THE BIRATIONALITY OF CUBIC SURFACES OVER A GIVEN FIELD

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Let V be a nonsingular cubic surface in 3-dimensional projective space, and assume that V is defined over a given algebraic number field k. It is well known that over the complex numbers any such V is birationally equivalent to a projective plane. The problem of finding necessary and sufficient conditions for V to be birationally equivalent to a projective plane over k was first raised by B. Segre; and partial answers to it have been given by Segre [3] and J. I. Manin [1]. In this paper, I use Segre's methods to give a complete answer to the problem; for the reader's convenience, I have developed the argument *ab initio,* rather than quote intermediate results from [3].

Following Segre, we denote by S_n any subset of the 27 straight lines on V that satisfies the conditions below:

(i) S_n consists of n lines, no two of which meet.

(ii) If S_n contains a line L, then S_n also contains all the conjugates of L over k.

Because of (i), we have that $n \leq 6$. We call an S_n *irreducible* if it consists of a line and its conjugates over k. We shall prove the following result.

THEOREM. *A necessary and sufficient condition that* V *should be birationally equivalent to a projective plane over* k *is that* V *should contain a point defined over* k *and that* V *should have at least one* S_2, S_3, *or* S_6.

The condition that V should contain a point defined over k (which is clearly necessary) can be put into an equivalent form in which it can be more easily checked, if the other condition is satisfied. It follows from the construction below that if V has an S_2, it automatically contains points defined over k. Again, if V has an S_3 or an S_6, then it contains points defined over k if and only if it contains points defined over each $\mathfrak{p}$-adic field, where $\mathfrak{p}$ runs through all the primes of k; for a proof of this result, which was first discovered by Châtelet, see [1] or [4].

Let $\bar{k}$ denote the algebraic closure of k. In what follows, we have to distinguish between the geometric properties of V, which are defined over $\bar{k}$ or the complex numbers, and the arithmetic properties of V, which are defined over k. In the language of schemes, this is just the distinction between $V \otimes_k \bar{k}$ and V. For geometric purposes, we can obtain a model for V as follows. Choose six skew lines on V; each of these is an exceptional curve of the first kind and can therefore be blown down into a point. By blowing down all six of these lines, we birationally transform V (over $\bar{k}$) into a plane containing six distinguished points $P_1, \cdots, P_6$. No three of these points are collinear, and they do not all lie on a conic. The 27 lines on V correspond to the 6 points P_i, the 15 lines P_iP_j, and the 6 conics each of which passes through five of the P_i; from this correspondence, their incidence relations can easily be read off. The plane sections of V correspond to the cubic curves passing

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through all the P_i, and by this means, V can be recovered if one knows the P_i. We shall refer to this as the standard model for V; for a fuller account see for example [2].

Let $\overline{NS}$ denote the geometric Neron-Severi group of V. It is a free abelian group of rank 7; and we can take as its generators $\lambda_0, \lambda_1, \cdots, \lambda_6$, where λ_0 is the class of those curves on V that correspond to general straight lines in the standard model, and λ_i $(i > 0)$ is the class of ℓ_i, the image of the point P_i in V. The class

$$\alpha = \sum n_i \lambda_i$$

corresponds in the standard model to curves of degree n_0 having a $(-n_i)$-fold point at P_i, for $i = 1, 2, \cdots, 6$. If

$$\alpha = \sum n_i \lambda_i \quad \text{and} \quad \alpha' = \sum n'_i \lambda_i$$

are any two classes in $\overline{NS}$, it follows from the last remark that their intersection number is given by the expression

$$(\alpha \cdot \alpha') = n_0 n'_0 - n_1 n'_1 - \cdots - n_6 n'_6 .$$

This is a bilinear form of rank 7 and signature -5.

Now let σ be some element of G, the Galois group of $\bar{k}$ over k, and let $\mathfrak{a}$, $\mathfrak{b}$ be two linearly equivalent divisors on V, both defined over $\bar{k}$. Thus $\mathfrak{a} - \mathfrak{b} = (f)$, for some function f on V defined over $\bar{k}$; hence $\sigma\mathfrak{a} - \sigma\mathfrak{b} = (\sigma f)$, so that $\sigma\mathfrak{a}$ and $\sigma\mathfrak{b}$ are linearly equivalent. This induces a natural action of G on $\overline{NS}$; denote by NS the subgroup of $\overline{NS}$ consisting of those elements of $\overline{NS}$ that are fixed under every element of G. It is not necessarily true that each element of NS contains a divisor defined over k; but this is true provided V contains points defined over k. For let λ be in NS. By adding a sufficiently large multiple of the class of plane sections of V (which can certainly be realized over k), we can assume that λ contains positive divisors. We can uniquely specify a positive divisor in λ by requiring it to pass through a certain number of prescribed points of V; and if these points are all defined over k, the resulting divisor is also defined over k. We do not even need to use Segre's theorem that if V contains one point defined over k it contains infinitely many; for if P is defined over k, we can find as many points as we need infinitely close to P and defined over k.

To find NS, we use the following lemma, which is essentially due to Segre [3].

LEMMA 1. $NS \otimes_Z Q$, *as a vector space over* Q, *is generated by the class of plane sections of* V *and the classes of the* S_n *on* V, *if any.*

Certainly, all these classes lie in NS. If, for some V, they did not generate $NS \otimes Q$, we could find a nonzero class α in NS whose intersection number with each of the classes in the lemma was zero. Let $\mathfrak{a}$ be a divisor in α defined over $\bar{k}$. Replacing $\mathfrak{a}$ by the sum of all its conjugates over k (each of which lies in α), and replacing α by a suitable multiple of itself, we can assume that $\mathfrak{a}$ is defined over k. Among the 27 lines ℓ on V, let ℓ_0 be the one that maximizes $|(\mathfrak{a} \cdot \ell)|$, the absolute value of the intersection number of ℓ with $\mathfrak{a}$. Note that if ℓ_1 is conjugate to ℓ_0 over k, then

$$(\mathfrak{a} \cdot \ell_0) = (\mathfrak{a} \cdot \ell_1),$$

because $\mathfrak{a}$ is defined over k.

(i) If ℓ_0 belongs to some S_n, then $(\alpha \cdot \ell_0) = n^{-1}(\alpha \cdot S_n) = 0$, by the original choice of α.

(ii) If ℓ_0 does not belong to any S_n, there is a conjugate ℓ_1 of ℓ_0 that meets ℓ_0. Let ℓ_2 be the unique line that meets ℓ_0 and ℓ_1; then $\ell_0 + \ell_1 + \ell_2$ is a plane section of V, and therefore it has intersection number zero with α, by the original choice of α. But now

$$(\alpha \cdot \ell_2) = -2(\alpha \cdot \ell_0),$$

and the maximality property of ℓ_0 shows that $(\alpha \cdot \ell_0) = 0$.

Thus in either case, we have that $(\alpha \cdot \ell_0) = 0$, and hence $(\alpha \cdot \ell) = 0$ for each ℓ. By what we know of $\overline{NS}$, this shows that the class of α in $\overline{NS}$, and hence also in NS, is zero; and this contradiction proves the lemma.

It is not true that the classes in the lemma always generate NS as an abelian group. For suppose, in the language of the standard model, that there is just one S_n, which is the S_6 given by $\ell_1 + \cdots + \ell_6$. The class of plane sections of V is $3\lambda_0 - \lambda_1 - \cdots - \lambda_6$; and the group generated by these two classes contains $3\lambda_0$ but not λ_0.

LEMMA 2. *Suppose that* V *contains points defined over* k. *A necessary and sufficient condition for* V *to be birationally equivalent to a projective plane over* k *is that there should exist on* V *a linear system of irreducible curves of self-intersection* 1, *freedom* 2, *and arithmetic genus* 0, *and that the class of these curves should be in* NS.

We shall show below that the three arithmetic conditions on the curves of the system are not independent. The curves may of course be constrained to have prescribed multiplicities at certain base points, some of which may be infinitely near points.

Let Π denote a projective plane, and suppose first that there is a birational map $\phi: \Pi \to V$ defined over k. The straight lines on Π form a linear system of irreducible curves of self-intersection 1, freedom 2, and arithmetic genus 0; hence their images under ϕ have the same properties. Moreover, the image under ϕ of a straight line on Π defined over k is a curve on V defined over k, and therefore its class is in NS. This proves the necessity of the condition.

Now suppose that such a linear system exists. By the remarks before Lemma 1, we can choose a curve Γ_0 of the system defined over k. Consider the functions f defined over k such that $(f) + \Gamma_0$ is a curve of the system; because the system is linear and has freedom 2, these functions form a vector space over k of dimension 3. Let f_0, f_1, f_2 be a base for this space. The map $V \to \Pi$ given by the correspondence

$$P \to (f_0(P), f_1(P), f_2(P))$$

is defined over k and takes curves of the linear system on V onto straight lines on Π. Moreover, this map has degree 1; for let P_1 be in general position on V, and let P_2 be a point of V whose image in Π is the same as that of P_1. Then every curve of the linear system that passes through P_1 would have to pass through P_2, and this is impossible, because the general curve of the system is irreducible and has self-intersection 1. This completes the proof of Lemma 2.

Now consider an arbitrary linear system on V that has assigned base points Q_j with multiplicity m_j and whose class in $\overline{NS}$ is $\sum n_i\lambda_i$. By considering the standard model, or otherwise, it is easy to see that for this system the following conditions hold:

$$(1)\quad \begin{cases} \text{self-intersection} = n_0^2 - \sum n_i^2 - \sum m_j^2, \\ \text{arithmetic genus} = \frac{1}{2}(n_0 - 1)(n_0 - 2) - \sum \frac{1}{2} n_i(n_i + 1) - \sum \frac{1}{2} m_j(m_j - 1), \\ \text{freedom} \geq \frac{1}{2} n_0(n_0 + 3) - \sum \frac{1}{2} n_i(n_i - 1) - \sum \frac{1}{2} m_j(m_j + 1), \end{cases}$$

where the sums over i are taken for $i = 1, 2, \cdots, 6$. (We may not have equality in the equation for the freedom, because the constraints imposed by the Q_j may not be independent.)

LEMMA 3. (i) *If* V *has an* S_4 *or an* S_5, *it has an* S_2 *or an* S_6.

(ii) *If* V *has an* S_2, *then* V *is birationally equivalent to a plane over* k.

(iii) *If* V *has an* S_3 *or an* S_6 *and* V *contains a point defined over* k, *then* V *is birationally equivalent to a plane over* k.

There are just two lines on V that meet all the lines of a given S_4; these two lines are skew and therefore form an S_2. There are two possible types of S_5 on V. One type can be taken as the images of P_1, P_2, P_3, P_4, and $P_5 P_6$ in the standard model; it has just two transversals (the images of $P_1 P_2 P_3 P_4 P_5$ and $P_1 P_2 P_3 P_4 P_6$), and these must form an S_2. The other type can be taken as the images of P_1, P_2, P_3, P_4, and P_5; there is just one line that meets none of these (the image of P_6), and by adjoining it to the original S_5, we obtain an S_6. This completes the proof of (i); for more details see [2].

Now suppose that V has an S_2, and let Π be a plane in general position in the space in which V is embedded. If P is a point of V defined over k and in general position, there is a unique transversal through P to the lines of S_2; and this transversal is defined over k and meets Π in a point P' defined over k. If P' is in general position, it comes from a unique P; hence $P \to P'$ determines a birational map $V \to \Pi$ defined over k. This proves (ii).

If, in the standard model, the lines of the S_2 correspond to the images of $P_1 P_3 P_4 P_5 P_6$ and $P_2 P_3 P_4 P_5 P_6$, then the class of the S_2 is

$$4\lambda_0 - \lambda_1 - \lambda_2 - 2\lambda_3 - 2\lambda_4 - 2\lambda_5 - 2\lambda_6,$$

NS contains this class and the class of plane sections, and the linear system of Lemma 2 associated with this map is given by the conditions

$$n_0 = 2, \quad n_1 = n_2 = -1, \quad n_3 = n_4 = n_5 = n_6 = 0, \quad m_1 = 1.$$

This corresponds to the conics through P_1, P_2, and Q_1, in the standard model.

If V has an S_6 and a point defined over k, then the map that gives the standard model is a birational map from V to a plane. The linear system of Lemma 2 is given by the conditions

$$n_0 = 1, \quad n_1 = \cdots = n_6 = 0,$$

and it is the system of twisted cubics on V that meets no line of the S_6.

Suppose finally that V has an S_3 and contains a point defined over k. There is now no simple geometric description of the birational map we need, but we can again use the machinery of Lemma 2. Let the lines in the S_3 be ℓ_1, ℓ_2, ℓ_3; then the linear system given by the conditions

$$n_0 = 3, \quad n_1 = n_2 = n_3 = 0, \quad n_4 = n_5 = n_6 = -1, \quad m_1 = 2, \quad m_2 = 1$$

satisfies all conditions of Lemma 2. The curves in the standard model to which the system corresponds are cubics with a double point at Q_1 and simple points at P_4, P_5, P_6, and Q_2; and in general such a curve is irreducible.

This concludes the proof of Lemma 3.

To complete the proof of the theorem, we have to show that if V has no S_n with $n > 1$, then V is not birationally equivalent to Π over k. To do this, we show that there is no linear system of irreducible curves on V satisfying the conditions of Lemma 2. But the larger NS is, the easier it will be to construct such a system. Thus we may assume that NS is maximal, subject to the condition that V does not have an S_n with $n > 1$. Clearly this happens when NS is generated by three coplanar lines, each of which is defined over k. Choose the standard model so that these lines are the images of $P_1 P_4$, $P_2 P_5$, and $P_3 P_6$; thus the general element of NS is given by

$$n_0 = a_1 + a_2 + a_3, \quad n_1 = n_4 = -a_1, \quad n_2 = n_5 = -a_2, \quad n_3 = n_6 = -a_3.$$

By (1), the arithmetic conditions of Lemma 2 now become

(2) $$\sum m_j^2 = 2a_1 a_2 + 2a_2 a_3 + 2a_3 a_1 - a_1^2 - a_2^2 - a_3^2 - 1,$$

(3) $$\sum m_j = a_1 + a_2 + a_3 - 3,$$

and by considering the corresponding curves in the standard model, we have the inequalities

(4) $$a_1 \geq 0, \quad a_2 \geq 0, \quad a_3 \geq 0, \quad m_j \geq 0.$$

To complete the proof of the theorem, we have to show that every curve Γ that satisfies (2), (3), and (4) is reducible.

To do this, we construct for each such curve Γ a curve Γ' (which must be positive but may be reducible) such that the intersection number of Γ and Γ' is strictly negative. This implies that Γ and Γ' have common components, at least one of which has negative self-intersection; and this is impossible if Γ is irreducible, because Γ has self-intersection number 1. We define Γ' by means of values of a_i' and m_j', in the same way as Γ is defined by the a_i and m_j; here the points $Q_j = Q_j'$ are to be the same for both curves. The conditions for Γ' to be positive and realizable are

(5) $$a_1' \geq 0, \quad a_2' \geq 0, \quad a_3' \geq 0, \quad m_j' \geq 0,$$

(6) $$(a_1' + a_2' + a_3')(a_1' + a_2' + a_3' + 3) - 2\sum a_i'(a_i' + 1) \geq \sum m_j'(m_j' + 1),$$

the last of these coming from the inequality in (1) for the freedom of Γ'. The intersection number of Γ and Γ' is

(7) $$(a_1 + a_2 + a_3)(a_1' + a_2' + a_3') - 2 \sum a_i a_i' - \sum m_j m_j' < 0 .$$

We have therefore reduced the proof of the theorem to the proof of the following purely arithmetical lemma.

LEMMA 4. *To each pair of integers* a_i, m_j *satisfying* (2), (3), *and* (4), *there correspond integers* a_i', m_j' *satisfying* (5), (6), *and* (7).

We prove this by induction on the value of $a_1 + a_2 + a_3$. Let $a_i = \alpha_i$, $m_j = \mu_j$ be a solution of (2), (3), and (4), and suppose that the lemma is true for all sets a_i, m_j with

$$a_1 + a_2 + a_3 < \alpha_1 + \alpha_2 + \alpha_3 .$$

At least one μ_j is nonzero, for there are no solutions of (2), (3), and (4) for which all m_j vanish. Renumbering if necessary, we can therefore assume that

$$\alpha_1 \geq \alpha_2 \geq \alpha_3 , \qquad \mu_1 = \operatorname{Max} \mu_j > 0 .$$

Moreover, we need only consider the case where

(8) $$\alpha_1 \leq \alpha_2 + \alpha_3 ,$$

for if $\alpha_1 > \alpha_2 + \alpha_3$, we can satisfy (5) to (7) by taking $a_1' = 1$, $a_2' = a_3' = 0$, $m_j' = 0$. We next prove that

(9) $$\alpha_2 + \alpha_3 > \mu_1 > \alpha_2 + \alpha_3 - \alpha_1 .$$

The first of these inequalities follows from (2) in the form

$$\sum \mu_j^2 = (\alpha_2 + \alpha_3)^2 - (\alpha_2 - \alpha_3)^2 - (\alpha_2 + \alpha_3 - \alpha_1)^2 - 1 ;$$

and the second follows from (2) and (3) in the form

$$\mu_1 \geq \frac{\sum \mu_j^2}{\sum \mu_j} = \alpha_2 + \alpha_3 - \alpha_1 + \frac{2\alpha_2(\alpha_1 - \alpha_2) + 2\alpha_3(\alpha_1 - \alpha_3) + 3(\alpha_2 + \alpha_3 - \alpha_1) - 1}{\alpha_1 + \alpha_2 + \alpha_3 - 1} .$$

Here the numerator of the fraction on the right-hand side is strictly positive, except perhaps when $\alpha_2 + \alpha_3 = \alpha_1$, and in that case the desired inequality follows from the inequality $\mu_1 > 0$.

It follows from (8) and (9) that the a_i, m_j defined by

(10) $$\begin{cases} a_1 = \alpha_2 + \alpha_3 - \mu_1, & a_2 = \alpha_2 , \quad a_3 = \alpha_3 , \\ m_1 = \alpha_2 + \alpha_3 - \alpha_1 , & m_j = \mu_j \text{ for } j > 1 , \end{cases}$$

satisfy (4); and they satisfy (2) and (3) because the α_i, μ_j do. But by (9),

$$a_1 + a_2 + a_3 < \alpha_1 + \alpha_2 + \alpha_3 ,$$

and by the induction hypothesis, it follows that to these a_i, m_j there correspond values of a_i', m_j' satisfying (5), (6), and (7). Let these values be

$$a'_1 = \alpha'_1, \quad a'_2 = \alpha'_2, \quad a'_3 = \alpha'_3, \quad m'_j = \mu'_j. \tag{11}$$

Now, by (6), we have the relations

$$\begin{aligned}\sum \mu'_j(\mu'_j + 1) &\le (\alpha'_1 + \alpha'_2 + \alpha'_3)(\alpha'_1 + \alpha'_2 + \alpha'_3 + 3) - 2\sum \alpha'_i(\alpha'_i + 1)\\ &= (\alpha'_2 + \alpha'_3)(\alpha'_2 + \alpha'_3 + 2) - \left(\alpha'_2 + \alpha'_3 - \alpha'_1 + \frac{1}{2}\right)^2 - (\alpha'_2 - \alpha'_3)^2 + \frac{1}{4}\\ &< (\alpha'_2 + \alpha'_3 + 1)(\alpha'_2 + \alpha'_3 + 2),\end{aligned}$$

whence $\mu'_1 < \alpha'_2 + \alpha'_3 + 1$, and as both sides are integers, it follows that

$$\mu'_1 \le \alpha'_2 + \alpha'_3. \tag{12}$$

Suppose first that $\alpha'_2 + \alpha'_3 \ge \alpha'_1$, and write

$$\begin{cases} a'_1 = \alpha'_2 + \alpha'_3 - \mu'_1, & a'_2 = \alpha'_2, \quad a'_3 = \alpha'_3,\\ m'_1 = \alpha'_2 + \alpha'_3 - \alpha'_1, & m'_j = \mu'_j \quad \text{for} \quad j > 1. \end{cases} \tag{13}$$

I claim that these are the values we need, to dispose of our given set

$$a_1 = \alpha_1, \quad a_2 = \alpha_2, \quad a_3 = \alpha_3, \quad m_j = \mu_j. \tag{14}$$

In view of (12), they certainly satisfy (5). But (6) for the values (13) is precisely the same as (6) for the values (11), and (7) for the values (13) and (14) is precisely the same as (7) for the values (10) and (11). Hence (5), (6), (7) hold, and the lemma is proved in this case. If instead $\alpha'_1 > \alpha'_2 + \alpha'_3$, the only change we need to make in (13) is to choose $m'_1 = 0$ instead of $m'_1 = \alpha'_2 + \alpha'_3 - \alpha'_1$. Now (5) is satisfied, and since this change diminishes $m'_1(m'_1 + 1)$ and $-m_1 m'_1$, (6) and (7) are still satisfied, since they were satisfied before. This completes the proof of Lemma 4 and thus also of the theorem.

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