# CELLULARITY CRITERIA FOR MAPS

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In [13], D. R. McMillan gave a criterion for cellularity of a compact set A in a PL n-manifold M ($n \neq 4$). Clearly, any such criterion must say that A behaves topologically like a cell (a sphere can never be cellular), and it must also say something about the way A is embedded in M (an arc may fail to be cellular in euclidean space). This is not necessarily the case with cellularity criteria for maps.

It is known, for example, that a proper map $f: M \to N$ between topological n-manifolds ($n \geq 5$) is cellular provided for each $y \in N$ the space $f^{-1}(y)$ is cell-like (a topological property, defined below): thus there is no need for assumptions on the embeddings $f^{-1}(y) \subset M$ ($y \in N$). In the present paper, we relax the topological conditions on point-inverses in two situations: for self-maps of a PL-manifold, and for maps between topological manifolds. The general idea of the criteria is to assume that point-inverses behave like k-connected spaces (that is, have property $UV^k$), where k is almost n/2. Then properties of the induced map on homology, together with duality and a kind of Hurewicz theorem for UV-properties, imply that point-inverses actually behave like contractible spaces, which implies that they are cell-like and that each inclusion $f^{-1}(y) \subset M$ satisfies McMillan's criterion. Our conditions for maps are best possible codimensionally, and they are necessary as well as sufficient.

*Addendum.* In Section 7, we examine the case in which point-inverses have property $UV^{k-1}$ and M has dimension 2k. In this, the critical codimension, we show that all but a finite number of point-inverses must be cellular in M (assuming M is compact and PL, and $k \neq 2$). A result of L. C. Siebenmann then implies that M is homeomorphic to the connected sum of N and a finite number of closed, (k - 1)-connected manifolds.

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*Conventions.* $R^k$ is euclidean k-space, $B^k$ is the closed unit ball in $R^k$, and $S^{k-1} = \partial B^k$. The symbols $H_*$, $H^*$, and $\check{H}^*$ denote singular homology, singular cohomology, and Čech cohomology, each with integer (Z) coefficients. The symbol ~ over a (co)homology symbol indicates "reduced". See [20] for a general reference on algebraic topics.

ANR's are always assumed to be metrizable. When $A \subset X$, a *neighborhood* of A in X is always understood to be an open set of X containing A. A map $f: X \to Y$ is *proper* if and only if $f^{-1}(K)$ is compact for all compact sets $K \subset Y$.

*Convention on manifolds.* In all statements and proofs, a *manifold* will be taken to be a connected, locally euclidean metric space. (No boundary points are allowed.)

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## 1. STATEMENT OF RESULTS

We shall use, at various times, all of the following hypotheses on maps. The two criteria are stated in terms of property $UV^k$.

*Cellular maps.* A set A in the n-manifold M is *cellular* in M if it is the intersection of a sequence of topological n-cells $Q_1, Q_2, \cdots$, where $Q_{i+1} \subset \operatorname{Int} Q_i \subset M$ for all i.

A mapping f: M → Y, where M is a manifold, is called *cellular* if $f^{-1}(y)$ is cellular in M for each $y \in Y$.

*Cell-like maps.* A space A is *cell-like* if there exist a manifold M and an embedding $\phi$: A → M such that $\phi(A)$ is cellular in M.

A mapping f: X → Y is *cell-like* if $f^{-1}(y)$ is a cell-like space for each $y \in Y$.

$UV^k$-*maps.* An inclusion $A \subset X$ has *property* k-UV if for each neighborhood U of A in X there exists a neighborhood V of A in U such that each map $S^k \to V$ can be extended to a map $B^{k+1} \to U$. We say $A \subset X$ has *property* $UV^k$ if it has property q-UV for $0 \leq q \leq k$.

A mapping f: X → Y is $UV^k$-*trivial* (or is a $UV^k$-*map*) if $f^{-1}(y) \subset X$ has property $UV^k$ for each $y \in Y$.

THEOREM 1.1. *Let* f: $S^n \to S^n$ *be an onto* $UV^k$-*map. If* $2k + 2 \geq n$, *then* f *is cell-like. Hence, if* $n \neq 4$, f *is cellular.*

(Essentially the same result holds when $S^n$ is replaced by any compact PL n-manifold. See Section 7.)

Theorem 1.1 fails whenever $2k + 2 < n$. The condition $n \neq 4$ is not known to be necessary. By strengthening the assumption codimensionally, we obtain a generalization to topological manifolds, as follows.

THEOREM 1.2. *Let* M *and* N *be* n-*manifolds, and let* f: M → N *be a proper, onto* $UV^k$-*map. If* $2k + 1 \geq n$, *then* f *is cell-like. Hence, if* $n \neq 3, 4$, *then* f *is cellular.*

Theorem 1.2 fails whenever $2k + 1 < n$. Again, the condition $n \neq 3, 4$ is possibly unnecessary. In fact, if n = 3, f is cellular if M contains no fake cubes or if M and N are compact and homeomorphic. (See [13] and [14].)

Cell-like maps preserve many properties of spaces. To illustrate this point, we quote two theorems.

THEOREM 1.3. *Let* X *and* Y *be euclidean neighborhood retracts, and let* f: X → Y *be a proper, cell-like map. Then, for each open set* U *of* Y (*including* U = Y), $f \mid f^{-1}(U): f^{-1}(U) \to U$ *is a proper homotopy equivalence.*

THEOREM 1.4 (Siebenmann). *Let* M *and* N *be manifolds of dimension at least* 5, *and let* f: M → N *be a proper cell-like map. Then* f *is properly homotopic to a homeomorphism of* M *onto* N.

Theorem 1.3 is proved in [12]. Theorem 1.4 has recently been proved by Siebenmann [18] as stated above. The simply-connected case had previously been proved by D. Sullivan. (See [16], [21].)

The proofs of Theorems 1.1 and 1.2, as well as some appropriate examples, are given in Section 6. Generalizations of Theorems 1.1 and 1.2 (in which the image of f is not necessarily a manifold) follow from Section 5 below together with Section 4 of [12].

*Remarks.* 1. Property k-UV is an intrinsic topological property of spaces in the following sense: If A is a compact set in the ANR X, and if $A \subset X$ has property k-UV, then every embedding of A into any ANR has property k-UV (see [1]). Moreover, if A is a compact ANR, then A has property k-UV if and only if $\pi_k A = 0$ (see [3]). Thus $UV^k$ should be thought of as a kind of Čech k-connectivity.

2. A compact, finite-dimensional metric space is cell-like if and only if it has property $UV^k$ for all k (see [11]).

## 2. PRODUCING THE ZERO HOMOMORPHISM

Let R be a principal ideal domain.

When G is an R-module, we let Hom G = $\mathrm{Hom}_R(G; R)$ and Ext G = $\mathrm{Ext}_R(G; R)$. If $\psi: G \to H$ is a homomorphism of R-modules, we denote by

$$\psi^*: \mathrm{Hom}\, H \to \mathrm{Hom}\, G \quad \text{and} \quad \psi^\#: \mathrm{Ext}\, H \to \mathrm{Ext}\, G$$

the induced homomorphisms. When G is an R-module, let

$$T(G) = \{a \in G \mid ra = 0 \text{ for some nonzero } r \in R\}.$$

When $\psi: G \to H$ is a homomorphism of R-modules, let $T(\psi): T(G) \to T(H)$ be the restriction of $\psi$ to T(G). The following will be needed in Section 3.

THEOREM 2.1. *Let* $F \xrightarrow{\phi} G \xrightarrow{\psi} H$ *be homomorphisms of finitely generated* R-*modules. If* $\phi^* = 0$ *and* $\psi^\# = 0$, *then* $\psi\phi = 0$.

Theorem 2.1 follows from Lemma 2.2 and Corollary 2.5 below.

LEMMA 2.2. *Let* $\psi: G \to H$ *be a homomorphism of finitely generated* R-*modules. If* $\psi^* = 0$, *then* $\mathrm{Im}\, \psi \subset T(H)$.

*Proof.* Let $x \in G$, $y = \psi(x)$. Suppose that $y \notin T(H)$. Then there exists a homomorphism $\hat{y}: H \to R$ such that $\hat{y}(y) \neq 0$. But

$$[\psi^*(\hat{y})](x) = (\hat{y} \circ \psi)(x) = \hat{y}(y) = 0,$$

by hypothesis, so that $y \in T(H)$.

LEMMA 2.3. *In the category of finitely generated torsion* R-*modules, there exists a natural isomorphism* $G \simeq \mathrm{Ext}\,\mathrm{Ext}\, G$.

*Proof.* Let G be a torsion R-module, and let $C_1$ and $C_0$ be finitely generated free R-modules such that

$$0 \to C_1 \to C_0 \to G \to 0$$

is exact. Then Ext G makes the "Hom" sequence

$$0 \leftarrow \mathrm{Ext}\, G \leftarrow \mathrm{Hom}\, C_1 \leftarrow \mathrm{Hom}\, C_0 \leftarrow 0$$

exact. (Hom G = 0, since G is a torsion module.)

Since rank Hom $C_1$ = rank Hom $C_0$, Ext G is a torsion module. (In fact, Ext $G \simeq G$.) Thus, Ext Ext G makes the sequence

$$0 \to \text{Hom Hom } C_1 \to \text{Hom Hom } C_0 \to \text{Ext Ext } G \to 0$$

exact. It is well-known that there are natural isomorphisms $h_i$: $C_i \simeq$ Hom Hom $C_i$ $(i = 0, 1)$, since the $C_i$ are free. Thus a unique homomorphism $h$: $G \to$ Ext Ext $G$ is induced, and this is the desired isomorphism. The naturality of $h$ follows from that of the $h_i$.

COROLLARY 2.4. *In the category of finitely generated* R-*modules, there is a natural isomorphism* $T(G) \simeq$ Ext Ext G.

*Proof.* If G is a finitely generated R-module, then Ext G is naturally isomorphic to Ext T(G) under $i^{\#}$, where i: $T(G) \subset G$. Hence

$$T(G) \simeq \text{Ext Ext } T(G) \simeq \text{Ext Ext } G,$$

each isomorphism being natural.

COROLLARY 2.5. *Let* $\psi$: $G \to H$ *be a homomorphism of finitely generated* R-*modules. If* $\psi^{\#} = 0$, *then* $T(\psi) = 0$.

*Proof.* $\psi^{\#} = 0$ implies $\psi^{\#\#} = 0$; therefore $T(\psi) = 0$, by Corollary 2.4.

## 3. HOMOLOGICAL UV-PROPERTIES AND COACYCLICITY

In [15], McMillan introduced the concept of strong acyclicity.

*Definition.* Let A be a compact set in the ANR X. We say A has *property* k-uv if to each neighborhood U of A in X there corresponds a neighborhood V of A in U such that the inclusion-induced map $\tilde{H}_k V \to \tilde{H}_k U$ (on reduced singular homology) is zero. If A has properties q-uv for $0 \leq q \leq k$, we say it has property $uv^k$.

Property k-uv does not depend on the embedding $A \subset X$, as long as X is an ANR and A is compact. (See [15], and the argument that (a) ⇒ (d) in [11].)

*Remark.* If A is a finite-dimensional compactum, "A has property $uv^k$ for all k" is equivalent to "A is strongly acyclic" in the sense of [15].

THEOREM 3.1. *Let* A *be a compact set in the* ANR X. *If* $\check{H}^k A = \check{H}^{k+1} A = 0$, *then* $A \subset X$ *has property* k - uv.

*Proof.* We may assume that X is (separable) Hilbert space, since the properties do not depend on particular embeddings.

Let $\{U_1, U_2, \cdots\}$ be a sequence of neighborhoods of A in X with the properties

(1) $\overline{U}_{i+1} \subset U_i$ for each i, and $A = \bigcap U_i$,

(2) each $U_i$ is a finite union of open (round) balls.

By (1) and the continuity property of $\check{H}^*$, we see that $0 = \lim_{\to} H^{\ell} U_i$ for $\ell = k, k+1$. Thus, each element of $H^{\ell} U_i$ hits zero somewhere in the sequence $H^{\ell} U_i \to H^{\ell} U_{i+1} \to \cdots$. By (2), each $H^{\ell} U_i$ is finitely generated, and hence some finite composition $H^{\ell} U_i \to \cdots \to H^{\ell} U_{j(i)}$ is zero for $\ell = k, k+1$ and all i. Taking the subsequence indexed by 1, j(1), jj(1), $\cdots$, we may assume that the following is satisfied:

(3) The inclusion-induced maps $H^\ell U_i \to H^\ell U_{i+1}$ are zero for $\ell = k$, $\ell = k+1$, and $i \geq 1$.

We now apply the universal coefficient theorem to obtain commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Ext } H_{\ell-1} U_{i+1} & \longrightarrow & H^\ell U_{i+1} & \longrightarrow & \text{Hom } H_\ell U_{i+1} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \text{Ext } H_{\ell-1} U_i & \longrightarrow & H^\ell U_i & \longrightarrow & \text{Hom } H_\ell U_i & \longrightarrow & 0 \end{array}$$

in which the vertical arrows are induced by inclusion and the horizontal rows are exact ($\ell = k, k+1$). By (3), the middle vertical arrow is zero, hence all vertical arrows are zero. In particular, the inclusion-induced maps

$$\text{Ext } H_k U_i \to \text{Ext } H_k U_{i+1} \quad \text{and} \quad \text{Hom } H_k U_i \to \text{Hom } H_k U_{i+1}$$

are zero for all i. Applying Theorem 2.1, we obtain the following conclusion.

(4) The inclusion-induced maps $H_k U_{i+2} \to H_k U_i$ are zero for all i. Thus, $A \subset X$ has property k-uv.

THEOREM 3.2. *Let* A *be a compact set in the* ANR X. *If* $A \subset X$ *has properties* (k - 1)-uv *and* k-uv, *then* $\check{H}^k A = 0$.

*Proof.* Let $V_1, V_2, \cdots$ be neighborhoods of A in X such that

(1) $\overline{V}_{i+1} \subset V_i$ for each i, and $\bigcap \overline{V}_i = A$, and

(2) the inclusion-induced maps $H_\ell V_{i+1} \to H_\ell V_i$ are zero for $\ell = k-1$, $\ell = k$, and all i.

Applying again the universal coefficient theorem, we obtain commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Ext } H_{k-1} V_{i+1} & \longrightarrow & H^k V_{i+1} & \longrightarrow & \text{Hom } H_k V_{i+1} & \longrightarrow & 0 \\ & & \uparrow & & \phi_i \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \text{Ext } H_{k-1} V_i & \longrightarrow & H^k V_i & \longrightarrow & \text{Hom } H_k V_i & \longrightarrow & 0, \end{array}$$

with exact rows. Moreover, (2) implies that the vertical arrows on either end are zero. We deduce easily that $\text{Im } \phi_i \subset T(H^k V_{i+1})$ and $\phi_i \,|T(H^k V_i) = 0$. Hence $\phi_{i+1} \phi_i = 0$, and the proof is complete.

COROLLARY 3.3. *Let* A *be a finite-dimensional compact set in the* ANR X. *Then* A *is strongly acyclic* (*in the sense of* [15]) *if and only if* $\check{H}^* A = 0$.

*Remark.* The above results hold equally well for property k-uv(R) and $\check{H}(- ; R)$, where R is a principal ideal domain.

## 4. HUREWICZ THEOREMS FOR UV-PROPERTIES

THEOREM 4.1. *Let* A *be a compact set in the* ANR X. *If* $A \subset X$ *has property* $UV^k$, *then it has property* $uv^k$.

*Proof.* Let U be a neighborhood of A in X. Find open sets $U_0, \cdots, U_{k+1}$, with

$A \subset U_0 \subset \cdots \subset U_{k+1} \subset U$, such that each map $S^q \to U_q$ extends to a map $B^{q+1} \to U_{q+1}$ $(0 \leq q \leq k)$. Let $V = U_0$.

Let K be a complex of dimension at most k, and let C(K) denote the cone on K. If f: $K \to V$ is a map, we can extend f successively over $K \cup C(K^q)$, using the inclusion $U_q \subset U_{q+1}$, to obtain a map $C(K) \to U$. In this way, we can easily see that each singular q-cycle in V is null-homologous in U $(0 \leq q \leq k)$, so that $A \subset X$ has property $uv^k$.

THEOREM 4.2. *Suppose that* A *is a compact set in the* ANR X, *and that* $k \geq 2$. *If* $A \subset X$ *has properties* $UV^{k-1}$ *and* k-uv, *then it has property* $UV^k$.

*Proof.* (Compare with [8, pp. 483-485].) Since A is connected, it has arbitrarily small path-connected neighborhoods. Let $V \subset U_0 \subset \cdots \subset U_k \subset U$ be path-connected neighborhoods of A in X, chosen so that $H_k V \to H_k U_0$ is zero, $\pi_q U_q \to \pi_q U_{q+1}$ is zero for $0 \leq q \leq k-1$, and $\overline{U}$ is compact. We shall show that $\pi_k V \to \pi_k U$ is zero.

Let $\alpha$: $S^k \to V$ be a map. Then $[\alpha] = 0$ in $H_k U_0$. Therefore, there is a subdivision L of $S^k$ such that $\sum_i \alpha\tau_i = \partial c$ for some finite singular $(k+1)$-chain $c = \sum_j n_j \sigma_j$. (Here, the $\tau_i$ are the simplicial maps $\Delta^k \to L$ determined by some ordering of the vertices of L.) Letting K be the geometric realization of the (finite) singular complex determined by $\{\sigma_j\}$, we obtain a complex containing L and an extension $\beta$: $K \to U_0$ of $\alpha$. Let K' be the union of K and the cone on its $(k-1)$-skeleton. Since $\pi_q U_q \to \pi_q U_{q+1}$ is zero for $0 \leq q \leq k-1$, we can extend $\beta$ over successive skeleta to a map $\overline{\alpha}$: $K' \to U$. Now, $[L] = 0$ in $H_k K$, hence in $H_k K'$; therefore, by the classical Hurewicz theorem, $[S^k] = 0$ in $\pi_k |K'|$. Hence $[\overline{\alpha} \mid S^k] = [\alpha] = 0$ in $\pi_k U$, and the proof is complete.

*Remark.* If A is a compact set in the locally path-connected metric space X, then property 0-UV and property 0-uv are each equivalent to connectivity of A.

COROLLARY 4.3. *Let* A *be a compact set in the* ANR X. *Suppose* $A \subset X$ *has property* 1-UV. *Then* $A \subset X$ *has property* $UV^k$ *if and only if it has property* $uv^k$.

COROLLARY 4.4. *Let* A *be a compact set in the* ANR X.

1. *If* $A \subset X$ *has property* $UV^k$, *then* $\tilde{\check{H}}^q A = 0$ *for* $0 \leq q \leq k$.

2. *If* $A \subset X$ *has property* $UV^{k-1}$ *and* $\check{H}^k A = \check{H}^{k+1} A = 0$, *where* $k \geq 2$, *then* $A \subset X$ *has property* $UV^k$.

Corollary 4.3 follows from Theorems 4.1 and 4.2 together with the Remark. Corollary 4.4 requires, in addition, some results from Section 3.

## 5. CRITERIA FOR MAPS TO BE CELL-LIKE

LEMMA 5.1. *Let* X *and* Y *be connected, locally compact* ANR's, *and let* f *be a proper* $UV^k$*-map of* X *onto* Y. *Let* V *be an open set in* Y, *and let* $U = f^{-1}(V)$. *Then*

$$f_\#: \pi_q(X, U) \to \pi_q(Y, V) \quad \textit{and} \quad f_*: H_q(X, U) \to H_q(Y, V)$$

*are isomorphisms for* $0 \leq q \leq k$ *and epimorphisms for* $q = k+1$.

*Proof.* First assume $U = V = \emptyset$. Then $f_\#: \pi_q X \to \pi_q Y$ is an isomorphism for

$0 \leq q \leq k$ and an epimorphism for $q = k + 1$ (see [12, Corollary 2.4]). If $Z_f$ denotes the mapping cylinder of f, we see that $\pi_q(Z_f, X) = 0$ for $q \leq k + 1$. By the relative Hurewicz theorem, $H_q(Z_f, X) = 0$ for $q \leq k + 1$, and hence $f_*: H_q X \to H_q Y$ is an isomorphism for $q \leq k$ and an epimorphism for $q = k + 1$. The general cases now follow from a standard generalization of the five-lemma.

LEMMA 5.2. *If* $f: X \to Y$ *is a proper, onto map between euclidean neighborhood retracts, and if* f *is* $UV^k$*-trivial for all* k, *then* f *is cell-like.*

This is a special case of Theorem 2.1 of [12].

THEOREM 5.3. *Let* Y *be a compact* ANR *such that* $\tilde{H}_p(Y - \{y\}) = 0$ *for all* $p \leq n - k - 2$ *and all* $y \in Y$, *and let* $f: S^n \to Y$ *be an onto* $UV^k$*-map. If* $2k + 2 \geq n$, *then* f *is cell-like.*

*Proof.* ($\tilde{H}_*$ is reduced homology.) Let $A = f^{-1}(y)$ for some $y \in Y$. Then, by Alexander duality and Lemma 4.1,

$$\check{H}^q A \simeq \tilde{H}_{n-q-1}(S^n - A) \simeq \tilde{H}_{n-q-1}(Y - \{y\}) = 0,$$

provided $n - q - 1 \leq k$ and $n - q - 1 \leq n - k - 2$. Since $n - k - 2 \leq k$, we see that $\check{H}^q A = 0$ whenever $n - q - 1 \leq n - k - 2$, that is, whenever $q \geq k + 1$. Thus, at least when $k \geq 1$, the inclusion $A \subset S^n$ has property $UV^q$ for all q, by Corollary 4.4.2, so that f is cell-like, by Lemma 5.2.

If, on the other hand, $k = 0$ and $n = 2$, it is to be shown that $A \subset S^2$ has property $UV^1$ whenever $S^2 - A$ and A are connected. We leave this statement, as well as the case $k = 0$, $n = 1$, to the reader. (See [22].)

THEOREM 5.4. *Let* M *and* N *be* n*-manifolds. If* $f: M \to N$ *is a proper, onto* $UV^k$*-map, and* $2k + 1 \geq n$, *then* f *is cell-like.*

*Proof.* Let $A = f^{-1}(y)$ for some y. Then $A \subset M$ has property $UV^k$. Assuming $n \geq 2$, we can let V be an open n-cell of N containing y, so that $U = f^{-1}(V)$ is a simply connected (hence orientable) neighborhood of $f^{-1}(y)$ in M. Consider the composition

$$\check{H}^q A \xrightarrow{D_A} H_{n-q}(U, U - A) \xrightarrow{f_*} H_{n-q}(V, V - \{y\}),$$

where $D_A$ is the duality isomorphism of Theorem 6.2.17 of [20]. If $n - q \leq k$, then $f_*$ is an isomorphism, by Lemma 5.1. Hence, $\check{H}^q A = 0$ for $q \geq n - k$. Since $n - k \leq k + 1$, $\check{H}^q A = 0$ for $q \geq k + 1$. Again, if $k \geq 1$, we are through, by Corollary 4.4.2 and Lemma 5.2, and we leave the case $k = 0$, $n = 1$ to the reader.

(*Note.* The referee has pointed out that the triviality of $\check{H}^* A$ follows from Theorem 4 of [10].)

*Remark.* Theorem 5.4 has a generalization similar to Theorem 5.3.

## 6. PROOFS AND EXAMPLES

Theorems 1.1 and 1.2 follow from Section 5 together with results from [12] and [13]. In [12], it is shown that proper, cell-like maps between (unbounded) topological manifolds of dimension at least 5 are cellular, which takes care of Theorem 1.2. Also, from Theorem 1.3 (which is proved in [12]), we see that if $f: M \to N$ is a proper cell-like map between topological manifolds of dimension at least 3, then each inclusion $f^{-1}(y) \subset M$ satisfies McMillan's criterion [13], so that Theorem 1.1 follows. The cases $n \leq 2$ follow from classical results. (See [22].)

We now give some examples to show that the codimensional restrictions in Theorems 5.3 and 5.4 are best possible.

*Example* (compare with [2, p. 7]). Let $n = k + \ell + 1$, and write $S^n$ as the join $S_0^k * S_1^\ell$ of two spheres. (That is, let $S^n = (S_0 \times S_1 \times I)/_\sim$, where "$\sim$" identifies $S_0 \times y \times 1$ and $x \times S_1 \times 0$ to points, for all $x \in S_0$, $y \in S_1$.) If $0 < t < 1$, let $T_t$ be the copy of $S_0 \times S_1 \times t$ at level $t$ in the join, and let

$$W_t = (S_0 \times s_1 \times t \cup s_0 \times S_1 \times t)/_\sim,$$

where $s_i$ is a base point in $S_i$. Note that $T_t/W_t \approx S^{n-1}$. Making the homeomorphism "independent" of $t$, we see that there are maps $f_t\colon T_t \to S^{n-1}$ such that the only nondegenerate point-inverse of $f_t$ is $W_t$, and $f_t$ varies continuously with $t$ $(0 < t < 1)$. Now we map $S^n$ onto $S(S^{n-1}) \approx S^n$ by defining

$$f \mid T_t = f_t \times t \quad (0 < t < 1), \qquad f(S_0) = 0, \qquad f(S_1) = 1.$$

COROLLARY 6.1. *If* $2k + 3 \leq n$ *and* $k \geq -1$, *there exists a map* $f$ *of* $S^n$ *onto itself whose point-inverses are tame,* $k$-*connected polyhedra but that has some point-inverses that are not* $(k+1)$-*connected. Hence,* $f$ *is* $UV^k$-*trivial but not cell-like.*

*Example.* If Corollary 6.1 fails to show that Theorem 5.4 is best possible, then $2k + 3 > n$ while $2k + 2 \leq n$. In other words, $2k + 2 = n$. For this case, we can map $S^{k+1} \times S^{k+1}$ onto $S^n$ by a map whose only nondegenerate inverse set is $S^{k+1} \times s \cup s \times S^{k+1}$.

COROLLARY 6.2. *If* $2k + 2 \leq n$, *there exist closed, orientable,* $k$-*connected,* PL $n$-*manifolds* $M$ *and* $N$ *and an onto map* $f\colon M \to N$ *whose point-inverses are tame,* $k$-*connected polyhedra but not all of whose point-inverses are* $(k+1)$-*connected. Thus,* $f$ *is* $UV^k$-*trivial but not cell-like.*

## 7. THE NONCELLULAR POINTS OF A MAP BETWEEN EVEN-DIMENSIONAL MANIFOLDS

Let $M$ and $N$ be $n$-manifolds, and let $f\colon M \to N$ be an onto, proper $UV^{k-1}$-map. Define

$$C_f = \{y \in N \mid f^{-1}(y) \text{ is not cellular in } M\}.$$

As the "join" example in Section 6 shows, $C_f$ may be one-dimensional, whenever $2k < n$, even if $M = N = S^n$. Moreover, Theorem 1.2 shows (modulo certain plausible conjectures when $n = 3$ or $4$) that $C_f = \emptyset$ whenever $2k > n$. In this addendum, we consider the remaining case $2k = n$.

It is easy, but instructive, to see that in case $n = 2k$, $C_f$ can be a finite set with any number of points: For $p \geq 0$, let $T_p$ be the connected sum of $S^{2k}$ and $p$ copies of $S^k \times S^k$. Then, if $0 \leq q \leq p$, there exists a map of $T_p$ onto $T_q$ that has exactly $p - q$ nondegenerate point-inverses, each of which is a wedge of two $k$-spheres. (In the noncompact case, a similar construction shows that $C_f$ can be an infinite discrete set in $N$.) We show below that these examples are typical. (Compare with [14].)

Throughout this section, we assume that

(i) M and N are closed manifolds of even dimension 2k,

(ii) $f: M \to N$ is an onto $UV^{k-1}$-map,

(iii) for each $y \in N$, there is a neighborhood V of $f^{-1}(y)$ in M such that $V - f^{-1}(y)$ can be triangulated as an open PL-manifold, and

(iv) $k \neq 2$.

*Note.* When $k = 1$, assumption (ii) means simply that f is a monotone map of M onto N.

THEOREM 7.1. $C_f$ *is a finite set.*

*Proof.* Let $y \in N$ and $W = M - f^{-1}(y)$. Then W is an open 2k-manifold that is (k - 1)-connected at infinity (by Lemma 5.1).

Assume first that $k \geq 3$. Let

$$U = f^{-1}(\text{open } 2k\text{-cell containing } y).$$

U is (k - 1)-connected, hence orientable. We want to calculate the homology of W. If $q \leq k - 1$, then $H_q W \simeq H_q(N - \{y\})$. For $k + 1 \leq q \leq 2k - 1$, the homology sequence of the pair (M, W) implies that $H_q W \simeq H_q M$, since

$$H_q(M, W) \simeq H_q(U, U - f^{-1}(y)) \simeq \check{H}^{2k-q} f^{-1}(y) = 0$$

by excision, duality, and Corollary 4.4.1. Finally, the middle of the sequence of (M, W) looks like

$$H_{k+1}(M, W) \to H_k W \to H_k M,$$

where $H_{k+1}(M, W) = 0$, as we noted above. Thus

$$H_q W \simeq \begin{cases} H_q(N - \{y\}) & \text{for } q \leq k - 1, \\ \text{a subgroup of } H_k M & \text{for } q = k, \\ H_q M & \text{for } k + 1 \leq q \leq 2k - 1, \\ 0 & \text{for } 2k \leq q. \end{cases}$$

From this, we see that $H_* W$ is finitely generated. Therefore, we can apply the main result of [4] (for PL-manifolds; see [17]) to see that there exists a compact mainfold $\overline{W}$ such that $W = \overline{W} - \partial\overline{W}$. Since $\partial\overline{W}$ is (k - 1)-connected, it is a (2k - 1)-sphere, by the generalized Poincaré conjecture [6], [19]. It follows that there exists a compact set K in W such that W - K is homeomorphic to $S^{2k-1} \times R$.

In the case $k = 1$, W is an open 2-manifold with exactly one end. An easy argument shows that W is the connected sum of infinitely many closed manifolds. Since M is compact, almost all of these closed manifolds must be spheres. As above, it follows that W - K is homeomorphic to $S^1 \times R$ for some compact set K.

In terms of neighborhoods of $f^{-1}(y)$, we have proved the following result.

COROLLARY 7.2. *For each* $y \in N$, $f^{-1}(y)$ *has a neighborhood* V *in* M *such that*

$$V - f^{-1}(y) \approx S^{2k-1} \times R .$$

Now let

$$D = \{y \in N \mid f^{-1}(y) \text{ does not lie in a topological copy of } S^{2k-1} \times R \text{ in } M\} .$$

An easy argument with limit points proves that D is a finite set. Let $y \in N - D$. Then $f^{-1}(y)$ has two neighborhoods $V \subset U$, where

$$U \approx S^{2k-1} \times R \quad \text{and} \quad V - f^{-1}(y) \approx S^{2k-1} \times R .$$

Since the two-point compactification of U is a 2k-sphere, the generalized Schoenflies theorem [5] implies that $f^{-1}(y)$ is cellular in U, and hence in M. Thus $C_f = D$, and $C_f$ is finite. The proof of Theorem 7.1 is now complete.

Let $C_f = \{y_1, \cdots, y_t\}$. For each $y_j$, let $V_j$ be a neighborhood of $f^{-1}(y_j)$ such that

$$V_j - f^{-1}(y_j) \approx S^{2k-1} \times R .$$

Choose the $V_j$ so that they are pairwise disjoint. Let $M_j$ be the one-point compactification of $V_j$, and let M' be the end-point compactification of $M - f^{-1}(C_f)$.

Define $f'\colon M' \to N$ by letting

$$f' = f \text{ on } M - f^{-1}(C_f) \quad \text{and} \quad f'(\text{end-point determined by } f^{-1}(y_j)) = y_j .$$

COROLLARY 7.3. *M' is a closed manifold, and $f'\colon M' \to N$ is a cellular map. Moreover, M is homeomorphic to the connected sum of M' and the closed,* (k - 1)-*connected manifolds* $M_1, \cdots, M_t$.

The result of Siebenmann (Theorem 1.4) now implies that f' is homotopic to a homeomorphism. Thus we obtain the following result.

COROLLARY 7.4. M *is homeomorphic to the connected sum of* N *and a closed,* (k - 1)-*connected manifold.*

We can now prove the following generalization of Theorem 1.1.

THEOREM 7.5. *If $H_k M$ and $H_k N$ are isomorphic, then f is a cellular map.*

The proof is separated into two cases.

*Case* 1. k = 1. We have three two-manifolds, M, M', and N, where M' is homotopy-equivalent to N and $H_1 M \simeq H_1 N$. Thus $M \approx M'$. That is, M is homeomorphic to the connected sum of itself and $M_1, \cdots, M_t$. A Meyer-Vietoris argument shows that each $M_j$ is a sphere, and hence each $V_j$ is an open 2-cell.

*Case* 2. $k \geq 3$. In this case, we need only show that $H_k V_j = 0$ for $j = 1, \cdots, t$; for then f is $UV^{k-1}$-trivial and $uv^k$-trivial, and therefore it is $UV^k$-trivial (by Theorem 4.2) and hence cellular (by Theorem 1.2).

Applying the Meyer-Vietoris sequence in homology, we see that

$$H_k M \simeq H_k M' \oplus H_k V_1 \oplus \cdots \oplus H_k V_t .$$

Since f' is cellular, $H_k M' \simeq H_k N$; moreover, $H_k N \simeq H_k M$ by hypothesis. But $H_k M$ is finitely generated, and we conclude that $H_k V_j = 0$ for all j. This concludes the proof.

*Remark* 1. (a) Assume that either $k \geq 3$ or M is orientable. Then the manifolds $M_j$ are $(k-1)$-connected and orientable, and therefore the only nonvanishing homology groups of $M_j$ occur in dimensions 0, k, and 2k; hence $H_k M_j$ is a free abelian group of even rank. Moreover, we have isomorphisms

$$\check{H}^q f^{-1}(y_j) \simeq H^q M_j \simeq H_{2k-q} M_j$$

for $q < 2k$. Hence, for each $y \in N$, $\check{H}^q f^{-1}(y)$ is a free abelian group of even rank for $q = k$, and

$$\check{H}^q f^{-1}(y) \simeq \begin{cases} Z & \text{for } q = 0, \\ 0 & \text{for } q \neq 0, k. \end{cases}$$

Thus, $f^{-1}(y)$ can be a wedge of an *even* number of k-spheres (as in the standard examples) but never a wedge of an *odd* number of k-spheres.

(b) When $k = 1$ and M is not orientable, $\check{H}^1 f^{-1}(y)$ is a free abelian group, but *not* necessarily of even rank: the projective plane maps onto $S^2$ by shrinking the center line of a Möbius band to a point.

*Remark* 2. Assumption (iii) can be dropped, except possibly when $k = 4$. For, when $k \geq 5$, each $f^{-1}(y)$ has a 4-connected neighborhood with a (unique) PL-manifold structure, by the triangulation theorem of R. C. Kirby and L. C. Siebenmann [9]. When $k = 3$, duality and Theorem 5.1 imply that

$$\check{H}^4(f^{-1}(y); Z_2) \simeq H_2(M, M - f^{-1}(y); Z_2) = 0;$$

therefore some neighborhood of $f^{-1}(y)$ has a PL-manifold structure, by a new result of J. Hollingsworth and R. B. Sher [7]. In fact, duality and [7] yield the following (this was pointed out to me by McMillan).

PROPOSITION. *Let* X *and* Y *be (open or closed) topological* n*-manifolds, and let* $g: X \to Y$ *be a proper, onto map. Suppose that each inclusion* $g^{-1}(y) \subset X$ *has property* $uv^k(Z_2)$, *where* $k = \min\{4, n-4\}$ *and* $n \geq 5$. *Then each* $g^{-1}(y)$ *has a neighborhood in* X *that can be triangulated as a* PL*-manifold.*

(Property $uv^k(Z_2)$ is defined similarly to property $uv^k$, except that we use homology with $Z_2$-coefficients.)

*Remark* 3. There is a version of Theorem 7.1 for open manifolds. The proof is essentially the same as that of Theorem 7.1, but the proposition is used to complete assumption (iii).

THEOREM 7.6. *Let* X *and* Y *be* 2k*-manifolds (without boundary), and let* $g: X \to Y$ *be a proper, onto* $UV^{k-1}$*-map. Assume that* $k \neq 2$, *and if* $k = 4$, *that each* $g^{-1}(y)$ *has a neighborhood* V *in* X *such that* $V - g^{-1}(y)$ *has a* PL*-manifold structure. Then* $C_g$ *is a closed, locally finite subset of* Y.

*Remark* 4. The analogue of Theorem 7.5 for open manifolds is false: There exists a proper $UV^{k-1}$-map g of $T^{2k}_\infty$ onto itself for which $C_g$ is infinite.

## REFERENCES

1. S. Armentrout and T. M. Price, *Decompositions into compact sets with* UV *properties*. Trans. Amer. Math. Soc. 141 (1969), 433-442.

2. R. H. Bing, *Decompositions of* $E^3$. Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), pp. 5-21. Prentice-Hall, Englewood Cliffs, N. J., 1962.

3. K. Borsuk, *Concerning homotopy properties of compacta.* Fund. Math. 62 (1968), 223-254.

4. W. Browder, J. Levine, and G. R. Livesay, *Finding a boundary for an open manifold.* Amer. J. Math. 87 (1965), 1017-1028.

5. M. Brown, *A proof of the generalized Schoenflies theorem.* Bull. Amer. Math. Soc. 66 (1960), 74-76.

6. E. H. Connell, *A topological* H-*cobordism theorem for* $n \geq 5$. Illinois J. Math. 11 (1967), 300-309.

7. J. Hollingsworth and R. B. Sher, *Triangulating neighborhoods in topological manifolds* (to appear).

8. W. Hurewicz, *Homotopie, Homologie und lokaler Zusammenhang.* Fund. Math. 25 (1935), 467-485.

9. R. C. Kirby and L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung.* Bull. Amer. Math. Soc. 75 (1969), 742-749.

10. K. W. Kwun and F. Raymond, *Almost acyclic maps of manifolds.* Amer. J. Math. 86 (1964), 638-650.

11. R. C. Lacher, *Cell-like spaces.* Proc. Amer. Math. Soc. 20 (1969), 598-602.

12. ———, *Cell-like mappings, I.* Pacific J. Math. 30 (1969), 717-731.

13. D. R. McMillan, Jr., *A criterion for cellularity in a manifold.* Ann. of Math. (2) 79 (1964), 327-337.

14. ———, *Strong homotopy equivalence of 3-manifolds.* Bull. Amer. Math. Soc. 73 (1967), 718-722.

15. ———, *Compact, acyclic subsets of three-manifolds.* Michigan Math. J. 16 (1969), 129-136.

16. C. P. Rourke, *The Hauptvermutung according to Sullivan, Part II.* Lecture Notes, Institute for Advanced Study, Princeton, 1968.

17. L. C. Siebenmann, *On detecting open collars.* Trans. Amer. Math. Soc. 142 (1969), 201-227.

18. ———, *Approximating cellular maps by homeomorphisms.* Notices Amer. Math. Soc. 17 (1970), 532 (Abstract #674-40). (The complete paper is to appear.)

19. S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four.* Ann. of Math. (2) 74 (1961), 391-406.

20. E. H. Spanier, *Algebraic topology.* McGraw-Hill, New York, 1966.

21. D. Sullivan, to appear in the Proceedings of the 1969 Georgia Topology Conference.

22. G. T. Whyburn, *Analytic topology.* American Mathematical Society Colloquium Publications, Vol. 28. American Mathematical Society, New York, 1942.

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