SCARCITY OF ORIENTATION-REVERSING PL INVOLUTIONS OF LENS SPACES

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1. INTRODUCTION

For convenience, we shall not consider the 3-sphere as a lens space. The following theorem justifies the title of this paper.

THEOREM. (i) *No lens space other than the projective* 3*-space* P_3 *admits an orientation-reversing involution.* (ii) *Up to* PL*-equivalences, there exists exactly one orientation-reversing* PL *involution of* P_3.

Part (i) is not new, but we have included it for emphasis. It follows from [5, Theorem V], and it is a special case of the result in [2]. We remark that the unique involution of Part (ii) is the one induced by the reflection of S^3 about the equator. The fixed-point set is the disjoint union of a projective plane and a point. As a corollary, we obtain the following result.

COROLLARY. *There exists no* PL *action of* $Z_2 + Z_2$ *on* S^3 *that leaves a four-point set* A *invariant (as a set) and acts freely off* A.

By a four-point set, we mean a set consisting of four distinct points. The corollary restricts PL actions of $Z_2 + Z_2$ on S^3.

Henceforth, let h denote an orientation-reversing PL involution of P_3 with fixed-point set F. It is a consequence of the parity theorem and the Lefschetz fixed-point formula that $\dim F = 0$ or $\dim F = 2$. We shall rule out the case $\dim F = 0$ in Section 2, establish the uniqueness for the case $\dim F = 2$ in Section 3, and prove the corollary in Section 4.

2. THE CASE $\dim F = 0$

2.1. We shall prove that h fixes exactly two points. Suppose h fixes $x_1, x_2, \cdots, x_k \in P_3$ and no other point. It seems to be known (and it is fairly easy to prove) that a PL involution of a finite simplicial complex becomes simplicial after a suitable subdivision. Hence we may assume h is simplicial with vertices x_i. Further, we assume that the closed stars of x_i are mutually disjoint. Let X be obtained from P_3 by removing open stars of the x_i. Then $h' = h \mid X$ is a free involution of X reversing the orientation of each boundary component of the 3-manifold X.

The Lefschetz number of h' is

$$1 - 0 + (1 - k) = 2 - k.$$

Hence $k = 2$.

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2.2. Let Y denote the orbit space of Z_2-action on X generated by h'. We prove that $\pi_1(Y)$ is isomorphic to $Z_2 + Z_2$ rather than to Z_4. We let p: $S^3 \to P_3$ be the usual double covering, and we lift h to $\tilde{h}$: $S^3 \to S^3$.

Let $p^{-1}(x_i)$ consist of y_i and y_i' (i = 1, 2), and consider the diagram

$$\begin{array}{ccc} (S^3, y_1) & \xrightarrow{\tilde{h}} & (S^3, y_1) \\ \downarrow p & & \downarrow p \\ (P_3, x_1) & \xrightarrow{h} & (P_3, x_1) \end{array} .$$

Lift h to $\tilde{h}$ as in the diagram. Since $\tilde{h} \cdot \tilde{h}$ covers $h \cdot h$ = identity, and since $\tilde{h} \cdot \tilde{h}(y_1) = y_1$, it follows from the uniqueness of lifting that $\tilde{h}$ is a PL involution of S^3 fixing y_1 and no other point near y_1. Hence $\tilde{h}$ fixes y_1 and y_1' and no other point. Thus $\tilde{h}(y_2) = y_2'$.

Let α be a path from x_1 to x_2. Lift $\alpha \circ (h\bar{\alpha})$, where $\bar{\alpha}$ is the inverse path of α, starting at x_2. Since $\tilde{h}(y_1) = y_1$ and $\tilde{h}(y_2) = y_2'$, the lifting of $\alpha \circ (h\bar{\alpha})$ ends at y_1'. Hence $\alpha \circ (h\bar{\alpha})$ represents the nontrivial element of $\pi_1(P_3)$. Later this fact will play a crucial role. Let a denote the antipodal involution of S^3. Because $a\tilde{h}(y_1) = y_1'$ and $\tilde{h}a(y_1) = y_1'$, it follows from the uniqueness of the lifting of h that $a\tilde{h} = \tilde{h}a$. Let Q denote the space obtained from S^3 by removing open stars of y_1, y_1', y_2, and y_2'. (Assume S^3 is so triangulated that p is simplicial.) Restrictions of a and $\tilde{h}$ to Q generate a group, isomorphic to $Z_2 + Z_2$ and acting freely on Q. The orbit space may be identified with Y. This proves our contention.

2.3. We are ready to rule out the case dim F = 0. Let q: $X \to Y$ be the orbit map. Let z_i (i = 1, 2) be points on the boundaries B_i of open stars of x_i in P_3. Let α_i be a path in B_i from z_i to $h(z_i)$. Since $\pi_1 Y$ is abelian, we shall not worry about the base point. Now the $q\alpha_i$ represent nontrivial elements of $\pi_1 Y$. Let α be a path in X from z_1 to z_2. Since $X \subset P_3$ induces an isomorphism for fundamental groups, we can use a fact established in Section 2.2 to deduce that $\alpha \circ \alpha_2 \circ (h\bar{\alpha}) \circ \bar{\alpha}_1$ represents the nontrivial element of $\pi_1 X$. *Via* q, we find that $q\alpha_1$ and $q\alpha_2$ represent different nontrivial elements of $H_1(Y : Z) \simeq Z_2 + Z_2$. Let $\dot{Y}$ denote the boundary of the 3-manifold Y. Then $\dot{Y}$ is the disjoint union of the two projective planes. The induced homomorphism $H_1(\dot{Y}; Z_2) \to H_1(Y; Z_2)$ is an epimorphism, by the preceding observation. Hence, by the homology sequence of $(Y, \dot{Y})$ over Z_2,

$$H_1(Y, \dot{Y}; Z_2) \simeq Z_2 .$$

By Poincaré duality over Z_2, $H^2(Y; Z_2) \simeq Z_2$, and therefore $H_2(Y; Z_2) \simeq Z_2$ by the universal coefficient theorem (with Z_2 as ground ring). On the other hand, by the universal coefficient theorem with integers as ground ring, $H_2(Y; Z_2)$ contains $Z_2 + Z_2$ as a direct summand. This contradiction rules out the case dim F = 0. (One may also bring about a contradiction by comparing Euler characteristics of Y and $\dot{Y}$.)

3. THE CASE dim F = 2

Let A be a 2-dimensional component of F.

3.1. We show that A is a projective plane. We continue to use $p: S^3 \to P_3$ for the usual double covering. Choose $x_0 \in A$ and $y_0 \in p^{-1}(x_0)$. Lift h to $\tilde{h}: (S^3, y_0) \to (S^3, y_0)$. Then $\tilde{h}$ is again an involution fixing this time a 2-dimensional set (therefore a 2-sphere) A'. Actually, A' is a whole component of $p^{-1}(A)$. Hence, A is covered by a 2-sphere in two-to-one or one-to-one fashion. Therefore A is a 2-sphere or a projective plane. But A cannot be a 2-sphere, because then the complementary domains would be an open 3-cell and something that is not 1-connected, and h could not interchange these complementary domains. Hence, A is a projective plane. Assume that the triangulation of P_3 is such that h is simplicial and the simplicial neighborhood N of A is a regular neighborhood of A. Assume furthermore that $h \mid (N - A)$ is fixed-point-free.

3.2. In this section, we analyse h before we complete our proof in the next subsection. Note that A is one-sided in P_3, because P_3 is orientable and A is not. Hence (N, A) is homeomorphic to (M, A), where M is the mapping cylinder of a double covering $S^2 \to A$. Since P_3 is irreducible, $N' = \overline{P_3 - N}$ is a 3-cell. Therefore $h \mid \dot{N}'$ is fixed-point-free, and $h \mid N'$ is essentially the cone over $h \mid N'$. (See [4].) Hence the analysis of h reduces to that of $h \mid N$.

3.3. We now analyse $h \mid N$. Let S be the orbit space and $f: N \to S$ the orbit map. The space S is a compact 3-manifold with exactly two boundary components $f(\dot{N})$ and f(A). Let U be a regular neighborhood of f(A) in S, disjoint from $f(\dot{N})$. The set $V = f^{-1}(U)$ is a regular neighborhood of A in N, disjoint from $\dot{N}$. Note that $\overline{N - V}$ is homeomorphic to $S^2 \times [0, 1]$, and that h is free on this set. By a theorem of Livesay [4], the orbit space of $h \mid \overline{N - V}$ is homeomorphic to $P_2 \times [0, 1]$, where P_2 is the projective plane. Since U is a collar of f(A), it is homeomorphic to $P_2 \times [0, 1]$. Hence S itself is homeomorphic to $P_2 \times [0, 1]$, with $P_2 \times 0$ and $P_2 \times 1$ corresponding to f(A) and $f(\dot{N})$, respectively. Thus $h \mid N$ is equivalent to the following construction: Take $h \mid \dot{N}$ to be the nontrivial covering transformation g of some PL double covering $d: \dot{N} \to A$. Regard N as a PL mapping cylinder of d. Let g induce a PL involution on this mapping cylinder in the obvious way. Consider it as $h \mid N$. Every two such involutions are PL-equivalent. That is, for every two such PL involutions h_1 and h_2 of N, there exists a PL homeomorphism $t: N \to N$ such that $h_1 = t^{-1} h_2 t$. This can be seen as follows. Let $q_1, q_2 : N \to P_2 \times [0, 1]$ be orbit maps corresponding to h_1 and h_2, with $P_2 \times 0$ corresponding to the fixed-point set A. Since $q_i \mid (N - A)$ is a universal covering, there exists a PL homeomorphism $t: N - A \to N - A$ such that $q_1 = q_2 t$. This t can be uniquely extended to a PL homeomorphism $t: N \to N$ such that $q_1 = q_2 t$. But $th_1 = h_2 t$, since t respects covering translation. This is true on N - A, and by continuity also on N. Hence $h_1 = t^{-1} h_2 t$.

4. PROOF OF THE COROLLARY

Suppose there exists an action of $Z_2 + Z_2$, free off a four-point set $A = \{x_1, x_2, x_3, x_4\}$. It is well known that $Z_2 + Z_2$ cannot act freely on S^3. In fact, there exists no closed 3-manifold M with $\pi_1 M \simeq Z_2 + Z_2$ (for a proof of this, see [1]). Hence at least one nontrivial element $\alpha \in Z_2 + Z_2$ must have a fixed point, and therefore it must have exactly two fixed points. Suppose α fixes x_1 and x_2. If $\beta \in Z_2 + Z_2$ is another nontrivial element, β cannot fix one of x_1 and x_2 and one of x_3 and x_4, because if it did, it would follow that $\alpha\beta \neq \beta\alpha$. Moreover, β cannot fix x_1 and x_2, because if it did, $\alpha\beta \neq 1$ would fix all four points. Hence β is either fixed-point-free or fixes x_3 and x_4, and in the first case, $\alpha\beta$ will fix x_3 and x_4. In any case, there exist two distinct, nontrivial elements $\alpha, \beta \in Z_2 + Z_2$ such that α fixes x_1 and x_2, β fixes x_3 and x_4, and $\alpha\beta$ is fixed-point-free.

Now consider the free Z_2-action generated by $\alpha\beta$. The orbit space is a PL projective 3-space [3] on which the PL involutions induced by α and β are identical and have exactly two fixed points. This is the case ruled out in Section 2.

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