended to the mapping

$$\bar{i}\colon S^9 \underset{\alpha_1}{\cup} e^{17} = (B_4(5))^{(17)} \to E_6/F_4.$$

Now ${}^5\pi_{25}(E_6/F_4) = 0$, since

$${}^5\pi_{25}(E_6) \cong {}^5\pi_{25}(E_6/F_4) \oplus {}^5\pi_{25}(F_4)$$

by (1.3) of [6], and ${}^5\pi_{25}(E_6) = 0$, by [8]. Let ε be the attaching element of e^{26} in $B_4(5)$, and let x be the order of $\pi_{25}(E_6/F_4)$. Then $(x, p) = 1$. We put

$$B_4'(5) = (S^9 \underset{\alpha_1}{\cup} e^{17}) \underset{x\varepsilon}{\cup} e^{26}.$$

Obviously, $B_4'(5)$ is 5-equivalent to $B_4(5)$. The mapping $\bar{i}$ is now extended to $B_4(5)$. The desired map is obtained as the composition

$$B_4(5) \to B_4'(5) \to E_6/F_4,$$

which induces the isomorphisms of $H^*(\ ; Z_5)$.

Next we show that for every prime $p \geq 7$,

$$S^9 \times S^{17} \underset{p}{\simeq} E_6/F_4. \tag{4.5}$$

Let $\beta \in \pi_{25}((E_6/F_4)^{(17)})$ be the attaching element of e^{26} in

$$E_6/F_4 = S^9 \cup e^{17} \cup e^{26}.$$

Let

$$p\colon (E_6/F_4)^{(17)} = S^9 \cup e^{17} \to S^{17}$$

by the map shrinking S^9 to a point. Then $p^*(2\beta) = 0$, since $\pi_{25}(S^{17}) \simeq Z_2 \oplus Z_2$. Hence p can be extended to a mapping

$$P\colon X = (E_6/F_4)^{(17)} \underset{2\beta}{\cup} e^{26} \to S^{17}.$$

Clearly, for every odd prime p,

(4.6) $$X \underset{p}{\simeq} E_6/F_4.$$

We may regard P as a fibre map. Let Y be its fibre. It is easy to see that Y is homotopy-equivalent to S^9. Thus we get the exact sequence

$$\begin{array}{ccccccccc} \longrightarrow & \pi_i(Y) & \xrightarrow{i^*} & \pi_i(X) & \xrightarrow{P^*} & \pi_i(S^{17}) & \xrightarrow{\Delta} & \pi_{i-1}(Y) & \longrightarrow \\ & \downarrow \cong & & & & & & \downarrow \cong & \\ & \pi_i(S^9) & & & & & & \pi_{i-1}(S^9) . & \end{array}$$

The element $\Delta\iota_{17}$ is in $\pi_{16}(Y) \cong \pi_{16}(S^9) \cong Z_{240}$. Therefore there exists a map $f\colon S^{17} \to X$ such that the P_*-image of the homotopy class $\{f\}$ is $240\,\iota_7$. Since $240 = 2^4 \cdot 3 \cdot 5$, the induced homomorphism

$$f^*\colon H^*(X; Z_p) \to H^*(S^{17}; Z_p)$$

is epimorphic for every prime $p \geq 7$. The composite

$$A\colon S^9 \times S^{17} \longrightarrow Y \times S^{17} \underset{i\times f}{\longrightarrow} X \times X \longrightarrow (E_6/F_4) \times (E_6/F_4) \underset{w}{\longrightarrow} E_6/F_4$$

of the maps induces the isomorphisms

$$A^*\colon H^*(E_6/F_4; Z_p) \cong H^*(S^9 \times S^{17}; Z_p) \qquad \text{for } p \geq 7.$$

From the proof of Theorem 4.4, we also obtain the following result.

COROLLARY 4.5. *E_6/F_4 is p-regular for every prime $p \geq 7$.*

Kumpel [7, Theorem 2] proved that *if G is a classical group and G/K is an irreducible symmetric space different from a sphere, then each prime $p < (n_\ell + 1)/2$ is irregular for G/K.* Corollary 4.5 shows that we cannot extend this to *exceptional* Lie groups, and it thus settles a question raised in Section 4 of [7].

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