ON THE STABLE SUSPENSION HOMOMORPHISM

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1. INTRODUCTION

The suspension homomorphism $s^q\colon \pi_i X \to \pi_{i+q} S^q X$ has been studied and exploited for a long time. Many of its properties are known, such as the fact that s^q is an isomorphism if X is k-connected and $i \leq 2k$. Let $\pi_i^s X$ denote the stable group $\pi_{i+q} S^q X$, for large values of q, and let $s = s^q$. In this paper, we prove a refinement of a theorem about the homomorphism $s\colon \pi_i X \to \pi_i^s X$ when we are in the metastable range, that is, when X is k-connected and $i \leq 3k$. It is probably the best possible theorem of this type. Our result should not be surprising, because there is much that can be done in the so-called metastable range in general. The condition of being in that range appears naturally in numerous problems. For example, the metastable range plays an important role in the imbedding theorems of [5] and [10], in the computation of the homotopy groups of S^n and O_n (see [2] and [11]), and in the E-H-P sequence of G. Whitehead [17].

Throughout this paper, C_t will denote the class of finite abelian groups whose elements have orders that divide some power of the order of $\pi_1^s \oplus \cdots \oplus \pi_t^s$, where $\pi_i^s = \pi_i^s S^0$ is the ith stable homotopy group of the sphere (C_t is the zero class for $t \leq 0$). By C, we denote the class of all finite abelian groups. For simplicity, all spaces will be finite CW-complexes with base points, which, however, will frequently be ignored.

Our main theorem follows.

THEOREM 1. *Let* X *be a* (k - 1)*-connected space.*

(a) *If* $2k \geq n + 3$, *then* $s\colon \pi_{n+k+1} SX \to \pi_{n+k}^s X$ *is* C_{n-k}*-onto.*

(b) *Let* K *denote the kernel of the map* $s\colon \pi_{n+k} X \to \pi_{n+k}^s X$. *If* $2k \geq n + 4$, *then* $s^1(K) \subseteq \pi_{n+k+1} SX$ *belongs to* C_{n-k+1}; *in other words,* $s \mid s^1(\pi_{n+k} X)$ *is a* C_{n-k+1}*-monomorphism.*

Note. It was pointed out to the author that Theorem 1 is true without any connectivity or dimension hypotheses if we replace C_{n-k} by C. This follows from theorems in homotopy theory and knowledge of the structure of Hopf algebras (see [13]). However, Theorem 1 is of value for two reasons: First, we obtain more information, because we use the classes C_{n-k} and C_{n-k+1}; and second, differential topologists would consider our proof elementary and much simpler than the proof of the general result. (Our proof may seem somewhat long, but this is due to Lemma 4, which we need to overcome a technical problem. The basic idea is really contained in Lemma 5.)

We also note that it is not always true in (a) that $\pi_{n+k} X \to \pi_{n+k}^s X$ is C-onto (consider the Eilenberg-MacLane space $X = K(\mathbb{Z}, k)$), nor is the group K in (b) always finite ($\pi_{4q-1} S^{2q} \to \pi_{4q-1}^s S^{2q}$ has infinite kernel).

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Theorem 1 can be used to study properties of the Hurewicz homomorphism (see Theorem 3). Its proof uses Theorem 2, which is interesting in its own right. All manifolds are compact, oriented C^∞-manifolds.

Definition. Let C' be a class of abelian groups. A manifold V^m is said to be *C'-spherical* if $\pi_i(V, \partial V) \in C'$, for $0 < i < m$.

Note. This definition extends the one given in [8].

THEOREM 2. *Let V^{n+k} be a π-manifold such that $H_i V = 0$ for $i > n$, $H_n V$ is torsion-free, and $\pi_1 \partial V \to \pi_1 V$ is an isomorphism. Suppose $n \geq 2$ and $2k \geq n + 3$. Set $\alpha = [(n+k)/2]$. Then V imbeds in an $(\alpha - 1)$-connected C_{n-k}-spherical π-manifold W^{n+k+1} with 1-connected boundary and $H_i W = 0$, for $i > n$. If ∂V is a homotopy sphere, then we may also assume that $\partial V \subseteq \partial W$.*

2. PROOF OF THEOREM 2

Throughout this section, we shall assume that V^{n+k} is a π-manifold such that $H_i V = 0$ for $i > n$, $H_n V$ is torsion-free, and $\pi_1 \partial V \to \pi_1 V$ is an isomorphism. Furthermore, we assume that $n \geq 2$ and $2k \geq n + 3$; hence $n + k \geq 5$. Set $\alpha = [(n+k)/2]$, $\beta = [(n+k+1)/2]$, and $W_0 = V \times [0, 1]$.

It follows from the proof of Theorem 2.1 of [1] (by adding handles to ∂W_0 away from $V \times 1$) that W_0 imbeds in an $(\alpha - 1)$-connected π-manifold W^{n+k+1} such that

(a) ∂W is 1-connected,

(b) $H_i W = 0$ for $i > n$,

(c) $H_i W \approx H_i W_0$ for $i > \alpha$, and

(d) $H_i \partial W \to H_i W$ is onto for $i \leq \beta$.

One can easily see that ∂W is $(k - 1)$-connected, by considering the exact homology sequence of the pair $(W, \partial W)$ and observing that

$$H_i(W, \partial W) \approx H^{n+k+1-i} W = 0 \qquad \text{if } i \leq k,$$

which is a consequence of the universal coefficient theorem and the fact that $H_n V$ is torsion-free. We should like to kill the rest of the homology of W by continuing to attach handles to ∂W. Unfortunately, we encounter two problems: (1) we cannot realize all homology classes of W as embedded spheres in ∂W, and (2) even if we could, it is not possible to determine whether they have trivial normal bundles (they are always stably trivial). This is why we must be satisfied with making W only C_{n-k}-spherical.

LEMMA 1. *Let X be an ℓ-connected space. Then the Hurewicz map $h: \pi_i X \to H_i X$ is a $C_{i-\ell-1}$-isomorphism for $0 < i \leq 2\ell$, and it is $C_{i-\ell-1}$-onto for $i = 2\ell + 1$.*

We shall prove Lemma 1 in Section 3. There are many known proofs of this lemma if we replace $C_{i-\ell-1}$ by C.

LEMMA 2. *If $j: SO_q \to SO$ is the natural inclusion, then $j_\#: \pi_i SO_q \to \pi_i SO$ is a C_{i-q+1}-isomorphism for $q \leq i \leq 2q - 3$. If $i = q - 1$, then $j_\#$ is a C_1-isomorphism provided $i \not\equiv 3 \pmod 4$, and $j_\#$ has an infinite kernel if $i \equiv 3 \pmod 4$.*

Proof. If we consider the fibrations $SO_\ell \to SO_{\ell+1} \to S^\ell$, we see that $\pi_i SO_\ell$ and $\pi_i SO_{\ell+1}$ are $C_{i-\ell+1}$-isomorphic whenever $\ell < i < 2\ell - 2$. (The upper bound insures that the homotopy groups of S^{ℓ} are stable.) This shows that $\pi_i SO_q$ and $\pi_i SO_i$ are C_{i-q+1}-isomorphic for $q \leq i \leq 2q - 3$. Thus, the first part of the lemma follows from the well-known fact that $\pi_i SO_i \to \pi_i SO$ is a C_1-isomorphism for all i (see the tables of [6] and [14]). The second part of the lemma also follows from [6] and [14].

We shall use the following lemma frequently.

LEMMA 3. *Let* U^{n+k} *denote a* (k - 1)*-connected* π*-manifold, where* $2k \geq n + 3$. *Let* $\ell \leq n$, *and suppose* $x \in \pi_\ell U$ *can be represented by an imbedding* ϕ: $S^{\ell} \to U$ *with normal bundle* ν_ϕ, *which we shall consider as being an element of* $\pi_{\ell-1} SO_{n+k-\ell}$. *If* ν_ϕ *has order* $d \geq 1$, *then* $d \cdot x$ *can be represented by an imbedding* ϕ': $S^{\ell} \to U$ *with trivial normal bundle.*

Proof. This lemma is an easy consequence of [5]. Observe that we do not assert that ν_ϕ is uniquely determined by x. To make that claim, we should need that $2k > n + 3$. However, for each fixed ϕ, one can explicitly construct an imbedding ϕ': $S^{\ell} \to U$ that represents $d \cdot x$ and has trivial normal bundle.

Definition. If C' is a class of finite abelian groups, we let $\Pi(C')$ denote the set of primes that are relatively prime to the orders of every group in C'. We let I(C') denote the set of nonzero integers that are relatively prime to every prime in $\Pi(C')$. If G is an abelian group and p is a prime, then $r_p(G)$ will denote the p-rank of G, that is,

$$r_p(G) = \dim (Z_p \otimes (\text{torsion subgroup of } G))$$

(see [4]).

LEMMA 4. *The manifold* W_0 *imbeds in an* $(\alpha - 1)$*-connected* π*-manifold* W^{n+k+1} *with* 1*-connected boundary such that* $H_i W \approx H_i W_0$ *for* $i > \beta$ *and* $H_i W \in C_{n-k}$ *for* $i \leq \beta$.

Note. Throughout the proof of this lemma and the next, we shall frequently have need to replace the particular element under consideration at the time, say u, by some multiple of itself, say tu, where $t \in I(C_{n-k})$. We do not always state explicitly that such a replacement has been made, although it should be clear from the context. The justification of this procedure lies in the fact that we do everything only modulo C_{n-k} anyway. In the special case where the order of u belongs to $\Pi(C_{n-k})$, we choose t such that tu = u.

Proof. Let W be as in the second paragraph of this section, so that $H_i \partial W \to H_i W$ is C_{n-k}-onto (in fact, onto) for $i \leq \beta$. We consider two cases.

Case 1. n + k is odd, that is, $n + k = 2\alpha + 1$ and $\beta = \alpha + 1$.

Let $r = \text{rank}(H_\alpha W)$. We show first that we may assume $H_\alpha W$ is finite, by induction on r. If r = 0, we are done. Therefore, let r > 0, and let $u \in H_\alpha W$ be an element of infinite order.

Consider the exact sequence:

$$\cdots \to H_{\alpha+2}(W, \partial W) \xrightarrow{\partial} H_{\alpha+1} \partial W \to H_{\alpha+1} W \to H_{\alpha+1}(W, \partial W) \to H_\alpha \partial W \xrightarrow{i_*} H_\alpha W \to \cdots.$$

By assumption, i_* is C_{n-k}-onto, and thus there exists an $m \in I(C_{n-k})$ and a $u_1 \in H_\alpha \partial W$ such that $i_*(u_1) = mu$. Next, Poincaré duality implies that there exists a $v \in H_{\alpha+2}(W, \partial W)$ with $mu \cdot v = 1$ ($\cdot$ denotes the intersection pairing). Thus

$$1 = mu \cdot v = i_*(u_1) \cdot v = (-1)^\alpha u_1 \cdot \partial(v) .$$

Let $v_1 = \partial(v)$.

Now, since ∂W is (k - 1)-connected, we may assume by Lemma 1 that u_1 can be represented by an imbedding ϕ: $S^\alpha \times D^{\alpha+1} \to \partial W$. Let $W_1 = W \cup D^{\alpha+1} \times D^{\alpha+1}$, where we identify $x \in S^\alpha \times D^{\alpha+1}$ with $\phi(x) \in \partial W$, and let

$$N = \partial W - (\text{interior of } \phi(S^\alpha \times D^{\alpha+1})).$$

If necessary, modify ϕ in the standard way by an element of $\pi_{\alpha-1} SO_{\alpha+1}$ such that W_1 is a π-manifold (see [7]).

Consider Diagram I, which has exact horizontal and vertical sequences (see [7, Lemma 5.6] for some of the details).

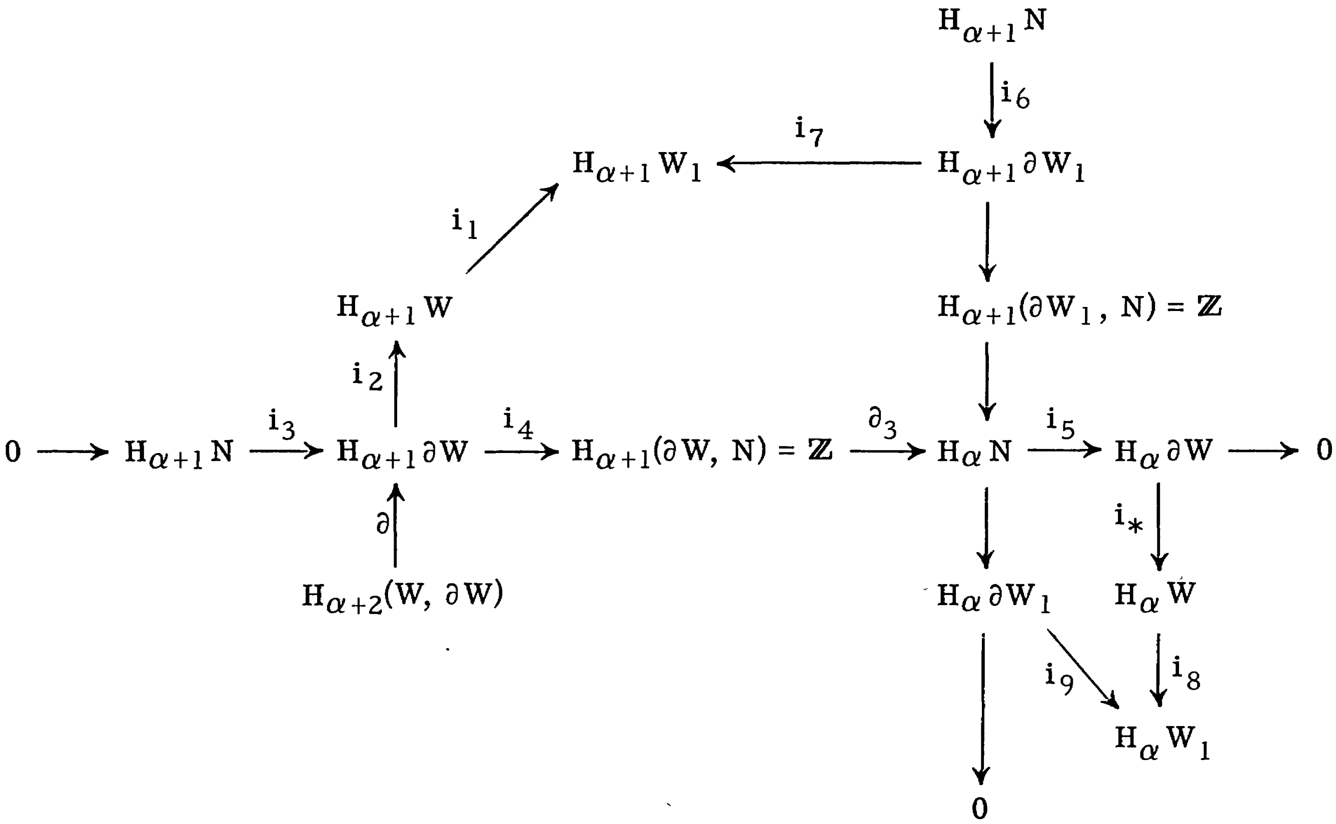


Diagram I

By construction, $i_8(mu) = 0$, and therefore we find that i_1 is an isomorphism by looking at the homology sequence of the pair (W_1, W). Furthermore,

$$i_4(v_1) = v_1 \cdot u_1 = \pm 1$$

(see [7]), and $i_2(v_1) = i_2 \partial(v) = 0$. Hence $i_1 i_2 i_3$ is C_{n-k}-onto, since by assumption i_2 is C_{n-k}-onto. This shows that $i_7 i_6$ is C_{n-k}-onto, that is, i_7 is C_{n-k}-onto. Similarly, i_9 is C_{n-k}-onto, because $i_8 i_* i_5$ is C_{n-k}-onto (i_5 is an isomorphism, i_* is C_{n-k}-onto by hypothesis, and clearly i_8 is onto). By construction,

$$\text{rank}(H_\alpha W_1) = r - 1.$$

Also, W_1 is $(\alpha - 1)$-connected and $H_i W_1 \approx H_i W_0$ for $i > \alpha$. We can now apply our inductive hypothesis, and we have proved that we may suppose $H_\alpha W$ is finite and $H_i \partial W \to H_i W$ is C_{n-k}-onto for $i \leq \beta$.

Next, let us reduce $H_{\alpha+1} W$ to a finite group. The proof is similar to the one above. We again use induction on rank$(H_{\alpha+1} W)$. Let $u \in H_{\alpha+1} W$ be an element of infinite order. Then, using Lemma 1 and the fact that $H_{\alpha+1} \partial W \to H_{\alpha+1} W$ is C_{n-k}-onto, we may assume as above that some multiple mu can be represented by an imbedding (see [5]) $\phi: S^{\alpha+1} \to \partial W$. The normal bundle of ϕ may not be trivial; however, a multiple of $[\phi] \in \pi_{\alpha+1} \partial W$ will have a trivial normal bundle, by Lemmas 2 and 3. Thus we may assume that ϕ extends to an imbedding

$$\phi: S^{\alpha+1} \times D^{\alpha} \to \partial W.$$

Define N and W_1 as before. Since $\pi_\alpha SO_\alpha \to \pi_\alpha SO$ is onto, we can again assume that W_1 is a π-manifold. Consider Diagram II, which is a commutative diagram whose row and columns are exact.

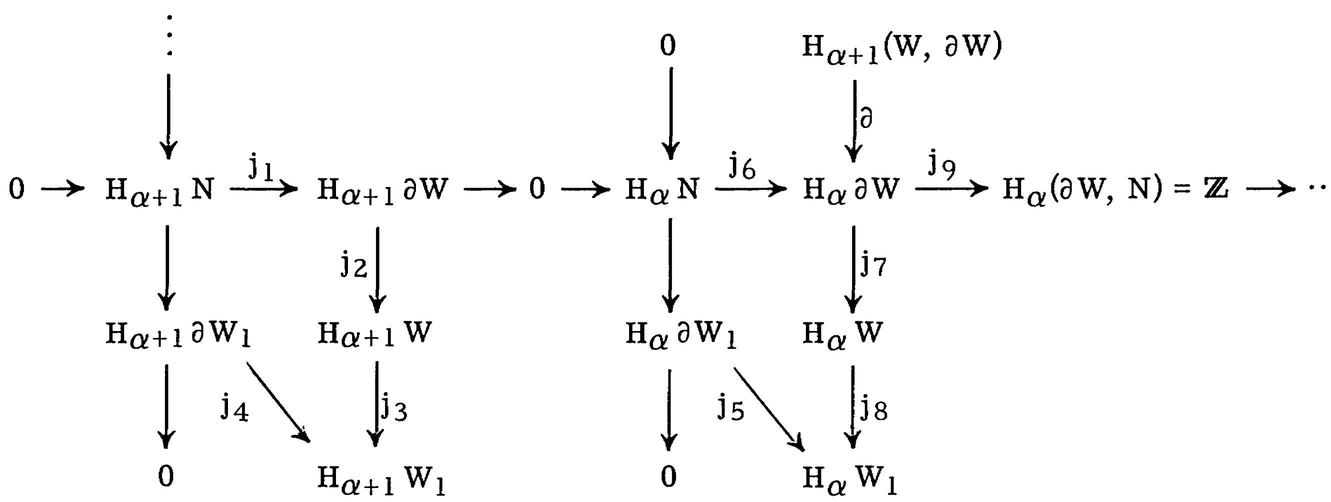


Diagram II

The map j_4 is C_{n-k}-onto because j_1 is an isomorphism, j_2 is C_{n-k}-onto by hypothesis, and j_3 is C_{n-k}-onto by construction. Assume that $j_2(u_1) = mu$ for $u_1 \in H_{\alpha+1} \partial W$ and that ϕ represents u_1. Use Poincaré duality to obtain a $v \in H_{\alpha+1}(W, \partial W)$ with $mu \cdot v = 1$. Then $j_9(\partial(v)) = \partial(v) \cdot u_1 = \pm 1$ and $j_7(\partial(v)) = 0$. Therefore, $j_7 j_6$ is C_{n-k}-onto, since j_7 was C_{n-k}-onto by hypothesis. But j_8 is an isomorphism, and hence it follows that j_5 is C_{n-k}-onto. This shows that we can inductively reduce $H_{\alpha+1} W$ to a finite group ($H_\alpha W$ is left undisturbed). Thus, since all other desired properties have been preserved by this construction, we have shown that we may assume initially in Case 1 that $H_\alpha W$ and $H_{\alpha+1} W$ are finite.

To kill the unwanted torsion in $H_\alpha W$ and $H_{\alpha+1} W$, we must make some changes in the proofs given above. Let

$$r = \sum_{p \in \Pi(C_{n-k})} r_p(H_\alpha W) .$$

Suppose $r > 0$, and let $u \in H_\alpha W$ be a nonzero element of order p^m, for some $p \in \Pi(C_{n-k})$. We shall use the notation of diagram I. Since i_* is C_{n-k}-onto, there exists a $u_1 \in H_\alpha \partial W$ such that $i_*(u_1) = u$. Furthermore, we can assume that u_1 can be represented by an imbedding $\phi: S^\alpha \times D^{\alpha+1} \to \partial W$. If $z \in H_{\alpha+1} \partial W$, then $i_4(z) = z \cdot u_1 = 0$ (the fact that i_4 can be interpreted as an intersection pairing is explained in [7, Lemma 5.6]); and therefore i_3 is an isomorphism. Consider the sequence

$$\cdots \longrightarrow H_{\alpha+1} W_1 \xrightarrow{i_{11}} H_{\alpha+1}(W_1, W) = \mathbb{Z} \xrightarrow{\partial_1} H_\alpha W \longrightarrow \cdots,$$

and let ε be a generator of $H_{\alpha+1}(W_1, W)$. Now $\partial_1(\varepsilon) = u$, by construction. It follows that there exists an $x \in H_{\alpha+1} W_1$ such that $i_{11}(x) = p^m \varepsilon$. But $i_1(H_{\alpha+1} W)$ and x generate $H_{\alpha+1} W_1$, and i_2 is C_{n-k}-onto by hypothesis. Thus, to prove that i_7 is C_{n-k}-onto, it is clearly sufficient to show that $x \in i_7(H_{\alpha+1} \partial W_1)$.

Let $Y = W_1$ - (interior of W), and think of the handle $D^{\alpha+1} \times D^{\alpha+1}$ as being contained in Y. Diagram III is a commutative diagram with exact row (a Mayer-Vietoris sequence) and column.

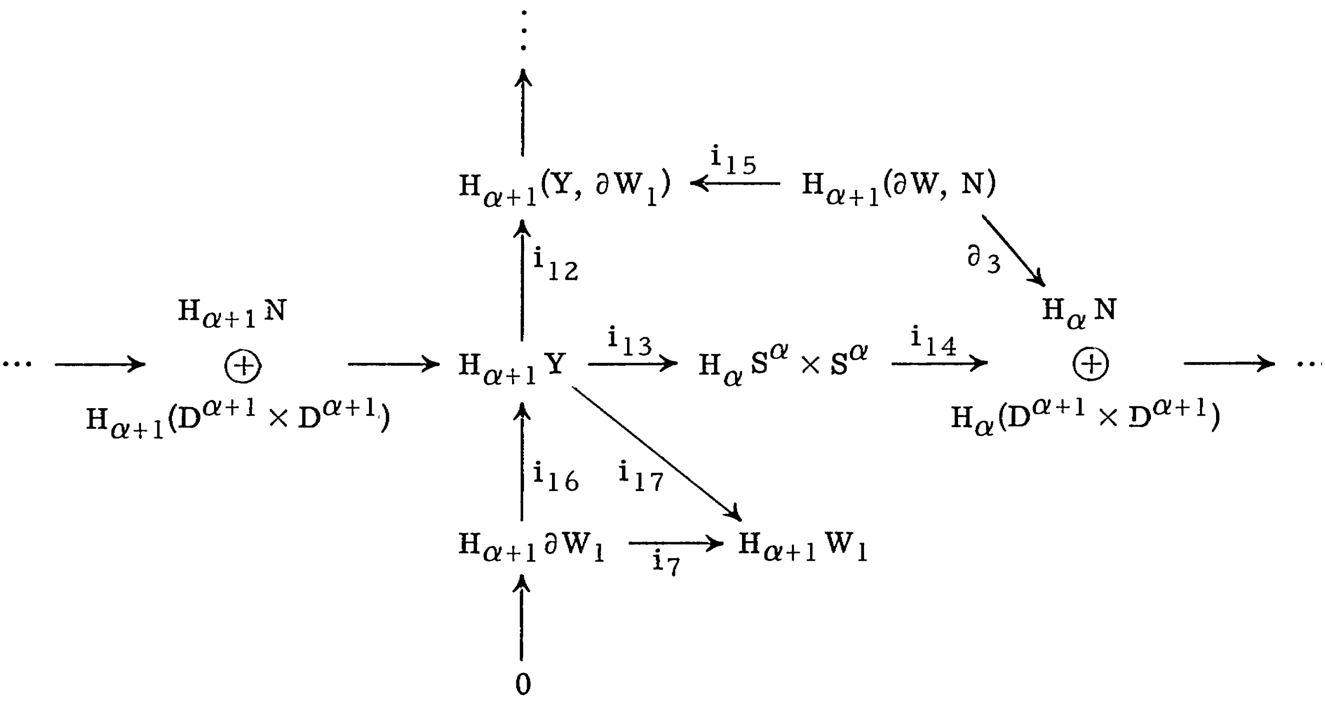


Diagram III

Since i_{15} is an isomorphism, since ∂_3 is one-to-one (because $i_4 = 0$), and since $i_{14} i_{13} = 0$, it follows that $i_{12} = 0$. Hence, i_{16} is an isomorphism, and we only need to show that $x \in i_{17}(H_{\alpha+1} Y)$. Now, $H_{\alpha+1}(W, \partial W) \approx H^{\alpha+1} W$ and

$$\operatorname{rank}(H^{\alpha+1} W) = \operatorname{rank}(H_{\alpha+1} W) = 0 .$$

Therefore, if we look at the exact sequence for the pair $(W, \partial W)$, we see that $H_\alpha \partial W$ is finite. This allows us to assume that u_1 is also finite and of order p^m. Consider Diagram IV, a commutative diagram, where the row is an exact Mayer-Vietoris sequence. A little inspection shows that we may choose x such that $i_{18}(x) = p^m \varepsilon_1$ for some $\varepsilon_1 \in H_{\alpha+1}(W_1, \partial W)$ with the property that $i_{19}(\varepsilon_1) = \varepsilon$ and $\partial_2(\varepsilon_1) = u_1$. Then

$$\partial_3(x) = \partial_2 i_{18}(x) = \partial_2(p^m \varepsilon_1) = p^m u_1 = 0 .$$

By exactness, there exist a $y_1 \in H_{\alpha+1} W$ and a $y_2 \in H_{\alpha+1} Y$ such that $i_{20}(y_1 + y_2) = x$. But $i_{19} i_{18} i_{20}(y_1) = 0$. Thus we may replace x by $i_{20}(y_2) = i_{17}(y_2)$, and we have shown that i_7 is C_{n-k}-onto. The map i_9 is C_{n-k}-onto, because i_5 and i_8 are onto and i_* is C_{n-k}-onto by hypothesis. Unfortunately, $H_{\alpha+1} W_1$ is no longer finite. In fact, $\operatorname{rank}(H_{\alpha+1} W_1) = 1$. However, we already have described a procedure whereby elements of infinite order in $H_{\alpha+1} W_1$ can be killed without disturbing

$$
\begin{array}{ccccccccc}
 & & & & & & & \swarrow & H_{\alpha+1}(X, \partial W) \\
 & & & & & & & & \downarrow \\
 & & & \nearrow^{i_{18}} & H_{\alpha+1}(W_1, \partial W) & \xrightarrow[i_{19}]{} & & & H_{\alpha+1}(W_1, W) \\
 & & & & \downarrow \partial_2 & & & & \downarrow \partial_1 \\
\cdots \longrightarrow & H_{\alpha+1} W \oplus H_{\alpha+1} Y & \xrightarrow{i_{20}} & H_{\alpha+1} W_1 & \xrightarrow{\partial_3} H_\alpha \partial W & \longrightarrow & & & H_\alpha W \oplus H_\alpha X \longrightarrow \cdots
\end{array}
$$

Diagram IV

$H_\alpha W_1$. This shows that by induction we can successively kill all torsion elements of $H_\alpha W$ until $H_\alpha W \in C_{n-k}$.

Finally, assume that $H_\alpha W \in C_{n-k}$ and $H_{\alpha+1} \partial W \to H_{\alpha+1} W$ is C_{n-k}-onto. Let $r = \sum_{p \in \Pi(C_{n-k})} r_p(H_{\alpha+1} W)$, and suppose $r > 0$. Let $u \in H_{\alpha+1} W$ be a nonzero element of order p^m, for some $p \in \Pi(C_{n-k})$. As before, we add a handle to kill u. It follows easily from diagram II that $H_{\alpha+1} \partial W_1 \to H_{\alpha+1} W_1$ is C_{n-k}-onto. Also, $H_\alpha W_1 \approx H_\alpha W \in C_{n-k}$. By induction, we can assume that $H_{\alpha+1} W \in C_{n-k}$, and Case 1 is proved, since all other desired properties have been preserved by our constructions.

Case 2. $n + k$ is even, that is, $n + k = 2\alpha$ and $\alpha = \beta$.

This case is also proved by induction. Let $r = \text{rank}\,(H_\alpha W)$, and suppose that $r > 0$. It is convenient to separate the discussion into two subcases.

(a) Assume α is odd. Let $u \in H_\alpha W$ be an element of infinite order, and consider the exact sequence

$$\cdots \longrightarrow H_{\alpha+1}(W, \partial W) \xrightarrow{\partial} H_\alpha \partial W \xrightarrow{i_*} H_\alpha W \longrightarrow \cdots .$$

Since i_* is C_{n-k}-onto, there exist an $m \in I(C_{n-k})$ and a $u_1 \in H_\alpha \partial W$ such that $i_*(u_1) = mu$. By Poincaré duality, we can find a $v \in H_{\alpha+1}(W, \partial W)$ with $mu \cdot v = 1$. Then

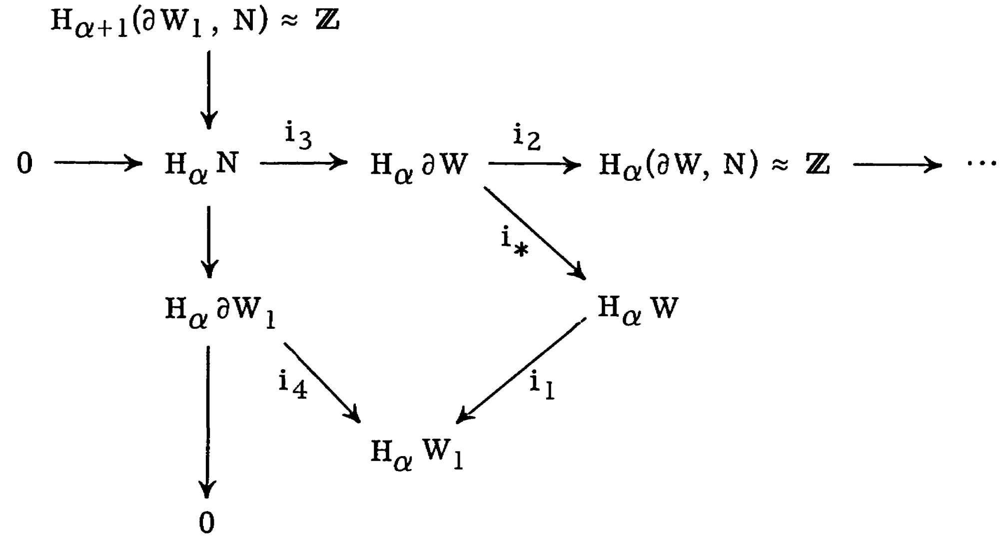
$$1 = mu \cdot v = i_*(u_1) \cdot v = (-1)^\alpha u_1 \cdot \partial(v) .$$

Let $v_1 = \partial(v) \in H_\alpha \partial W$.

By Lemma 1, we may assume that u_1 can be represented by an imbedding $\phi\colon S^\alpha \to \partial W$. Since α is odd, we can use Lemmas 2 and 3 to assume further that the normal bundle of ϕ is trivial, in other words, that ϕ extends to an imbedding $\phi\colon S^\alpha \times D^\alpha \to \partial W$. Let

$$N = \partial W - \text{interior}\,(\phi(S^\alpha \times D^\alpha)) \quad \text{and} \quad W_1 = W \cup D^{\alpha+1} \times D^\alpha ,$$

where we identify $x \in S^\alpha \times D^\alpha$ with $\phi(x) \in \partial W$. We have the commutative diagram (see [7, Lemma 7.1] for more details):



Note that i_* is C_{n-k}-onto by hypothesis, and i_1 is onto by construction. Since i_2 is just $\cdot u_1$, it follows that $i_1 i_* i_3$ is C_{n-k}-onto, because $v_1 \cdot u_1 = \pm 1$ and $i_*(v_1) = i_* \partial(v) = 0$. Therefore i_4 is C_{n-k}-onto. But $\pi_{\alpha-1} SO_\alpha \to \pi_{\alpha-1} SO$ is onto, and hence we may assume that W_1 is a π-manifold, by modifying ϕ if necessary. Clearly, $\operatorname{rank}(H_\alpha W_1) = r - 1$, and we have given an inductive procedure for making $H_\alpha W$ finite.

(b) Assume α is even. The proof of (a) does not apply here, because now the kernel of $\pi_{\alpha-1} SO_\alpha \to \pi_{\alpha-1} SO$ is not always finite (see Lemma 2), and we cannot assert that the normal bundle of a sphere representing some multiple of u_1 will be trivial. However, ∂W is a boundary, and thus the signature of ∂W is zero. Therefore we can find $x_1, \cdots, x_s, y_1, \cdots, y_s \in H_\alpha \partial W$ such that

$$x_i \cdot x_j = 0 = y_i \cdot y_j, \qquad x_i \cdot y_j = \delta_{ij};$$

and if G is the subgroup of $H_\alpha \partial W$ generated by $x_1, \cdots, x_s, y_1, \cdots, y_s$, then $H_\alpha \partial W/G$ is finite (see [12, Lemma 9]). Since i_* is C_{n-k}-onto, either $i_*(x_t)$ or $i_*(y_t) \in H_\alpha W$ will be an element of infinite order for some t, if $\operatorname{rank}(H_\alpha W) > 0$. But if $\psi: S^\alpha \to \partial W$ represents $z \in H_\alpha \partial W$, then the normal bundle of ψ is trivial if and only if $z \cdot z = 0$ (see [12, Lemma 7]). This shows that if $\operatorname{rank}(H_\alpha W) > 0$, then we can always find an element $u_1 \in H_\alpha \partial W$ that can be represented by an imbedding $\phi: S^\alpha \to \partial W$ with trivial normal bundle such that $u = i_* u_1$ has infinite order. The proof of (b) then proceeds as in (a).

Now assume that $H_\alpha W$ is finite and $H_\alpha \partial W \to H_\alpha W$ is C_{n-k}-onto. Let $r = \sum_{p \in \Pi(C_{n-p})} r_p(H_\alpha W) > 0$, and let $u \in H_\alpha W$ be a nonzero element of order p^m, for some $p \in \Pi(C_{n-k})$. Since

$$H_{\alpha+1}(W, \partial W) \approx H^\alpha W \quad \text{and} \quad \operatorname{rank}(H^\alpha W) = \operatorname{rank}(H_\alpha W) = 0\,,$$

we see from the exact sequence of the pair $(W, \partial W)$ that $H_\alpha \partial W$ is finite. Choose $u_1 \in H_\alpha \partial W$ such that $i_*(u_1) = u$. We may assume that u_1 can be represented by an imbedding $\phi: S^\alpha \to \partial W$ with trivial normal bundle. This is also possible if α is even, since u_1 has finite order and hence $u_1 \cdot u_1 = 0$. Define W_1 as before. Consider the diagram in (a). The map i_3 is now an isomorphism, because i_2 can be interpreted as $\cdot u_1$, which is zero since u_1 is finite. Therefore, i_4 is C_{n-k}-onto. This shows that by induction we can arrange to have $H_\alpha W \in C_{n-k}$. The proof of

Case 2 is now complete, since all other desired properties are again preserved. Lemma 4 is established.

Remark. The proof of Lemma 4 given above is technically complicated. It would be nice if it were possible to simplify it.

LEMMA 5. ∂W_0 *imbeds in an* $(\alpha - 1)$-*connected* C_{n-k}-*spherical* π-*manifold* W^{n+k+1} *with* 1-*connected boundary such that* $H_i W = 0$ *for* $i > n$.

Proof. Let W be as in Lemma 4, and set

$$r = \sum_{j=\beta+1}^{n} \operatorname{rank}(H_j W) + \sum_{\substack{j=\beta+1 \\ p \in \Pi(C_{n-k})}}^{n} r_p(H_j W)\,.$$

We shall prove Lemma 5 by induction on r.

If $r = 0$, then clearly $H_i(W, \partial W) \in C_{n-k}$ for $0 < i < n+k+1$. Therefore, by the generalized Hurewicz theorem (see [16, page 511]), the Hurewicz homomorphism $h\colon \pi_{n+k+1}(W, \partial W) \to H_{n+k+1}(W, \partial W)$ is a C_{n-k}-isomorphism; in other words, W is C_{n-k}-spherical (compare [8]).

Assume that $r > 0$ and Lemma 5 is true for $r - 1$. Let s denote the smallest positive integer such that $H_s W \notin C_{n-k}$. Then $\beta + 1 \leq s \leq n$. Let $u \in H_s W$ be an element whose order is either infinite or a power of some $p \in \Pi(C_{n-k})$. Consider the exact sequence

$$\cdots \longrightarrow H_s \partial W \xrightarrow{i_*} H_s W \longrightarrow H_s(W, \partial W) \longrightarrow \cdots .$$

Note that $H_s(W, \partial W) \approx H^{n+k+1-s} W$, which belongs to C_{n-k} by Lemma 4 and the universal coefficient theorem, since $n + k + 1 - s \leq \alpha$. Therefore i_* is C_{n-k}-onto, and we may assume that there exist an $m \in I(C_{n-k})$ and a $u_1 \in H_s \partial W$ such that $i_*(u_1) = mu$. But ∂W is $(k - 1)$-connected and $2(k - 1) + 1 = 2k - 1 \geq (n + 3) - 1 \geq s$. Hence, by Lemma 1, may assume that u_1 can be represented by a map $\phi\colon S^s \to \partial W$, which, by [5], we can take to be an imbedding (we need $2k \geq n + 3$ for this, although $2k \geq n + 2$ would suffice for the rest). Lemmas 2 and 3 and the fact that $2k \geq n + 3$ imply that we may further suppose that the normal bundle of ϕ is trivial, that is, ϕ extends to an imbedding $\phi\colon S^s \times D^{n+k-s} \to \partial W$. Let $W_1 = W \cup D^{s+1} \times D^{n+k-s}$, where we identify $x \in S^s \times D^{n+k-s}$ with $\phi(x) \in \partial W$. Since $2k \geq n + 3$, it follows from [2] that $\pi_{s-1} SO_{n+k-s} \to \pi_{s-1} SO$ is onto (actually only the case $n + k - s \geq 13$ is handled in [2], but the other low-dimensional cases are also known). Hence, as usual, we can assume that W_1 is a π-manifold by changing the imbedding ϕ via an element of $\pi_{s-1} SO_{n+k-s}$, if necessary. Now W_1 has all the properties of W except that

$$\sum_{j=\beta+1}^{n} \operatorname{rank}(H_j W_1) + \sum_{\substack{j=\beta+1 \\ p \in \Pi(C_{n-k})}}^{n} r_p(H_j W_1) = r - 1 < r\,.$$

This proves Lemma 5, by the induction hypothesis.

Since $V \subseteq W_0$, Lemma 5 also proves the first part of Theorem 2. Assume next that ∂V is a homotopy sphere. In Lemma 4, we may still suppose that $V \times 1 \subseteq \partial W$. This follows from the construction in [1]. If we could do the surgery, which is necessary in Lemma 5, away from $\partial V \times 1 \subseteq \partial W$, then the second part of Theorem 2 would be proved. In fact, we shall show that we can do the surgery away from $U = \text{closure}(\partial W - V \times 1)$. The composition $H_i(V \times 1) \to H_i W_0 \to H_i W$ is an isomorphism for $i > \beta$, and $V \times 1$ is $(k-1)$-connected. Therefore we can start the surgery in Lemma 5 on $V \times 1$, and it is easy to check that we can continue to stay away from U. This completes the proof of Theorem 2.

3. PROOF OF THEOREM 1

We begin with some generalities.

Recall that X/A denotes the space obtained from X by collapsing A to a point, the base point of X/A. We denote the union of X and a base point by $X/\emptyset$. If X, Y are spaces with base points x_0, y_0, respectively, then

$$X \wedge Y = X \times Y/(X \times y_0 \cup x_0 \times Y) .$$

Let $S^q X = S^q \wedge X$ denote the qth iterated (reduced) suspension of X. We write SX for $S^1 X$.

Following [18], one can define a generalized homology theory h_* by setting

$$h_i(X, A) = \lim_q \pi_{i+q}(S^q(X/A)) .$$

The reduced group $\tilde{h}_i X = h_i(X, x_0)$ is then just the ith stable homotopy group $\pi_i^s X$ of X. In fact, imitating [3], one can characterize $h_i(X, A)$ as the set of framed cobordism classes of maps $f: (V^i, \partial V^i) \to (X, A)$, where V is an oriented π-manifold. (For example, if $[g] \in h_i X$, where $g: S^{i+q} \to S^q(X/\emptyset)$, then V is essentially $g^{-1}(X)$. One can justify this using techniques of [9, page 6] and observing that $S^q(X/\emptyset)$ is homeomorphic to the Thom space of the trivial q-plane bundle over X.) There is also a natural map $h: h_i(X, A) \to H_i(X, A)$ that sends the element $[f, V] \in h_i(X, A)$ determined by (f, V) into $f_*(\mu_V) \in H_i(X, A)$, where $\mu_V \in H_i(V, \partial V)$ is the orientation class of V. The map h induces a homomorphism $\tilde{h}: \tilde{h}_i X \to \tilde{H}_i X$ ($\tilde{H}_i X$ is the reduced homology group of X).

LEMMA 6. *Suppose that* X *is* ℓ*-connected. Then, for all* i, $\tilde{h}: \tilde{h}_i X \to \tilde{H}_i X$ *is a* $C_{i-\ell-1}$*-isomorphism.*

Proof. Let X^j denote the j-skeleton of X. Let $t \geq \ell + 1$. Then the triple $(X^{t+1}, X^t, X^{\ell+1})$ induces the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & h_i(X^t, X^{\ell+1}) & \longrightarrow & h_i(X^{t+1}, X^{\ell+1}) & \longrightarrow & h_i(X^{t+1}, X^t) & \longrightarrow \cdots \\
& \downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & \\
\cdots \longrightarrow & H_i(X^t, X^{\ell+1}) & \longrightarrow & H_i(X^{t+1}, X^{\ell+1}) & \longrightarrow & H_i(X^{t+1}, X^t) & \longrightarrow \cdots ,
\end{array}$$

where the ϕ_i are induced by h. If $t = \ell + 1$, then ϕ_1 is clearly an isomorphism for all i. Assume that for some $t \geq \ell + 1$, ϕ_1 is a $C_{i-\ell-2}$-isomorphism for all i. Since X^{t+1}/X^t is just a wedge of $(t+1)$-spheres, the map ϕ_3 is a $C_{i-\ell-2}$-monomorphism and is onto. (Observe that $h_i(X^{t+1}, X^t)$ is a sum of π^s_{i-t-1}.) A slight generalization of the Five Lemma then proves that, for all i, ϕ_2 is a $C_{i-\ell-2}$-isomorphism. It follows by induction on t that the map $\phi: h_i(X, X^{\ell+1}) \to H_i(X, X^{\ell+1})$ induced by h also is a $C_{i-\ell-2}$-isomorphism for all i.

Next, consider the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & h_i(X^{\ell+1}, x_0) & \longrightarrow & h_i(X, x_0) = \tilde{h}_i X & \longrightarrow & h_i(X, X^{\ell+1}) & \longrightarrow & \cdots \\ & & \downarrow \phi_4 & & \downarrow \tilde{h} & & \downarrow \phi & & \\ \cdots & \longrightarrow & H_i(X^{\ell+1}, x_0) & \longrightarrow & H_i(X, x_0) = \tilde{H}_i X & \longrightarrow & H_i(X, X^{\ell+1}) & \longrightarrow & \cdots \end{array}$$

induced by the triple $(X, X^{\ell+1}, x_0)$. Since X is ℓ-connected, it is easy to show that $X^{\ell+1}$ has the homotopy type of a wedge of $(\ell + 1)$-spheres; hence ϕ_4 is a $C_{i-\ell-1}$-isomorphism. Therefore, another application of the Five Lemma, together with the facts obtained above about ϕ, establishes Lemma 6.

Proof of Lemma 1. It is well known that if X is ℓ-connected, then $s: \pi_i X \to \pi_i^s X = h_i X$ is an isomorphism for $0 < i \leq 2\ell$, and s is onto for $i = 2\ell + 1$ (see [16, page 458]). Lemma 1 is thus an immediate consequence of Lemma 6.

Proof of Theorem 1. (a): Let $x \in \pi_{n+k}^s X = \tilde{h}_{n+k} X$. Then there exist an oriented π-manifold V^{n+k} and a map $f: (V, \partial V) \to (X, x_0)$ such that $f_*(\mu_V) = \tilde{h}(x)$. By taking the double of V and deleting an $(n+k)$-disk, we may assume that $\partial V = S^{n+k-1}$. Define

$$F: U^{n+k+1} = [0, 1] \times V \to SX$$

by $F(t, v) = p(t, f(v))$, where $p: [0, 1] \times X \to S^1 \times X \to SX$ is the natural projection. Since X is (k - 1)-connected, we may assume that V has the same property (otherwise, do surgery on V to make it (k - 1)-connected). Therefore it follows from Poincaré duality and the universal coefficient theorem that $H_i V = 0$ for $i > n$ and that $H_n V$ is torsion-free. Clearly, ∂V is 1-connected, because $n + k - 1 \geq 2$.

Since the case $n = 1$ is trivial, assume $n \geq 2$. The proof of Theorem 2 shows that U imbeds in a C_{n-k}-spherical π-manifold W^{n+k+1}. Define $g: W \to SX$ by $g \mid U = F$ and $g(W - U) = x_0$. It is easy to see that $[g, W] = [F, U] \in \tilde{h}_{n+k+1}(SX)$. But W is C_{n-k}-spherical, and we may assume that W and ∂W are 1-connected. Hence, by the generalized Hurewicz and Whitehead theorem (see [16, pages 508-512]), there exists a map $\phi: (D^{n+k+1}, S^{n+k}) \to (W, \partial W)$ that induces a C_{n-k}-isomorphism

$$\phi_1: h_{n+k+1}(D^{n+k+1}, S^{n+k}) \to h_{n+k+1}(W, \partial W) .$$

Let $g_1: h_{n+k+1}(W, \partial W) \to h_{n+k+1}(SX, x_0)$ be the homomorphism induced by g. Then

$$s([g\phi]) = [g\phi, D^{n+k+1}] = g_1 \phi_1([\text{identity}, D^{n+k+1}])$$

$$= g_1(m[\text{identity}, W]) = m[g, W],$$

for some $m \in I(C_{n-k})$. This proves (a).

(b): Let $g: S^{n+k} \to X$ be a map such that $s([g]) = 0$. Then there exist a π-manifold V^{n+k+1} and a map $G: V \to X$ such that $\partial V = S^{n+k}$ and $G \mid \partial V = g$. Since X is (k - 1)-connected, we may assume that V is (k - 1)-connected. By Poincaré duality and the universal coefficient theorem, we have that $H_i V = 0$ for $i > n + 1$ and that $H_{n+1} V$ is torsion-free. Again, the proof of Theorem 2 shows that $[0, 1] \times V$ imbeds in a 1-connected C_{n-k+1}-spherical π-manifold W^{n+k+2} with 1-connected boundary in such a way that $[0, 1] \times \partial V \subseteq \partial W$. Therefore there exists a map $\phi: (D^{n+k+2}, S^{n+k+1}) \to (W, \partial W)$ such that $\phi_*(\mu) = m\mu_W$, for some nonzero integer m as above. Let $\psi = \phi \mid S^{n+k+1}$. Extend G to a map $G'\colon W \to SX$ as in (a). It is easy to check that $0 = [G' \psi] = ms^1([g]) \in \pi_{n+k+1} SX$. This proves (b) and completes the proof of Theorem 1.

Finally, we state an easy corollary of Theorem 1 and Lemma 6 about the Hurewicz homomorphism.

THEOREM 3. *Let* X *be a* (k - 1)*-connected space* ($k \geq 1$).

(a) *If* $2k \geq n + 3$, *then the Hurewicz homomorphism* $h'\colon \pi_{n+k+1} SX \to H_{n+k+1} SX$ *is* C_n*-onto.*

(b) *Let* K *denote the kernel of the Hurewicz homomorphism*

$$h'\colon \pi_{n+k} X \to H_{n+k} X .$$

If $2k \geq n + 4$, *then* $s^1(K) \subseteq \pi_{n+k+1} SX$ *belongs to* C_n.

REFERENCES

1. M. K. Agoston, S-*tangential maps and imbeddings in the conditionally stable range*. Topology 9 (1970), 13-23.
2. M. G. Barratt and M. E. Mahowald, *The metastable homotopy of* O(n). Bull. Amer. Math. Soc. 70 (1964), 758-760.
3. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*. Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Band 33. Springer-Verlag, Berlin, 1964.
4. L. Fuchs, *Abelian groups*. Pergamon Press, New York-Oxford-London-Paris, 1960.
5. A. Haefliger, *Plongements différentiables de variétés dans variétés*. Comment. Math. Helv. 36 (1961), 47-82.
6. M. A. Kervaire, *Some nonstable homotopy groups of Lie groups*. Illinois J. Math. 4 (1960), 161-169.
7. M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres*. I. Ann. of Math. (2) 77 (1963), 504-537.
8. K. W. Kwun and F. Raymond, *Spherical manifolds*. Duke Math. J. 34 (1967), 397-401.
9. R. Lashof, *Some theorems of Browder and Novikov on homotopy equivalent manifolds with an application*. Notes, University of Chicago.
10. J. Levine, *On differentiable imbeddings of simply-connected manifolds*. Bull. Amer. Math. Soc. 69 (1963), 806-809.
11. M. Mahowald, *The metastable homotopy of* S^n. Memoirs of the Amer. Math. Soc., No. 72. American Mathematical Society, Providence, R.I., 1967. 81 pp.
12. J. W. Milnor, *A procedure for killing homotopy groups of differentiable manifolds*. Proc. Sympos. Pure Math., Vol. III, pp. 39-55. American Mathematical Society, Providence, R.I., 1961.
13. J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*. Ann. of Math. (2) 81 (1965), 211-264.

14. G. F. Paechter, *The groups* $\pi_r(V_{n,m})$. I. Quart. J. Math. Oxford Ser. (2) 7 (1956), 249-268.

15. S. Smale, *On the structure of manifolds*. Amer. J. Math. 84 (1962), 387-399.

16. E. H. Spanier, *Algebraic topology*. McGraw-Hill, New York, 1966.

17. G. W. Whitehead, *A generalization of the Hopf invariant*. Ann. of Math. (2) 51 (1950), 192-237.

18. ———, *Generalized homology theories*. Trans. Amer. Math. Soc. 102 (1962), 227-283.

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