A NOTE ON MULTIVALUED MONOTONE OPERATORS

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1. INTRODUCTION

Let E and F be two real vector spaces in duality with respect to a bilinear form $\langle x, u \rangle$ for $x \in E$ and $u \in F$. A (generally multivalued) mapping $T: E \to F$ is called a *monotone operator* if

$$\langle x - y, u - v \rangle \geq 0$$

whenever $u \in Tx$ and $v \in Ty$; the *domain* of T is defined by

$$D(T) = \{x \in E; Tx \text{ nonempty}\}.$$

The purpose of this note is to show, roughly, that a monotone operator that is actually multivalued admits no continuous selection (Proposition 1) and is not lower-semicontinuous (Proposition 3). We give applications to duality mappings (Proposition 2) and to subdifferentials of convex functions (Proposition 4).

2. SELECTION

A *selection* for a multivalued mapping $T: E \to F$ is a (singlevalued) mapping $\tilde{T}: D(T) \to F$ such that $\tilde{T}x \in Tx$ for every $x \in D(T)$. A selection $\tilde{T}$ is said to be *hemicontinuous* at $x \in D(T)$ if it is continuous (in the $\sigma(F, E)$-topology of F) at x, on each line segment in $D(T)$ with endpoint x.

We shall say that a point x of a subset D of E is *quasi-internal* to D if the convex cone generated by the set of y for which the line segment $[x, y]$ is contained in D is $\sigma(E, F)$-dense in E. Thus each internal point of D, or each point of D if D is a $\sigma(E, F)$-dense subspace of E or an open subset of E (for some vector-space topology on E), is quasi-internal to D.

PROPOSITION 1. *Let* $T: E \to F$ *be a monotone operator that is not single-valued at* $x \in D(T)$. *If* x *is quasi-internal to* $D(T)$, *then* T *admits no selection that is hemicontinuous at* x.

Proof. Suppose that T admits a selection $\tilde{T}: D(T) \to F$, hemicontinuous at x. Since T is not singlevalued at x, there exists $u \in Tx$ with $u \neq \tilde{T}x$. Take y such that $x + ty \in D(T)$ for all $t \in [0, 1]$. The monotonicity of T implies that

$$\langle (x + ty) - x, \tilde{T}(x + ty) - u \rangle \geq 0 \qquad \forall\, t \in [0, 1],$$

so that

$$\langle y, \tilde{T}(x + ty) - u \rangle \geq 0 \qquad \forall\, t \in]0, 1];$$

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if we let $t \to 0$, it follows by hemicontinuity that

$$\langle y, \tilde{T}x - u \rangle \geq 0. \tag{1}$$

Since the set of such y generates in E a $\sigma(E, F)$-dense convex cone, (1) implies that $\tilde{T}x = u$, a contradiction. ∎

Remark. An assumption of the kind that x is quasi-internal to D(T) is needed, in Proposition 1: take

$$E = F = R, \quad D(T) = [0, 1], \quad Ty = 0 \ \forall\, y \in\,]0, 1[, \quad T0 =]-\infty, 0], \quad T1 = [0, +\infty[, \quad x = 0.$$

This example also shows the insufficiency of the weaker assumption that x is almost internal to D(T) (see below).

Consider now the (multivalued) duality mapping J_ϕ of a normed space X into its dual X^*, defined by

$$J_\phi(x) = \{u \in X^*;\ \langle x, u \rangle = \|x\|\ \|u\| \text{ and } \|u\| = \phi(\|x\|)\},$$

where ϕ is a strictly increasing continuous function from R^+ to R^+ with $\phi(0) = 0$ and $\phi(r) \to +\infty$ as $r \to +\infty$. It is easy to see that $J_\phi: X \to X^*$ is monotone and that $D(J_\phi) = X$. Consequently, we have the following proposition.

PROPOSITION 2. *Let* X *be a normed space. If the duality mapping is not singlevalued at* x, *then it admits no selection that is hemicontinuous at* x.

Several fixed-point theorems for nonexpansive mappings in a Banach space X have been proved under the assumption that J_ϕ admits a selection that is sequentially continuous on X, $\sigma(X, X^*)$ into X^*, $\sigma(X^*, X)$ (F. E. Browder [3], Z. Opial [7], $\cdots$). Proposition 2 implies that a necessary condition for this assumption to be satisfied is that J_ϕ be singlevalued. The fact that J_ϕ is singlevalued is equivalent to the Gâteaux differentiability of the norm of X (S. Mazur [5]), and (when X is reflexive) to the strict convexity of X^* (Šmulian [8]).

Applications of Proposition 2 to fixed-point problems are given in [4].

3. LOWER-SEMICONTINUITY

A multivalued mapping $T: E \to F$ is said to be *hemi-lower-semicontinuous* (hemi-L.S.C.) at $x \in D(T)$ if it is L.S.C. at x on each line segment in D(T) with endpoint x, in the $\sigma(F, E)$-topology of F.

We shall say that a point x of a subset D of E is *almost internal* to D if the set of y for which the line segment [x, y] is contained in D distinguishes the points of F. A quasi-internal point is almost internal, but the converse is not true (see the remark above).

PROPOSITION 3. *Let* $T: E \to F$ *be a monotone operator that is not singlevalued at* $x \in D(T)$. *If* x *is almost internal to* D(T), *then* T *is not hemi-L.S.C. at* x.

Proof. Translating T, if necessary, we can assume that $x = 0$. By assumption, there exist u and v in T0 with $u \neq v$. Since 0 is almost internal to D(T), there exists y in D(T) such that the line segment [0, y] is contained in D(T) and

$$\langle y, u - v \rangle \neq 0. \tag{2}$$

Let us denote by G the line $\{ry;\ r \in R\}$, by i the injection mapping of G into E, and by i^* the adjoint projection of F onto G^*. The multivalued mapping $S: G \to G^*$ with domain $D(S) = [0, y]$, defined by $Sz = i^*Tiz\ \ \forall z \in [0, y]$ is easily verified to be monotone; it is not singlevalued at 0, since $i^*u \in S0$, $i^*v \in S0$, and, by (2), $i^*u \neq i^*v$. Giving a suitable orientation to the lines G and G^*, we obtain the relation

$$i^*u < i^*v \leq Sz \qquad \forall z \in \,]0, y].$$

This clearly shows that S is not hemi-L.S.C. at 0. Consequently, T is not hemi-L.S.C. at 0. ■

Remark. An assumption of the kind that x is almost internal to D(T) is needed in Proposition 3: take $D(T) = \{y \in E:\ \langle y, u\rangle = 0\}$, with u in F, $u \neq 0$, and $Ty = \{ru;\ r \in R\}\ \ \forall y \in D(T)$.

Consider now a locally convex vector space X with a Hausdorff topology, and let f be a proper convex function on X, that is, a convex function from X to $]-\infty, +\infty]$ not identically $+\infty$. The subdifferential of f is the (multivalued) mapping $\partial f: X \to X^*$ defined by

$$\partial f(x) = \{u \in X^*;\ f(y) \geq f(x) + \langle y - x, u\rangle \ \ \forall y \in X\}.$$

It is easy to see that $\partial f: X \to X^*$ is monotone. As a corollary of Proposition 3, we have the following result, which may be compared with a theorem of E. Asplund and R. T. Rockafellar [1, p. 460].

PROPOSITION 4. *Let* f *be a lower-semicontinuous, proper, convex function on* X. *Suppose that* f *is (finite and) continuous at a point* x. *Then* f *is Gâteaux-differentiable at* x *if and only if* $\partial f: X \to X^*$ *is hemi-L.S.C. at* x.

Proof. It is well-known [2, p. 92] that a finite convex function on an open convex set V is continuous throughout V if it is continuous at one point of V. Thus, in our case f is continuous on the (nonempty) interior of $\{y \in X;\ f(y) < +\infty\}$. Since $\partial f(y)$ is nonempty at the points y where f is continuous (a consequence of the Hahn-Banach theorem), x is interior (hence almost internal) to $D(\partial f)$. Consequently, by Proposition 3, if $\partial f: X \to X^*$ is hemi-L.S.C. at x, then ∂f is singlevalued at x, and it follows [6, p. 66] from the continuity of f at x that f is Gâteaux-differentiable at x. This proves the first part of the proposition. The converse implication is a consequence of the fact [6, p. 79] that, since f is continuous at x, $\partial f: X \to X^*$, $\sigma(X^*, X)$ is U.S.C. at x. ■

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