RANGES OF NORMAL AND SUBNORMAL OPERATORS

C. R. Putnam

1. Let T be a bounded operator on a Hilbert space H, and denote its spectrum by $sp(T)$ and its range by $R(T)$. (Only bounded operators will be considered.) For each set E of complex numbers, let $S(T; E)$ be the subset of H defined by

$$S(T; E) = \bigcap_{t \in E} R(T - tI), \qquad S(T; \text{empty set}) = H. \tag{1.1}$$

Denote the interior of E by $\text{int}(E)$ and the complement of E by $C(E)$. Clearly, S is a decreasing function of E in the sense that $S(T; E_1) \subset S(T; E_2)$ if $E_1 \supset E_2$. Also, since $R(T - tI) = H$ whenever t does not belong to $sp(T)$,

$$S(T; sp(T)) \subset S(T; E) \qquad \text{for each } E. \tag{1.2}$$

If T is normal and has the spectral resolution

$$T = \int z\, dK_z, \tag{1.3}$$

let $K(E)$ denote the associated projection measure defined on the Borel sets E of the plane. We shall prove the following result.

THEOREM 1. *If* T *is normal and has the spectral resolution* (1.3), *and if* E *is any Borel set of the plane, then*

$$S(T; C(E)) \subset R(K(E)) \subset S(T; \text{int}(C(E))). \tag{1.4}$$

Consequently,

$$S(T; sp(T) - E) = R(K(E)) \textit{ if } E \textit{ is a closed subset of } sp(T), \tag{1.5}$$

and, in particular,

$$S(T; sp(T)) = 0. \tag{1.6}$$

To obtain (1.5) from (1.4), note that now $C(E) = \text{int}(E)$ and hence, by (1.4), $R(K(E)) = S(T; C(E)) = S(T; sp(T) \cap C(E)) = S(T; sp(T) - E)$.

We see that if T is normal, then $S(T; sp(T)) = 0$ but $S(T; E) \neq 0$ whenever E is small relative to $sp(T)$, more precisely, whenever the closure of E is a proper subset of $sp(T)$. In case T is not normal, simple examples show that even (1.6) can be false. We need only consider an operator $T \neq 0$ for which $sp(T)$ is the single point 0.

Received May 21, 1970.
This work was supported by a National Science Foundation research grant.

Michigan Math. J. 18 (1971).

Recall that T is *subnormal* if it has an extension T_1 that is normal on a Hilbert space $H_1 \supset H$, and if moreover T_1 leaves H invariant and $T_1 = T$ on H. (See P. R. Halmos [3, pp. 100 ff.] for properties of such operators.) It is known that if T_1 is the minimal extension of T, then $\mathrm{sp}(T_1) \subset \mathrm{sp}(T)$ (Halmos [2]) and, in fact, that $\mathrm{sp}(T)$ is obtained from $\mathrm{sp}(T_1)$ by filling in some of the holes of the latter set (J. Bram [1]). Further, by (1.6),

$$S(T; \mathrm{sp}(T)) \subset S(T; \mathrm{sp}(T_1)) \subset S(T_1; \mathrm{sp}(T_1)) = 0. \tag{1.7}$$

For example, let A denote the unilateral shift defined on the Hilbert space of sequences $\{x_n\}_{n=1}^{\infty}$ satisfying the condition $\sum |x_n|^2 < \infty$. Then A has the matrix representation $A = (a_{ij})$ $(i, j = 1, 2, \cdots)$, where $a_{i+1,i} = 1$ and $a_{ij} = 0$ $(j \neq i - 1)$. It is known that A is subnormal and that $\mathrm{sp}(A)$ is the closed unit disk. Further, the minimal normal extension of A is the unitary operator $B = (b_{ij})$ $(i, j = 0, \pm 1, \cdots)$ on the Hilbert space of sequences $\{y_n\}_{n=-\infty}^{\infty}$ satisfying $\sum |y_n|^2 < \infty$, where the elements b_{ij} are defined by $b_{i+1,i} = 1$ and $b_{ij} = 0$ $(j \neq i - 1)$. The spectrum of B is the circle

$$C = \{z: |z| = 1\}. \tag{1.8}$$

It is clear from (1.7) that if T is subnormal, in contrast to the situation where T is normal, then it is possible that $S(T; E) = 0$ when E is small relative to $\mathrm{sp}(T)$. Thus, when $T = A$, $S(A; C) = 0$, where C is the circle of (1.8). In fact, we shall prove the following proposition.

THEOREM 2. *Let* A *be the unilateral shift defined above. Then*

$$S(A; E) = 0 \tag{1.9}$$

if either

(1.10) E *is an infinite set having a limit point inside* C,

or

$$m(E \cap C) > 0, \tag{1.11}$$

where m *denotes ordinary Lebesgue measure on* C.

2. *Proof of Theorem* 1. Let E denote any Borel set of the plane, and let $f \in S(T; C(E))$, so that whenever c belongs to $C(E)$, $f = (T - cI)g$ for some $g = g_c$. Let $c = a + ib$ (a, b real), and let D_s denote the disk $0 \leq |z - c| \leq s$ for $s > 0$. It follows from (1.3) that

$$s^{-2} \|K(D_s)f\|^2 = s^{-2} \int_{D_s} [(x - a)^2 + (y - b)^2] d\|K_z g\|^2 \leq \int_{D_s - \{0\}} d\|K_z g\|^2 \to 0,$$

as $s \to 0$. Thus, the symmetric derivate of the set function $\|K(X)f\|^2$ (S. Saks [4, p. 149]) is 0 at all points of $C(E)$, and hence [4, p. 155] $K(C(E))f = 0$. Thus, if $f \in S(T; C(E))$, then $f \in R(K(E))$, so that the first relation of (1.4) is proved.

If $c \in \mathrm{int}(C(E))$, then, for each f in H, $K(E)f = (T - cI)g_c$, where

$$g_c = \int_E (z - c)^{-1} dK_z f.$$

This establishes the second part of (1.4), and the proof of Theorem 1 is complete.

3. *Proof of Theorem* 2. We see that each vector $u(z) = \{1, z, z^2, \cdots\}$ ($|z| < 1$) is in the eigenspace of A^* belonging to z. If $f = \{f_1, f_2, \cdots\}$ belongs to $S(A; E)$, where E satisfies (1.10), then $f = (A - zI)g_z$ for each z in E and some vector g_z, and therefore

$$(f, u(\bar{z})) = \sum_{k=0}^{\infty} f_k z^k = 0 \qquad \text{for each } z \text{ in } E.$$

It follows from (1.10), by a basic property of power series, that $f_k = 0$ for all k, that is, $f = 0$, and so (1.9) is proved.

It remains to prove (1.9) under the assumption (1.11). Suppose then that $f = (A - zI)g$, where $g = g_z$ for z in $E \cap C$, where C is defined by (1.8). It is easily verified that if $g = \{g_1, g_2, \cdots\}$, then

$$g_n = -(1/z)^{n+1} \sum_{k=1}^{n} f_k z^k. \tag{3.1}$$

Note that $g_n = g_n(z)$, and let $z = e^{i\theta}$ $(0 \leq \theta < 2\pi)$. For each fixed z in E, $|g_n| = |\sum_{k=1}^{n} f_k e^{ik\theta}|$, and since $\sum |g_n|^2 < \infty$, $g_n \to 0$ as $n \to \infty$.

Let $F(\theta)$ denote the function in $L^2(0, 2\pi)$ defined by

$$F(\theta) \sim \sum_{k=1}^{\infty} f_k e^{ik\theta}. \tag{3.2}$$

If $F_n(\theta) = \sum_{k=1}^{n} f_k e^{ik\theta}$, then $F_n(\theta) \to 0$ as $n \to \infty$, for each θ in the set $Q = \{\theta: 0 \leq \theta < 2\pi,\ e^{i\theta} \in E\}$. Clearly,

$$\int_0^{2\pi} |F(\theta) - F_n(\theta)|^2 \, d\theta \to 0 \qquad \text{as } n \to \infty. \tag{3.3}$$

In view of (1.11), Q has positive one-dimensional Lebesgue measure. By Egoroff's theorem, there exists a subset Q_1 of Q such that Q_1 has positive measure and such that $F_n(\theta) \to 0$ as $n \to \infty$, uniformly for θ in Q_1. Since $|F - F_n|^2 \leq (|F| + \text{const.})^2$ on Q_1 and $(|F| + \text{const.})^2 \in L(0, 2\pi)$, it follows from (3.3) and Lebesgue's dominated-convergence theorem that, as $n \to \infty$,

$$\int_{Q_1} |F(\theta) - F_n(\theta)|^2 \, d\theta \to \int_{Q_1} |F(\theta)|^2 \, d\theta = 0. \tag{3.4}$$

Thus, $F(\theta) = 0$ almost everywhere on Q_1, and hence, by the theorem of F. and M. Riesz (see Halmos [3, p. 82] for example), $F(\theta) = 0$ almost everywhere on $L^2(0, 2\pi)$. Hence $f_k = 0$ for all k, and again (1.9) holds.

REFERENCES

1. J. Bram, *Subnormal operators.* Duke Math. J. 22 (1955), 75-94.

2. P. R. Halmos, *Spectra and spectral manifolds.* Ann. Soc. Polon. Math. 25 (1952), 43-49.

3. ———, *A Hilbert space problem book.* Van Nostrand, Princeton, N.J., 1967.

4. S. Saks, *Theory of the integral.* (Second revised edition) English translation by L. C. Young. G. E. Stechert and Co., New York, 1937.

Purdue University
Lafayette, Indiana 47907