AN INFINITE-DIMENSIONAL VERSION OF LIAPUNOV'S CONVEXITY THEOREM

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The classical theorem of Liapunov asserts that the range of a finite measure with values in a finite-dimensional vector space is convex and closed (see [1], [2], [3], [4]). In his later paper [5], Liapunov gives an example of an L_1-valued measure whose range is compact but not convex. In this note, we prove a weaker version of Liapunov's theorem, where the measure takes values in a Hilbert space and is absolutely continuous with respect to a numerical measure.

Let $(S, \mathscr{F}, \mu)$ denote a measure space, where μ is a positive, nonatomic measure with $\mu(S) = 1$, and let H denote a real Hilbert space with the inner product (x, y) and norm $\|x\|$.

THEOREM. *Let* $f\colon S \to H$ *be an integrable function (that is,* $\int \|f\|\, d\mu < \infty$*), and let* $R = R(f)$ *be the set of all vectors of the form* $\int_E f\, d\mu$ $(E \in \mathscr{F})$. *Then* $\overline{R}$ *is convex.*

The proof is motivated by a method due to Halkin [2] who considered the finite-dimensional case only. We need several lemmas.

LEMMA 1. *Let* $\{x_1', x_2', \cdots, x_N'\}$ *be a collection of* N *vectors in* H *such that* $\sum x_i' = 0$. *Then the* x_j' *can be rearranged to form a set* $\{x_1, x_2, \cdots, x_N\}$ *such that*

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 \leq \sum_{i=1}^{N} \|x_i\|^2 \qquad (1 \leq n \leq N).$$

Proof. We choose x_1 arbitrarily. Having chosen $x_1, x_2, \cdots, x_n$, we select x_{n+1} to be one of the remaining vectors with the property that

$$(x_1 + x_2 + \cdots + x_n,\ x_{n+1}) \leq 0.$$

Such a choice is always possible, because

$$0 = \left(\sum_1^N x_i', \sum_1^N x_i'\right) = \left(\sum_1^n x_i, \sum_1^n x_i\right) + 2\sum_{j=n+1}^{N}\left(\sum_1^n x_i, x_j'\right) + \left(\sum_{n+1}^{N} x_j', \sum_{n+1}^{N} x_j'\right).$$

Since the first and the last inner products are nonnegative, at least one summand in the middle term must be nonpositive. Our arrangement of the x_j gives us the equations

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$$\left\|\sum_1^{n+1} x_i\right\|^2 = \left(\sum_1^{n+1} x_i, \sum_1^{n+1} x_i\right) = \left\|\sum_1^{n} x_i\right\|^2 + \|x_{n+1}\|^2 + 2\left(\sum_1^{n} x_i, x_{n+1}\right).$$

The result now follows by induction.

The following lemma was proved by P. R. Halmos [3].

LEMMA 2. *For every set* $E \in \mathscr{F}$, *there exists a function* ϕ: $E \to [0, 1]$ *such that*

$$\mu(\{x \in E: \phi(x) < \lambda\}) = \lambda\mu(E).$$

The next result is crucial.

LEMMA 3. *Let* g: $X \to H$ *be an integrable function* (*that is,* $\int \|g\|\, d\mu < \infty$). *Then, for every* $\varepsilon > 0$, *there exists a function* Φ: $X \to [0, 1]$ *such that*

i) $\left\| \int_{E(\lambda)} g\, d\mu - \lambda \int_S g\, d\mu \right\| < \varepsilon$, *where* $E(\lambda) = \{x: \Phi(x) < \lambda\}$, *and*

ii) $\mu(\{x: \Phi(x) < \lambda\}) = \lambda$.

(We denote the collection of such functions Φ by $K(g, \varepsilon)$.)

Proof. We may assume $\int_S g\, d\mu = 0$, since otherwise we could apply the result to $g - \int_S g\, d\mu$. Choose an integer N such that if $\mu(E) \leq \frac{1}{N}$, then

(1) $$\int_E \|g\|\, d\mu < \min\left\{\frac{1}{2}\varepsilon, \frac{1}{4}\varepsilon^2\left[\int_X \|g\|\, d\mu\right]^{-1}\right\} = \eta.$$

Select a function ϕ: $X \to [0, 1]$ as in Lemma 2, so that $\mu\{x: \phi(x) < \lambda\} = \lambda$. Let

$$A_i' = \left\{x: \frac{i-1}{N} \leq \phi(x) < \frac{i}{N}\right\} \quad (i = 1, 2, \cdots, N).$$

Then $\mu(A_i') = \frac{1}{N}$, $\sum_i \int_{A_i'} g\, d\mu = 0$, and $\int_{A_i'} \|g\|\, d\mu < \eta$.

By Lemma 1, A_i' can be rearranged into $\{A_1, A_2, \cdots, A_N\}$, say, such that

$$\left\|\sum_{i=1}^n \int_{A_i} g\, d\mu\right\|^2 \leq \sum_1^N \left\|\int_{A_i} g\, d\mu\right\|^2 \leq \sum_1^N \eta \int_{A_i} \|g\|\, d\mu \leq \frac{1}{4}\varepsilon^2 \quad (1 \leq n \leq N).$$

Hence each partial sum satisfies the inequality

(2) $$\left\|\sum_{i=1}^n \int_{A_i} g\, d\mu\right\| \leq \frac{1}{2}\varepsilon.$$

For each index i $(i = 1, 2, \cdots, N)$, we choose a function $\phi_i: A_i \to [0, 1]$ as in Lemma 2. Set

$$\Phi(x) = \sum_{i=1}^{N} \frac{i-1}{N} I_{A_i}(x) + \sum_{i=1}^{N} \frac{1}{N} \phi_i(x),$$

where I_F is the characteristic function of F and $\phi_i(x) = 0$ for $x \notin A_i$. Now we can write the set $E(\lambda)$ as

$$E(\lambda) = \{x: \Phi(x) < \lambda\}$$

$$(3) \qquad = \bigcup_{\frac{i-1}{N} \leq \lambda} A_i \cup \left\{A_{[N\lambda]+1} \cap \left\{x: \phi_{[N\lambda]+1}(x) < N\left(\lambda - \frac{[N\lambda]}{N}\right)\right\}\right\},$$

where $[\alpha]$ denotes the greatest integer not exceeding α.

The sets whose union we take in (3) are disjoint, so that

$$\left\| \int_{E(\lambda)} g\,d\mu \right\| \leq \left\| \sum_{i=1}^{[N\lambda]} \int_{A_i} g\,d\mu \right\| + \int_{A_{[N\lambda]+1}} \|g\|\,d\mu .$$

The first partial sum is less than $\varepsilon/2$, by (2). The integral is less than $\varepsilon/2$, by (1). The result now follows.

We proceed to prove the theorem. It is enough to show that if E and F are two measurable subsets of X, then for every $\lambda \in [0, 1]$ and every $\varepsilon > 0$, there exists a measurable set $C(\lambda)$ such that

$$(4) \qquad \left\| \int_{C(\lambda)} f\,d\mu - \lambda \int_E f\,d\mu - (1-\lambda) \int_F f\,d\mu \right\| < \varepsilon .$$

We select $\Phi \in K(fI_{E-F}, \varepsilon/2)$ and $\psi \in K(fI_{F-E}, \varepsilon/2)$ (the sets K are defined by Lemma 3) and put

$$C(\lambda) = \{E \cap F\} \cup \{x \in E - F: \Phi(x) < \lambda\} \cup \{x \in F - E: \psi(x) < 1 - \lambda\} .$$

Since the sets above are disjoint, we obtain the inequalities

$$\left\| \int_{C(\lambda)} - \lambda\left[\int_{E\cap F} + \int_{E-F} \right] + (1-\lambda)\left[\int_{E\cap F} + \int_{F-E} \right] \right\|$$

$$\leq \left\| \int_{\{\Phi < \lambda\}} fI_{E-F}\,d\mu - \lambda \int fI_{E-F}\,d\mu \right\|$$

$$+ \left\| \int_{\{\psi < 1-\lambda\}} fI_{F-E}\,d\mu + (1-\lambda) \int fI_{F-E}\,d\mu \right\| < \varepsilon .$$

This completes the proof.

REFERENCES

1. D. Blackwell, *The range of certain vector integrals.* Proc. Amer. Math. Soc. 2 (1951), 390-395.

2. H. Halkin, *Some further generalizations of a theorem of Lyapounov.* Arch. Rational Mech. Anal. 17 (1964), 272-277.

3. P. R. Halmos, *The range of a vector measure.* Bull. Amer. Math. Soc. 54 (1948), 416-421.

4. A. A. Lyapunov, *Sur les fonctions-vecteurs complètement additives.* (Russian. French summary) Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4 (1940), 465-478.

5. ———, *Sur les fonctions-vecteurs complètement additives.* (Russian. French summary) Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10 (1946), 277-279.

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