ON WEYL'S THEOREM

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Let $\mathscr{B}(H)$ be the algebra of all bounded operators on an infinite-dimensional complex Hilbert space H, and let $\mathscr{K}$ be the closed ideal of compact operators. L. Coburn [3] has defined the Weyl spectrum $\omega(A)$ by

$$\omega(A) = \bigcap \sigma(A+K),$$

where $\sigma(A)$ denotes the spectrum of A in $\mathscr{B}(H)$ and the intersection is taken over all K in $\mathscr{K}$. A celebrated theorem of H. Weyl [9] asserts that if A is normal, then $\omega(A)$ consists precisely of all points in $\sigma(A)$ except the isolated eigenvalues of finite multiplicity.

In [3], Coburn proved that Weyl's theorem holds for two large classes of generally nonnormal operators, namely, the class of hyponormal operators and the class of Toeplitz operators. In this paper, we shall show that Weyl's theorem holds for yet another class of operators.

Recall that an operator A is a Fredholm operator if it has a closed range and both a finite-dimensional kernel and cokernel. The class $\mathscr{F}$ of Fredholm operators constitutes a multiplicative open semigroup in $\mathscr{B}(H)$. In fact [1], if π is the natural quotient map from $\mathscr{B}(H)$ to $\mathscr{B}(H)/\mathscr{K}$, then A is in $\mathscr{F}$ if and only if $\pi(A)$ is invertible. For any A in $\mathscr{F}$, the index i(A) is defined by the formula

$$i(A) = \dim[\ker A] - \dim[\operatorname{coker} A],$$

and it is known that i is a continuous integer-valued function on $\mathscr{F}$.

Let L^2 and L^∞ denote the Lebesgue spaces of square-integrable and essentially bounded functions with respect to normalized Lebesgue measure on the unit circle in the complex plane. Let H^2 and H^∞ denote the corresponding Hardy spaces. If $\phi \in L^\infty$, the Toeplitz operator induced by ϕ is the operator T_ϕ on H^2 defined by $T_\phi f = P(\phi f)$; here P stands for the orthogonal projection in L^2 with range H^2. Recall that the linear span $H^\infty + C$ of H^∞ and C is a closed subalgebra of L^∞ [5, Theorem 2], where C stands for the space of continuous, complex-valued functions on the unit circle. This algebra can also be characterized as the subalgebra of L^∞ generated by H^∞ and the function $\bar{z}$. It is well-known [4] that if $\phi \in H^\infty + C$, then T_ϕ is a Fredholm operator if and only if ϕ is an invertible function of $H^\infty + C$.

The relation between the index and the invertibility of Toeplitz operators is described by the following result of Coburn [3].

LEMMA A. *If* $\phi \in L^\infty$, *then either* $\ker T_\phi = (0)$ *or* $\operatorname{coker} T_\phi = (0)$.

For ϕ in L^∞, let $R(\phi)$ denote the essential range of ϕ. Suppose that $u \in H^\infty + C$, that $|u| = 1$ a.e., and that T_u is invertible; it is easy to show that then the spectrum $\sigma(T_u)$ of T_u is R(u). In fact, we can use the same argument as in

Received May 28, 1970.

Michigan Math. J. 18 (1971).

Theorem 4 in [6]. *In this paper, we shall always assume that* u *is a nonconstant function in* $H^\infty + C$, *and that* $|u| = 1$ a.e. *and* T_u *is invertible.*

LEMMA 1. $T_u - \lambda I$ *has a trivial kernel, for each complex number* λ.

Proof. If λ is not in $\sigma(T_u)$, then $T_u - \lambda I$ is invertible, so that $T_u - \lambda I$ has a trivial kernel.

Assume now that $\lambda \in \sigma(T_u)$. Since $\sigma(T_u) = R(u)$, $|\lambda| = 1$. Suppose $T_u - \lambda I$ has a nontrivial kernel; then λ is an eigenvalue of T_u. Because T_u is a contraction, an elementary argument shows that $\bar{\lambda}$ is an eigenvalue of T_u^*. Since

$$T_u^* - \bar{\lambda} I = (T_u - \lambda I)^*,$$

this contradicts Lemma A. The proof is complete.

Let $\mathscr{W}$ be the collection of all rational functions w(z) that are analytic on $\sigma(T_u)$. Since the Lebesgue area of $\sigma(T_u) = R(u)$ equals zero, $\mathscr{W}$ is dense in the space $C(\sigma(T_u))$ of continuous complex functions on $\sigma(T_u)$. We shall show that *Weyl's theorem holds for the class of operators* $\{w(T_u): w(z) \in \mathscr{W}\}$. First we shall prove the following lemma.

LEMMA 2. *If* $w(z) \in \mathscr{W}$, *then* $w(T_u)$ *is invertible if and only if* $w(T_u)$ *is a Fredholm operator and* $i(w(T_u)) = 0$.

Proof. Let $w(z) = (z - \alpha_1) \cdots (z - \alpha_n)/(z - \beta_1) \cdots (z - \beta_m)$ be a nonconstant function in $\mathscr{W}$. Since β_i is not in $\sigma(T_u)$ for $i = 1, 2, \cdots, m$, each factor of $w(T_u)$ is a one-to-one operator in $\mathscr{B}(H^2)$, by Lemma 1. Hence $w(T_u)$ has a trivial kernel, for every nonzero function w(z) in $\mathscr{W}$. Therefore we may conclude that $w(T_u)$ is invertible if and only if $w(T_u)$ is a Fredholm operator and $i(w(T_u)) = 0$. The proof is complete.

Our main result is an easy consequence of Lemma 2.

THEOREM 1. $\omega(w(T_u)) = \sigma(w(T_u))$ *for each* w(z) *in* $\mathscr{W}$.

Proof. It is known [7] that $A + K$ is a Fredholm operator, and that $i(A + K) = i(A)$ for every compact operator K in $\mathscr{K}$, if A is a Fredholm operator. Hence $\sigma(w(T_u)) \subset \sigma(w(T_u) + K)$ for every compact K in $\mathscr{K}(H^2)$, by Lemma 2. Therefore $\omega(w(T_u)) = \sigma(w(T_u))$, by the definition of the Weyl spectrum. The proof is complete.

COROLLARY 1.1. *If* $w(z) \in \mathscr{W}$, *then the spectral radius of* $w(T_u)$ *is less than* $\|w(T_u) + K\|$, *for each* K *in* $\mathscr{K}(H^2)$.

The Weyl spectrum $\omega(A)$ is in general not empty; but we can say more in our special case.

COROLLARY 1.2. *If* $w(z) \in \mathscr{W}$, *then* $\sigma(w(T_u)) = \sigma(T_{w(u)})$, *hence* $\sigma(w(T_u))$ *is connected.*

Proof. It is known [10] that $\sigma(T_\phi)$ is connected, for every ϕ in L^∞. Further, $w(T_u) = T_{w(u)} + K_1$ for some compact K_1 (see J. G. Stampfli [8]) and $\sigma(T_{w(u)} + K) \supset \sigma(T_{w(u)})$ for every compact K in $\mathscr{K}(H^2)$ (see Coburn [3]). Hence $\omega(w(T_u)) = \sigma(T_{w(u)})$, and the desired result follows from Theorem 1. The proof is complete.

It is easy to see that the class $\{w(T_u): w(z) \in \mathscr{W}\}$ contains nonnormal operators, and the following theorem shows that not all of its elements are hyponormal, and that not all of them are Toeplitz operators.

THEOREM 2. *There exist a function* $w_1(z)$ *in* $\mathscr{W}$ *and a nontrivial compact operator* K_1 *in* $\mathscr{K}(H^2)$ *such that* $\|w_1(T_u)\| > \|w_1(T_u) + K_1\|$.

Proof. If $\|w(T_u) + K\| \geq \|w(T_u)\|$ for every K in $\mathscr{K}(H^2)$ and for every w(z) in $\mathscr{W}$, then

$$\|w(T_u)\| = \|\pi(w(T_u))\| \leq \|w(z)\|_\infty ,$$

where $\|w(z)\|_\infty = \sup\{|w(z)| : z \in \sigma(\pi(T_u)) = \sigma(T_u)\}$, since $\pi(T_u)$ is normal in $\mathscr{B}(H^2)/\mathscr{K}(H^2)$ [8, Lemma 3]. Hence T_u has the unit circle as a spectral set. It is well known that under this condition T_u is a unitary operator, that is, u is a constant of modulus 1, by a corollary in [2]; but this is not the case. Hence there exist a function $w_1(z)$ in $\mathscr{W}$ and a nontrivial compact operator K_1 in $\mathscr{K}(H^2)$ such that $\|w_1(T_u)\| > \|w_1(T_u) + K_1\|$. The proof is complete.

COROLLARY 2.1. *The class of operators* $\{w(T_u): w(z) \in \mathscr{W}\}$ *is contained neither in the class of hyponormal operators nor in the class of Toeplitz operators.*

Proof. By Theorem 2, there exist a $w_1(z)$ in $\mathscr{W}$ and a nontrivial compact operator K_1 in $\mathscr{K}(H^2)$ such that $\|w_1(T_u)\| > \|w_1(T_u) + K_1\|$. Hence $\|w_1(T_u)\|$ is strictly greater than the spectral radius of $w_1(T_u)$, by Corollary 1.1, but this is not the case for hyponormal operators and Toeplitz operators. The proof is thus complete.

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