NONLINEAR HYPERBOLIC PROBLEMS WITH SOLUTIONS ON PREASSIGNED SETS

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1. INTRODUCTION AND SUMMARY

This paper is concerned with initial-value problems and mixed boundary problems for hyperbolic systems of quasi-linear partial differential equations in two independent variables. The setting is classical: we shall assume that the coefficients of the equations and the data possess continuous first derivatives, and the solutions will be C^1-functions satisfying the equations everywhere on their domains of definition.

We consider the hyperbolic initial-value problem

$$z_t + A(x, t, z)z_x = f(x, t, z) \qquad ((x, t) \in \mathscr{R}(T)), \tag{1.1}$$

$$z(x, 0) = \phi(x) \qquad (x \in R), \tag{1.2}$$

where R is the set of real numbers, $\mathscr{R}(T)$ is the strip $R \times [0, T]$, the function $z = z(x, t)$ takes values in R_m (the real euclidean m-dimensional space), A is a matrix-valued function, and f, ϕ are vector-valued functions. Our main object is to consider the following problem.

Problem I. Find conditions on A and f that guarantee the existence of a class of initial conditions for each $T > 0$ such that for every ϕ in such a class, the Cauchy problem (1.1), (1.2) has a solution z on the strip $\mathscr{R}(T)$.

Results of this type are useful, in particular, in the control theory of hyperbolic equations; see [2].

It is known (see for instance P. Lax [9], P. Hartman and A. Wintner [7, p. 855], A. Douglis [6, p. 149], M. Cinquini-Cibrario and S. Cinquini [1, Chapter V]) that if (1.1) is hyperbolic and A, f, ϕ are prescribed C^1-functions that are bounded, together with their first derivatives, then there exist some real number $T > 0$ and a (unique) C^1-function $z = z(x, t)$ on $\mathscr{R}(T)$ satisfying (1.1), (1.2).

However, even if $\phi = 0$, the local solution z cannot be extended to an arbitrarily prescribed strip $\mathscr{R}(T')$ under the above hypotheses. We give a scalar example in Section 3 to illustrate this fact. On the other hand, we shall prove (Theorem 3.II) that if A, f satisfy the conditions above and if in addition $f_x = 0$, then for each $T' > 0$, there is a class of initial data (namely C^1-functions with sufficiently small first derivatives) such that if ϕ is chosen in that class, the local solution z can be extended to $\mathscr{R}(T')$.

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This result becomes obvious as soon as one realizes that if $f_x = 0$, it is possible to prove (Theorem 2.IV) that for some real number N, the x-derivative of the solution satisfies, on its domain of definition, the a priori bound

$$|z_x| \leq N|\phi'| .$$

Thus, under the additional hypothesis that $f_x = 0$, the solution of (1.1), (1.2) for the case where ϕ is a constant exists on the whole plane. Hence the continuation theorem mentioned above is essentially a perturbation result. Similar a priori bounds and continuation results hold also for the mixed boundary problem.

We shall now introduce some notation and definitions. By $R_{m\times n}$, we denote the usual space of real matrices with m rows and n columns; $C(\Omega, R)$, where $\Omega \subset R_m$, is the set of real-valued continuous functions on Ω, and $C^1(\Omega, R)$ is its subset of continuously differentiable functions; $C^1(\Omega, R_m)$ and $C^1(\Omega, R_{m\times m})$ denote the natural elementwise generalizations of $C^1(\Omega, R)$. We reserve the symbol $|\cdot|$ for sup-norms.

Definition. We define $|h|$ as follows:

$$|h| = \begin{cases} \text{absolute value of } h & (h \in R), \\ \max\{|h_i|: i = 1, \cdots, m\} & (h = (h_i) \in R_m), \\ \max\left\{\sum_{j=1}^{n} |h_{ij}|: i = 1, \cdots, m\right\} & (h = (h_{ij}) \in R_{m\times n}), \\ \sup\{|h(x)|: x \in X\} & (h \text{ is a function defined on a set } X \text{ with values in } R_m \text{ or } R_{m\times m}). \end{cases}$$

Suppose $\alpha \in [0, \infty]$ and $\mathscr{R} \subset R_2$; the set $\mathscr{R}_\alpha$ is defined by the condition

$$\mathscr{R}_\alpha = \{(x, t, w): (x, t) \in \mathscr{R},\ w \in R_m,\ |w| \leq \alpha\}.$$

Definition (the class $\Sigma(\mathscr{R}, m, \alpha)$). Fix $\mathscr{R}$ ($\mathscr{R} \subset R_2$), m, α ($0 \leq \alpha \leq \infty$); we say that $A \in \Sigma(\mathscr{R}, m, \alpha)$ if the following holds:

$A = A(x, t, w)$ belongs to $C^1(\mathscr{R}_\alpha, R_{m\times m})$, and there exists an

$$S \in C^1(\mathscr{R}_\alpha, R_{m\times m})$$

such that for some $\delta > 0$ and $k > 0$, the following three conditions are satisfied:

(i) $|\det S(x, t, w)| > \delta$ for all (x, t, w);

(ii) $D(x, t, w) = S(x, t, w)A(x, t, w)S^{-1}(x, t, w)$ is a diagonal matrix for every (x, t, w); let $d_i = d_i(x, t, w)$ $(i = 1, \cdots, m)$ denote its diagonal elements;

(iii) if β denotes S or D or any of their first partial derivatives, then $|\beta(x, t, w)| \leq k$, for all (x, t, w).

Also, $(A, f) \in \Sigma = \Sigma(\mathscr{R}, m, \alpha)$ means that $A \in \Sigma$, $f = f(x, t, w)$ belongs to $C(\mathscr{R}_\alpha, R_m)$, the partial derivatives $f_x, f_{w^1}, \cdots, f_{w^m}$ exist and are continuous on $\mathscr{R}_\alpha$, and if β is f or any of these derivatives, then $|\beta(x, t, w)| \leq k$ for all (x, t, w)

and some $k > 0$. If $\delta > 0$, $k > 0$ are prescribed, we write $(A, f) \in \Sigma(\mathscr{R}, m, \alpha, \delta, k)$ if $(A, f) \in \Sigma(\mathscr{R}, m, \alpha)$ and the inequalities above hold for the prescribed δ and k.

Note that (i) and (ii) amount to the definition of the statement "the system (1.1) is hyperbolic on $\mathscr{R}_\alpha$," and that in view of (i), condition (iii) implies that S^{-1} and A are bounded, together with their first partial derivatives, where S^{-1} is the map $(x, t, w) \to (S(x, t, w))^{-1}$. Also, if $\mathscr{R}$ is compact and α is real, then in the definition of $A \in \Sigma(\mathscr{R}, m, \alpha)$, condition (iii) is redundant, and it is enough that (i) holds for $\delta = 0$.

2. INITIAL-VALUE PROBLEM: A PRIORI ESTIMATES

It is known (see Lax [8, p. 97], [9, p. 242], Douglis [6, p. 127], Cinquini-Cibrario and Cinquini [1, Theorem 6, p. 397]) that if z solves (1.1), (1.2), then, roughly speaking, the first derivatives of z satisfy initial-value problems in diagonal form and hence certain systems of integral equations. The next lemma recalls this fact.

LEMMA 2.I. *Let* $\mathscr{R} = R \times [0, \infty)$, $(A, f) \in \Sigma(\mathscr{R}, m, \infty, \delta, k)$, $\phi \in C^1(R, R_m)$, *and suppose there exists* $z \in C^1(\mathscr{R}(T), R_m)$ *satisfying* (1.1) *and* (1.2). *Define* v *on* $\mathscr{R}(T)$ *by the condition*

$$v(x, t) = S(x, t, z(x, t))\, z_x(x, t)\,,$$

and for $(x, t) \in \mathscr{R}(T)$, *let* $\xi_i(s) = \xi_i(s; x, t)$ $(i = 1, \cdots, m)$ *be defined by the conditions*

$$\frac{d}{ds}\,\xi_i(s) = d_i(\xi_i(s), s, z(\xi_i(s), s)) \quad (s \in [0, t]), \qquad \xi_i(t) = x\,.$$

Under the hypotheses above, there exist bounded continuous functions $h = h(x, t, w)$ *from* $\mathscr{R}_\infty$ *to* R_m *and* $H = H(x, t, w)$, $H^i = H^i(x, t, w)$ $(i = 1, \cdots, m)$ *from* $\mathscr{R}_\infty$ *to* $R_{m\times m}$ *such that for each* $(x, t) \in \mathscr{R}(T)$ *and each integer* i $(i = 1, \cdots, m)$, *we have the relation*

$$v^i(x, t) = v^i(\xi_i(0), 0) + \int_0^t [\langle v, H^i v\rangle + (Hv)^i + h^i]\, ds\,. \tag{2.1}$$

Here H^i, H, *and* h^i *are evaluated at* $(\xi_i(s), s, z(\xi_i(s), s))$ [*thus, for instance,* H^i *stands for* $H^i(\xi_i(s), s, z(\xi_i(s), s))$]; w^i *denotes the* i*th component of* w $(w \in R_m)$; v *stands for* $v(\xi_i(s), s)$; *and* $\langle u, w\rangle = \sum_{i=1}^m u^i w^i$, *provided* $u, w \in R_m$. *Furthermore,* h *is given by the expression*

$$h(x, t, w) = S(x, t, w) f_x(x, t, w)\,.$$

Remark 2.II. If z is twice continuously differentiable, the content of the lemma amounts essentially to the computation in [8, p. 97], the difference being that here we seek the diagonal system satisfied by the derivatives of Sz_x rather than that satisfied by the derivatives of Sz_t. The procedure for obtaining such a diagonal system can be summarized as follows: multiply the identity (1.1) by the left eigenvectors S of $A(x, t, z)$, introduce v and $u = S(x, t, z)\, z_t$, and differentiate with respect to the geometric variable x; in the result, replace z_x, z_w, and u_x by their obvious equivalent in terms of v, namely, by $S^{-1} v$, by $f - S^{-1} Dv$, and by the expression for u_x

obtained by differentiating u and v and using the identity $z_{xt} = z_{tx}$. The result thus obtained has the form

$$v_t^i + d_i v_x^i = \langle v, H^i v \rangle + (Hv)^i + h^i \qquad (i = 1, \cdots, m;\ (x, t) \in \mathscr{R}(T)),$$

$$v(x, 0) = S(x, 0, \phi(x))\phi'(x) \qquad (x \in R),$$

where the functions H^i, H, h are independent of v, and, together with the eigenvalue d_i, are evaluated at (x, t, z(x, t)). Hence v satisfies a quadratic initial-value problem in diagonal form. Specializing (x, t) to $(\xi_i(s), s)$, using the chain rule, and integrating, we obtain (2.1). These integral identifies hold even if z is only assumed to be of class C^1; this is proved in [1, Theorem 6, p. 397] and can be proved as indicated in [6, p. 127]. As for the functions appearing in the diagonal system above, it is useful to retain that each entry of h(x, t, w), H(x, t, w), $H^i(x, t, w)$ $(i = 1, \cdots, m)$ is a finite sum of finite products of the entries of

$$S^{-1},\ S,\ D,\ f,\ S_t,\ S_x,\ D_x,\ f_x,\ S_{w^i},\ D_{w^i},\ f_{w^i} \qquad (i = 1, \cdots, m) \tag{2.2}$$

evaluated at (x, t, w).

Thus the functions h, H, H^i are defined and continuous, whenever the functions (2.2) are defined and continuous, and this is true if $(A, f) \in \Sigma = \Sigma(\mathscr{R}, m, \infty, \delta, k)$. Moreover, for such (A, f), one can easily see that there exists a real number $M = M(m, \delta, k) > 0$, depending solely on m, δ, and k, such that

$$\max(|S^{-1}|, |h|, |H|) \leq M, \qquad |H^i| \leq \frac{M}{m} \qquad (i = 1, \cdots, m), \tag{2.3}$$

$$|\langle u, H^i(x, w, t)u \rangle| \leq M|u|^2, \text{ for all } (x, t, w, u). \tag{2.4}$$

Our a priori estimates will be based on the following simple lemma asserting that all nonnegative solutions of a certain integral inequality are uniformly bounded by the solution of the corresponding integral equation. Suppose T $(T > 0)$ and N are real numbers, and $M = M(t)$ is a nonnegative real-valued function defined and Lebesgue-integrable on [0, T]. For $t \in [0, T]$, define $\|M\|(t)$, w(t), $\|M\|$ by the conditions

$$\|M\|(t) = \int_0^t M(s)\,ds; \qquad \|M\| = \|M\|(T); \qquad w(t) = \frac{N}{-N + (N+1)\exp\{-\|M\|(t)\}}.$$

LEMMA 2.III. *Let* T, N, M, w(t) *be defined as above. Assume*

$$0 \leq N \leq \exp(-\|M\|),$$

and suppose $u = u(t)$ *is a continuous real-valued function on* [0, T] *satisfying the inequalities*

$$0 \leq u(t) \leq N + \int_0^t M(s)[u^2(s) + u(s)]\,ds, \qquad \textit{for all } t \in [0, T].$$

Then

$$u(t) \leq w(t) \leq w(T) \leq N\exp(2\|M\|), \qquad \textit{for all } t \in [0, T].$$

We omit the proof of Lemma 2.III, because it is simple and of well-known type (see for instance [13, p. 12]); indeed, the conclusion follows almost immediately from the monotonicity of the kernel of the integral appearing in the hypotheses. Some applications of similar comparison results to nonlinear hyperbolic equations can be found in [10] and [1, Chapter IV].

The following theorem gives an a priori estimate on the derivative z_x of the solution of (1.1), (1.2); it asserts that if the usual hypotheses on A and f insuring the existence of a local solution are satisfied and if in addition $f_x = 0$, then, on a preassigned strip and for all initial data ϕ whose derivative ϕ' is sufficiently small, z_x cannot be too large and indeed can be made arbitrarily small by taking ϕ' sufficiently small.

THEOREM 2.IV. *Fix* $\delta > 0$, $k > 0$, $T' > 0$; *let* m *be a positive integer, and put* $\mathscr{R} = R \times [0, \infty)$, $\mathscr{R}(T) = R \times [0, T]$. *Then there exist positive numbers* c *and* N, *where* c *depends only on* m, δ, k, *and* T', *such that if*

$$
(2.5)\quad \begin{cases} (A, f) \in \Sigma = \Sigma(\mathscr{R}, m, \infty, \delta, k), \quad f_x = 0, \quad \phi \in C^1(R), \quad |\phi'| \leq c, \\ 0 < T \leq T', \quad \textit{and the function } z \in C^1(\mathscr{R}(T)) \textit{ satisfies the Cauchy problem} \\ \qquad z_t + A(x, t, z) z_x = f(t, z) \qquad (\textit{on } \mathscr{R}(T)), \\ \qquad z(x, 0) = \phi(x) \qquad (\textit{on } R), \end{cases}
$$

then

$$
|z_x| \leq N |\phi'| .
$$

Remark 2.V. In the proof, we shall show that if c is fixed so that $0 < ck \leq e^{-MT'}$, where $M = M(m, \delta, k)$ is as in (2.3) and (2.4), then

$$
|z_x(x, t)| \leq \frac{|\phi'|\ |S|\ |S^{-1}|}{|\phi'|\ |S|(e^{-Mt} - 1) + e^{-Mt}} \leq kMe^{2MT'} |\phi'|
$$

for all $(x, t) \in \mathscr{R}(T)$, whenever (2.5) holds. Note that f_x has been assumed to vanish everywhere (whence f is independent of x) only to obtain $h = 0$ in (2.1).

Proof. Fix c so that

$$
0 < ck \leq e^{-MT'} .
$$

Let $A = S^{-1}DS$, f, ϕ, T, z be as in the hypotheses, and define $v = v(x, t)$ on $\mathscr{R}(T)$ by the relation

$$
(2.6)\qquad v = S(x, t, z) z_x .
$$

Fix $(x, t) \in \mathscr{R}(T)$, and for $s \leq t$, define $\xi_i = \xi_i(s)$ $(i = 1, \cdots, m)$ by the conditions

$$
(2.7)\qquad \frac{d}{ds}\xi_i(s) = d_i(\xi_i(s), s, z(\xi_i(s), s)) ,
$$

$$
(2.8)\qquad \xi_i(t) = x .
$$

Since d_i is bounded, it follows from a simple result in ordinary differential equations (see for instance [3, p. 15]) that the solution ξ_i of (2.7), (2.8) can be continued until it reaches the boundary of $\mathscr{R}(T)$; hence ξ_i is a C^1-function defined on $[0, t]$. The trivial case $0 = t$ is not excluded.

Let $H^i(\xi_i(s))$ and $H(\xi_i(s))$ denote

$$H^i(\xi_i(s), s, z(\xi_i(s), s)) \quad \text{and} \quad H(\xi_i(s), s, z(\xi_i(s), s)),$$

respectively. Since $f_x = 0$, it follows from Lemma 2.I and Remark 2.II that for all i $(i = 1, \cdots, m)$, the components $v^i(x, t)$ satisfy the condition

$$v^i(x, t) = v^i(\xi_i(0), 0) + \int_0^t [\langle v(\xi_i(s), s), H^i(\xi_i(s)) v(\xi_i(s), s)\rangle + (H(\xi_i(s)) v(\xi_i(s), s))^i] ds .$$

Hence, in view of the properties of the functions H^i and H (see Remark 2.II),

$$|v^i(x, t)| \leq |S| \, |\phi'| + M \int_0^t (|v(\xi_i(s), s)|^2 + |v(\xi_i(s), s)|) ds . \tag{2.9}$$

Let

$$x_0 = x - tk , \quad x_1 = x + tk ,$$

and let τ be the triangle with vertices $(x_0, 0)$, (x, t), $(x_1, 0)$; then $\tau \subset \mathscr{R}(T)$. If $(\tilde{x}, \tilde{t}) \in \tau$ and $\tilde{\xi}_i$ $(1 \leq i \leq m)$ is the solution of (2.7) on $[0, \tilde{t}]$ satisfying $\tilde{\xi}_i(\tilde{t}) = \tilde{x}$, then $(\tilde{\xi}_i(s), s) \in \tau$ for all $s \in [0, \tilde{t}]$; this is true because k bounds $|d_i|$. In other words, characteristics that start in τ remain in τ until they reach the lower boundary $(x, 0)$ of τ. Hence (2.9) holds also for each $(\tilde{x}, \tilde{t}) \in \tau$, the integrand being evaluated on the corresponding set $(\tilde{\xi}_i(s), s)$ $(s \in [0, \tilde{t}])$.

For $0 \leq s \leq t$, define

$$|v|(s) = \max \{ |v(\tilde{x}, \tilde{t}) : (\tilde{x}, \tilde{t}) \in \tau, \ 0 \leq \tilde{t} \leq s \} .$$

Then $|v|(s)$ is a nondecreasing, continuous function on $[0, t]$, and by (2.9) and what has just been established, it follows that

$$|v|(\tilde{t}) \leq |S| \, |\phi'| + M \int_0^{\tilde{t}} [(|v|(s))^2 + |v|(s)] ds , \qquad \text{for all } \tilde{t} \in [0, t] .$$

By Lemma 2.III, we have that

$$|v(x, t)| \leq \frac{|S| \, |\phi'|}{|\phi'| \, |S| (e^{-Mt} - 1) + e^{-Mt}} .$$

Hence the conclusion of the theorem follows from (2.6), the bound on $|S^{-1}|$, and the arbitrariness of $(x, t) \in \mathscr{R}(T)$.

3. INITIAL-VALUE PROBLEMS WITH SOLUTIONS ON A PREASSIGNED SET

From the standard existence theorem for the initial-value problem (1.1), (1.2) follows that the possibility of continuing the local solution z to a preassigned strip $\mathscr{R}(T')$ depends solely on the availability of an a priori bound for $|z_x|$ on $\mathscr{R}(T')$. Theorem 2.IV makes one such bound available. Hence the conditions on A and f in this theorem give an answer to Problem I.

We shall now formalize these remarks. The following well-known result is the standard existence theorem mentioned previously.

THEOREM 3.I. *Fix* m, δ, *and* k, *put* $\mathscr{R} = R \times [0, \infty)$, $\mathscr{R}(T) = R \times [0, T]$, *and let* $r > 0$. *Then there exists a positive number* T *such that if*

$$(A, f) \in \Sigma(\mathscr{R}, m, \infty, \delta, k) \quad \textit{and} \quad \phi \in C^1(R) \textit{ with } |\phi'| \leq r,$$

then there exists a (unique) $z \in C^1(\mathscr{R}(T), R_m)$ *that satisfies the Cauchy problem* (1.1), (1.2).

Proofs may be found in the papers by Hartman and Wintner [7, Theorem VI, p. 855] and Douglis [6, Theorem 8, p. 149]; see also Lax [9] and Cinquini-Cibrario and Cinquini [1, Chapter V]. Actually, in [7] and [6], the initial data are given on a compact interval, and the solution is proved to exist in a trapezoidal region. Thus, to obtain Theorem 3.I, it suffices to cover the initial line with compact intervals of fixed length and fixed amount of overlapping, and then apply the result in [7] or [6] to each such interval.

The following answer to our main question is an immediate consequence of Theorems 2.IV and 3.I.

THEOREM 3.II. *Fix* m, δ, *and* k, *put* $\mathscr{R} = R \times [0, \infty)$, *and let* $T' > 0$. *Then there exist two numbers* $c > 0$ *and* $N > 0$ *such that if*

$$(A, f) \in \Sigma(\mathscr{R}, m, \infty, \delta, k), \quad f_x = 0, \quad \phi \in C^1(R), \quad |\phi'| \leq c, \tag{3.1}$$

then the local solution z *given by Theorem* 3.I *can be (uniquely) continued to a solution* z *of* (1.1), (1.2) *on* $\mathscr{R}(T') = R \times [0, T']$; *moreover,* z *satisfies the inequality*

$$|z_x(x, t)| \leq N|\phi'|$$

for all $(x, t) \in \mathscr{R}(T')$.

Proof. Fix c and N so that the conclusion of Theorem 2.IV holds. Then

$$|z_x(x, t)| \leq Nc, \quad \text{for all } (x, t) \in \mathscr{R}(T),$$

whenever (3.1) holds, $0 < T \leq T'$, and $z \in C^1(\mathscr{R}(T))$ satisfies the initial-value problem (1.1), (1.2). Now the conclusion follows by applying Theorem 3.I finitely often with initial data on the lines $t = 0$, $t = T$, $t = 2T$, and so forth, where $T > 0$ corresponds to $r = Nc$.

COROLLARY 3.III. *If* $(A, f) \in \Sigma(\mathscr{R}, m, \infty)$, $f_x = 0$, *and* $\phi' = 0$, *the solution of* (1.1), (1.2) *exists on the half-plane* $\mathscr{R}$.

This corollary has little to do with partial differential equations, since if ϕ is constant, Theorem 3.II implies that z_x must be identically zero; hence (1.1) reduces to an ordinary differential equation with bounded right-hand side, for the restriction of z to any line $x = \text{constant}$.

We now restate Theorem 3.II for the case in which A, f, ϕ are defined on compact sets. Let $a < b$ and $A = S^{-1} DS \in \Sigma(\mathscr{R}, m, \alpha)$, and let τ be the triangular region

$$\tau = \left\{ (x, t): a + |D|t \leq x \leq b - |D|t, \quad 0 \leq t \leq \frac{b-a}{|D|} \right\}.$$

In view of [7, (VI)] and the proof of Theorem 2.IV, it is clear that the following analogue of Theorem 3.II holds.

THEOREM 3.IV. *Let* $a < b$, $T > 0$, $0 < c_0 < \alpha$, *and put* $\mathscr{R} = [a, b] \times [0, T]$. *Suppose* $(a, f) \in \Sigma(\mathscr{R}, m, \alpha)$, $f_x = 0$, *and let* τ *be defined as above. Then there exist two positive numbers* c *and* N *such that if* $\phi \in C^1([a, b])$, $|\phi| \leq c_0$, *and* $|\phi'| \leq c$, *then there exists a (unique)* $z \in C^1(\tau \cap \mathscr{R})$ *that satisfies the conditions*

$$z_t + A(x, t, z)z_x = f(t, z), \qquad \textit{on } \tau \cap \mathscr{R},$$

$$z(x, 0) = \phi(x), \qquad \textit{on } [a, b],$$

and moreover,

$$|z| \leq \min(\alpha, 2c_0) \qquad \textit{and} \qquad |z_x| \leq N|\phi|.$$

Let us note that, except for the requirement that $f_x = 0$, the hypotheses in Theorem 3.II are exactly those of the standard existence theorem. If this extra condition is dropped, the continuation assertion in Theorem 3.II may fail, and hence the remaining conditions on A and f do not solve Problem I. The kind of difficulty that may arise is shown in the following example.

Example. Suppose T is a positive real number. Consider the scalar initial-value problem

$$u_t + d(u)u_x = f(x), \tag{3.2}$$

$$u(x, 0) = 0 \qquad (x \in R). \tag{3.3}$$

We shall show that there exist real-valued functions d and f defined and bounded on the real line R, together with their first derivatives, such that the local solution $u = u(x, t)$ of (3.2), (3.3) cannot be extended to $\mathscr{R}(T) = R \times [0, T]$. To this end, fix $\varepsilon > 0$, subject to the condition $20\sqrt{\varepsilon} \leq T$, and define

$$f(x) = 1 - \frac{x}{\sqrt{\varepsilon + x^2}} \qquad (x \in R),$$

$$d(z) = 1 + \frac{z}{\sqrt{\varepsilon + z^2}} \qquad (z \in R).$$

Observe that for every $x \in R$ and $z \in R$, the following relations hold:

$$0 < f(x) < 2, \qquad f'(x) = \frac{-\varepsilon}{(\varepsilon + x^2)^{3/2}}, \qquad \int_0^x f(s)\,ds \leq \sqrt{\varepsilon}\,; \tag{3.4}$$

$$0 < d(z) < 2, \qquad d'(z) = \frac{\varepsilon}{(\varepsilon + z^2)^{3/2}}, \qquad \max\{d'(z): z \in R\} = \frac{1}{\sqrt{\varepsilon}}.$$

We assert that (3.2), (3.3), with f and d defined as above, have no solution on $\mathscr{R}(T)$. To see this, assume it is false. Then there exists a C^1-function u satisfying (3.2) and (3.3) on $\mathscr{R}(T)$; moreover $u \in C^2(\mathscr{R}(T))$, because f and d are of class C^2 (see [6, Lemma 7.3, p. 141]). Define $\xi \in C^1([0, T])$ as the integral

$$\xi(t) = \int_0^t d(u(\xi(s), s))\,ds \qquad (t \in [0, T]).$$

In view of the standard existence and uniqueness theorem for ordinary differential equations, the function $\xi = \xi(t)$ is well defined; it exists on [0, T], because d is bounded. Put

$$\underline{u}(t) = u(\xi(t), t) \qquad (t \in [0, T]).$$

Observe that for each $t \in [0, T]$,

$$0 \le \underline{u}(t) \le \sqrt{\varepsilon}, \qquad d'(\underline{u}(t)) \ge \frac{1}{3\sqrt{\varepsilon}}; \tag{3.5}$$

in fact, by (3.2), (3.3), the chain rule, and the positivity of f, it follows that

$$0 \le \underline{u}(t) \le \int_0^t f(\xi(s))\,ds.$$

Also, if $0 \le z < \infty$, then $1 \le d(z) < 2$. In view of the definition of ξ, we have that

$$t \le \xi(t) \le 2t \qquad (t \in [0, T]), \tag{3.6}$$

and, since f is decreasing,

$$\underline{u}(t) \le \int_0^t f(s)\,ds \le \sqrt{\varepsilon}.$$

Thus (3.5) is proved, for its second part is an immediate consequence of this last bound.

Put

$$v(s) = u_x(\xi(s), s) \qquad (s \in [0, T]). \tag{3.7}$$

Since $u \in C^2$, differentiation of the identities (3.7), (3.2) gives the identity

$$v'(s) = -d'(\underline{u}(s))v^2(s) + f'(\xi(s)) \qquad (s \in [0, T]).$$

Hence, by (3.5) and (3.4),

$$v'(s) \le -\frac{v^2(s)}{3\sqrt{\varepsilon}} - \frac{\varepsilon}{(\varepsilon + \xi^2(s))^{3/2}} \qquad (s \in [0, T]).$$

By (3.6), $\xi(s) \le \sqrt{\varepsilon}$ if $0 \le s \le t_0 = \sqrt{\varepsilon}/2$; thus

$$v(t_0) = \int_0^{t_0} v'(s)\,ds \leq -\int_0^{t_0} \frac{\varepsilon}{(\varepsilon + \xi^2(s))^{3/2}} \leq -\frac{1}{6},$$

and hence

$$v(t) \leq -\frac{1}{6} - \frac{1}{3\sqrt{\varepsilon}} \int_{t_0}^{t} v^2(s)\,ds \qquad (t \in [t_0, T]).$$

Put

$$t_{cr} = \frac{\sqrt{\varepsilon}}{2} + 18\sqrt{\varepsilon},$$

and define

$$y(t) = \frac{3\sqrt{\varepsilon}}{-18\sqrt{\varepsilon} + t - t_0} \qquad (t_0 \leq t < t_{cr});$$

it is easy to check that the real-valued function y satisfies the relation

$$y(t) = -\frac{1}{6} - \frac{1}{3\sqrt{\varepsilon}} \int_{t_0}^{t} y^2(s)\,ds \qquad (t_0 \leq t < t_{cr}),$$

and hence

$$v(s) \leq y(s) < 0 \qquad (s \in [t_0, t_{cr})).$$

Thus the restriction v of u_x to the characteristic $(\xi(s), s)$ is not bounded on $[0, T]$, since y is not bounded on $[t_0, t_{cr})$ and $t_{cr} < T$. This contradicts the continuity of u_x and proves that problem (3.2), (3.3) has no solution on $\mathscr{R}(T)$.

4. MIXED BOUNDARY PROBLEM: A PRIORI ESTIMATES

We shall show that results analogous to those proved for the initial-value problem hold for the mixed boundary problem; here too the main hypothesis is that f (in (4.2)) does not depend on x. We first introduce some notation.

Let $m, \overline{m}, \underline{m}$ be fixed integers satisfying the conditions

$$m = \overline{m} + \underline{m}, \qquad \overline{m} > 0, \qquad \underline{m} > 0.$$

If $h = (h_i) \in R_m$, then $\overline{h}$ and $\underline{h}$ are the points of $R_{\overline{m}}$ and $R_{\underline{m}}$ defined by the conditions

$$\overline{h}_i = h_i \quad (i = 1, \cdots, \overline{m}); \qquad \underline{h}_i = h_{\overline{m}+i} \quad (i = 1, \cdots, \underline{m}).$$

If $h \in R_{m\times m}$, then $\overline{h}$ is the submatrix formed by the first $\overline{m}$ rows of h and $\underline{h}$ is that formed by the last $\underline{m}$ rows.

If $0 \leq b \leq \infty$ and $0 \leq T \leq \infty$, then $\mathscr{R} = \mathscr{R}(b, T)$ is the set

$$\mathscr{R} = \{(x, t)\colon 0 \leq x \leq b,\ x \neq \infty,\ 0 \leq t \leq T,\ t \neq \infty\}.$$

Definition. Suppose T and α are positive reals, $0 \leq b \leq \infty$, and $m = \overline{m} + \underline{m}$. Put $\mathscr{R} = \mathscr{R}(b, T)$. We write $A \in \overline{\Sigma}(\mathscr{R}, m, \alpha)$ if and only if $A = S^{-1}DS \in \Sigma(\mathscr{R}, \overline{m}, \alpha)$

and, moreover, for some $\delta > 0$ and all $(x, t, w) \in \mathscr{R}_\alpha$, the following two conditions hold:

the diagonal elements of D satisfy the inequalities

$$d_i(x, t, w) > \delta \quad (i = 1, \cdots, \overline{m}); \qquad d_i(x, t, w) < -\delta \quad (i = \overline{m}+1, \cdots, m);$$

and

$$(4.1) \quad \begin{cases} \text{the submatrices of } S(x, t, w) \text{ defined by} \\ \overline{S}(x, t, w) = [\nabla_1(x, t, w), \nabla_2(x, t, w)] \quad (\nabla_1(x, t, w) \in R_{\overline{m}\times\overline{m}}), \\ \underline{S}(x, t, w) = [\Delta_1(x, t, w), \Delta_2(x, t, w)] \quad (\Delta_2(x, t, w) \in R_{\underline{m}\times\underline{m}}) \\ \text{satisfy the inequalities} \\ |\det \nabla_1(x, t, w)| > \delta, \qquad |\det \Delta_2(x, t, w)| > \delta. \end{cases}$$

Similarly, we write $(A, f) \in \overline{\Sigma} = \overline{\Sigma}(\mathscr{R}, m, \alpha)$ if and only if $A \in \overline{\Sigma}$ and $f = f(t, w)$ is a C^1-map from $\{(t, w): 0 \leq t \leq T,\ w \in R_m,\ |w| \leq \alpha\}$ to R_m.

It is clear that if b is finite and A satisfies the conditions above with $\delta = 0$, then $A \in \overline{\Sigma}(\mathscr{R}, m, \alpha)$.

Let $0 < b \leq \infty$, $0 < T' < \infty$, $\mathscr{R} = \mathscr{R}(b, T')$, and for $a \in (0, b] \cap R$ and $T \in (0, T']$, consider the mixed boundary problem

$$(4.2) \qquad z_t + A(x, t, z)z_x = f(t, z) \qquad ((x, t) \in \mathscr{R}(a, T)),$$

$$(4.3) \qquad z(x, 0) = \phi(x) \qquad (x \in [0, a]),$$

$$(4.4) \qquad \overline{z}(0, t) = \overline{u}(t), \quad \underline{z}(a, t) = \underline{u}(t) \qquad (t \in [0, T]),$$

where

$$(4.5) \quad \begin{cases} \text{the functions } \phi = \phi(x),\ \overline{u} = \overline{u}(t),\ \underline{u} = \underline{u}(t) \text{ belong to } C^1([0, b] \cap R, R_m), \\ C^1([0, T'], R_{\overline{m}}),\ C^1([0, T'], R_{\underline{m}}), \text{ respectively, and satisfy the compatibility conditions} \\ \text{(i) } \overline{u}(0) = \overline{\phi}(0), \quad \overline{u}'(0) + \overline{A}(0, 0, \phi(0))\phi'(0) = f(0, \phi(0)), \\ \text{(ii) } \underline{u}(0) = \underline{\phi}(a), \quad \underline{u}'(0) + \underline{A}(a, 0, \phi(a))\phi'(a) = f(0, \phi(a)). \end{cases}$$

The following a priori estimates are analogous to those proved in Theorem 2.IV.

THEOREM 4.I. *Fix* $m = \overline{m} + \underline{m}$, $0 < b \leq \infty$, $0 < T' < \infty$, $\mathscr{R} = \mathscr{R}(b, T')$, *and* $(A, f) \in \overline{\Sigma}(\mathscr{R}, m, \alpha)$. *Suppose* $\lim_{w \to 0} |f(t, w)|/|w| = 0$ *for each* $t \in [0, T']$, *and choose* ε $(0 < \varepsilon < b)$, N $(0 < N < \infty)$, *and* $C_0 > 0$. *Then there exist numbers* c_0 *and* c $(0 < c_0 \leq C_0,\ c > 0)$ *such that if*

$$(4.6) \quad \begin{array}{l} a\ (a \in [\varepsilon, b] \cap R),\ \phi,\ \overline{u},\ \textit{and } \underline{u} \textit{ satisfy } (4.5);\ 0 < T \leq T'; \\ \textit{and } z \in C^1(\mathscr{R}(a, T), R_m) \textit{ satisfies } (4.2) \textit{ to } (4.4); \end{array}$$

$$(4.7) \qquad |\phi| \leq c_0 \textit{ and } \max(|\phi'|, |\overline{u}'|, |\underline{u}'|) \leq c;$$

then

$$|z| \leq 2c_0 \quad \text{and} \quad |z_x| \leq N .$$

Remark. From the proof it will be clear that for fixed a $(0 < a < \infty)$, the hypotheses on A can be weakened by requiring (4.1) to hold only for all (0, t, w), (a, t, w) $\in \mathscr{R}_\alpha$.

Proof. A basic inequality—(4.10) below—will be proved first on

$$\mathscr{R}(a, T_1) \subset \mathscr{R}(a, T),$$

where, loosely speaking, $T_1 > 0$ is chosen so that the characteristics starting in $\mathscr{R}(a, T_1)$ are reflected at most once by the lateral boundary of $\mathscr{R}(a, T_1)$. If T and a are as in (4.6), $z \in C^1(\mathscr{R}(a, T))$, and g is one of S, D, S^{-1}, let g(x, t) stand for g(x, t, z(x, t)), and define v on $\mathscr{R}(a, T)$ and $|v|(\cdot)$ on [0, T] by the relations

$$v(x, t) = S(x, t)z_x(x, t); \qquad |v|(t) = \max\{|v(x, s)|: (x, s) \in \mathscr{R}(a, t)\}.$$

We assert that there exists a positive number M such that if

$$\text{(4.8)} \qquad a,\ \phi,\ \bar{u},\ \underline{u},\ T,\ z \text{ are chosen as in (4.6)},$$

$$\text{(4.9)} \qquad 0 < t \leq T_1, \text{ where } T_1 = \min\{\varepsilon|D|^{-1}, T\},$$

then

$$\text{(4.10)} \quad |v|(t) \leq M(|\bar{u}'| + |\phi'| + |\underline{u}'| + |f(\cdot, z)|(t)) + M\int_0^t [(|v|(s))^2 + |v|(s)]ds,$$

where

$$|f(\cdot, z)|(t) = \max\{|f(s, z(x, s))|: (x, s) \in \mathscr{R}(a, t)\}.$$

To see this, suppose that (4.8) holds; the corresponding v is then a continuous function, and by (4.2), (4.3) it satisfies the conditions

$$\text{(4.11)} \qquad v(x, 0) = S(x, 0)\phi'(x) \qquad (x \in [0, a]),$$

$$\text{(4.12)} \qquad v(0, t) = D^{-1}(0, t)S(0, t)(f(t, z(0, t)) - z_t(0, t)) \qquad (t \in [0, T]),$$

$$v(a, t) = D^{-1}(a, t)S(a, t)(f(t, z(a, t)) - z_t(a, t)) \qquad (t \in [0, T]).$$

Using (4.4) and the partitioning (4.1) of S, we have, by (4.12), that for each $t \in [0, T]$,

$$\bar{D}v = -\nabla_1 \bar{u}' - \nabla_2 \Delta_2^{-1}(\underline{D}v + \Delta_1 \bar{u}' - \underline{S}f) + \bar{S}f,$$

where all matrix-valued functions are evaluated at (0, t, z(0, t)), and where $\bar{v}$, $\bar{u}'$, f stand for v(0, t), $\bar{u}'(t)$, f(t, z(0, t)); from the hypotheses on S, D, ∇_1, Δ_2, it follows that there exists an M' independent of the particular choice in (4.8) such that

$$\text{(4.13)} \qquad |\bar{v}(0, t)| \leq M'(|\bar{u}'| + |\underline{v}| + |f(t, z(0, t))|), \quad \text{for all } t \in [0, T].$$

An analogous bound obtains for $|\underline{v}(a, t)|$.

For $(x, t) \in \mathscr{R}(a, T)$, define $\xi_i(s) = \xi_i(s; x, t)$ $(i = 1, \cdots, m)$ by the conditions

$$\frac{d\xi_i(s)}{ds} = d_i(\xi_i(s), s, z(\xi_i(s), s)) \quad (s \leq t), \qquad \xi_i(t) = x, \tag{4.14}$$

where d_i is the ith diagonal element of D. In view of Lemma 2.I and Remark 2.II, there exist continuous functions $H^i = H^i(x, t, w)$ and $H = H(x, t, w)$ from $\mathscr{R}_\alpha$ to $R_{m\times m}$ such that for some real M",

$$|H| \leq M",$$

$$|\langle \tilde{w}, H^i(x, t, w)\tilde{w}\rangle| \leq M" |\tilde{w}|^2, \qquad \text{for all } (x, t, w) \in \mathscr{R}_\alpha,\ \tilde{w} \in R_m.$$

Also, if $\xi_i(s)$ exists on $[\tilde{t}, t]$, then

$$v_i(x, t) = v_i(\xi_i(\tilde{t}), \tilde{t}) + \int_{\tilde{t}}^{t} [\langle v, H^i(\xi_i(s), s, z)v\rangle + (H(\xi_i(s), s, z)v)_i]ds,$$

where, in the integrand, z and v stand for $z(\xi_i(s), s)$ and $v(\xi_i(s), s)$. Thus

$$|v_i(x, t)| \leq |v_i(\xi_i(\tilde{t}), \tilde{t})| + M" \int_{\tilde{t}}^{t} (|v(\xi_i(s), s)|^2 + |v(\xi_i(s), s)|)ds, \tag{4.15}$$

whenever (4.8) holds, $(x, t) \in \mathscr{R}(a, T)$, $\tilde{t} \leq t$, and $\xi_i(s)$ $(i = 1, \cdots, m)$ exists on $[\tilde{t}, t]$.

To complete the proof of our assertion, suppose (4.8), (4.9) hold and $(x, t) \in \mathscr{R}(a, t)$. Since d_i is bounded, the solution ξ_i of the ordinary differential equation (4.14) exists up to the boundary of $\mathscr{R}(a, t)$. Also, by (4.9), $t|d_i| \leq a$; fix i $(1 \leq i \leq \overline{m})$; then $d_i > 0$, and hence either (case I) ξ_i exists on $[0, t]$, or (case II) there is $0 < \tilde{t} \leq t$ such that ξ_i exists on $[\tilde{t}, t]$, satisfies the condition $\xi_i(\tilde{t}) = 0$, and for some $j > \overline{m}$, $\xi_j(s) = \xi_j(s; 0, \tilde{t})$ exists on $[0, \tilde{t}]$. In other words, from (x, t), the lower boundary (x, 0) of $\mathscr{R}(a, t)$ can be reached either along the ith characteristic through (x, t) or, if this is not possible, along the ith characteristic through (x, t) up to the lateral boundary $x = 0$ of $\mathscr{R}(a, t)$ and along any of the last $\underline{m}$ characteristics from there on. An analogous situation obtains if i were assumed to satisfy the inequality $\overline{m} < i \leq m$. By (4.15), we have in case I that

$$|v_i(x, t)| \leq |S|\ |\phi'| + M" \int_0^t [(|v|(s))^2 + |v|(s)]ds;$$

in case II, using (4.15) and (4.13), we obtain the inequalities

$$|v_i(x, t)| \leq |v_i(0, \tilde{t})| + M" \int_{\tilde{t}}^{t} (|v(\xi_i(s), s)|^2 + |v(\xi_i(s), s)|)ds$$

$$\leq M'(|\overline{u}'| + |S|\ |\phi'| + |f(\cdot, z)|(t)) + (M' + M") \int_0^t [(|v|(s))^2 + |v|(s)]ds.$$

Thus there exists a number M such that if (4.8), (4.9) hold and $1 \leq i \leq \overline{m}$, then the inequality

$$|v_i(x, t)| \leq M(|\overline{u}'| + |\phi'| + |f(\cdot, z)|(t)) + M \int_0^t [(|v|(s))^2 + |v|(s)]ds$$

holds for all $(x, t) \in \mathscr{R}(a, t)$. In an analogous manner, one sees that there exists an M such that this inequality holds also for $\overline{m} < i \leq m$. Thus (4.10) is established, since $|v|(t)$ is $|v_i(x, s)|$, for some $(x, s) \in \mathscr{R}(a, t)$ and some i $(1 \leq i \leq m)$.

In view of what has been proved, it is easy to see that an inequality similar to (4.10) holds for $kT_1 \leq t \leq \min(k\varepsilon|D|^{-1}, T)$, if $k > 1$ is an integer and $kT_1 < T$. Indeed, the only difference is that now $|\phi'|$ should be replaced by $|S^{-1}|\ |v|(kT_1)$ and the lower limit of the integral by kT_1. Thus, using this inequality iteratively, we conclude that there exists a number $M > 0$ such that if (4.8) is satisfied, then (4.10) holds for all $0 \leq t \leq T$.

To complete the proof of the theorem, observe that in view of the hypotheses on f, there exists a c_0 $(0 < c_0 \leq C_0)$ such that

$$\frac{|f(t, w)|}{|v|} < \min\left\{\frac{1}{4T'}, \frac{1}{4T'|S^{-1}|\ |D|\ Me^{2MT'}}\right\} \quad (0 \leq t \leq T',\ |w| \leq 2c_0).$$

Define

$$c = \frac{c_0}{8T'|S^{-1}|\ |D|\ Me^{2MT'}},$$

and fix c_0 so that the following inequalities also hold:

$$c_0 \leq 2T'|S^{-1}|\ |D|\ e^{MT'}, \qquad 4|S^{-1}|\, cMe^{2MT'} \leq N. \tag{4.16}$$

Then

$$0 < 4Mc \leq e^{-MT'}, \tag{4.17}$$

$$|f(t, w)| \leq \min\left(\frac{c_0}{2T'}, c\right) \quad (0 \leq t \leq T',\ |w| \leq 2c_0). \tag{4.18}$$

For this choice of c_0 and c, we now prove that if (4.6) and (4.7) hold, then

$$|z(x, t)| \leq 2c_0, \quad |z_x(x, t)| \leq N, \qquad \text{for all } (x, t) \in \mathscr{R}(a, T). \tag{4.19}$$

Indeed, suppose (4.6) and (4.7) hold. Then (4.10) holds for every $t \in [0, T]$. Since z is continuous and bounded by c_0 on the compact initial interval, there exists t $(0 < t \leq T)$ such that $|z(x, s)| \leq 2c_0$ for all $(x, s) \in \mathscr{R}(a, t)$. If there exists an $(x, t) \in \mathscr{R}(a, T)$ such that

$$|z(x, t)| = 2c_0, \tag{4.20}$$

then $t = T$. Suppose not. Then, by the continuity of z, there is a smallest t, call it t' $(0 < t' < T)$, for which (4.20) holds. Hence, by (4.7) and (4.18),

$$M(|\overline{u}'| + |\phi'| + |\underline{u}'| + |f(\cdot, z)|(t')) \leq 4Mc.$$

Since $|v|(t)$ is a continuous function of t, and in view of (4.10) and (4.17), Lemma 2.III implies that

$$|v|(t') \leq 4cMe^{2Mt'},$$

which yields the contradiction

$$|z(x, t')| \leq |\phi| + t'(|S'|\ |D|\ |v|(t') + |f(\cdot, z)|(t')) < 2c_0 .$$

We conclude that

$$|z(x, t)| \leq 2c_0, \qquad \text{for all } (x, t) \in \mathscr{R}(a, T).$$

Hence (4.19) is established, since its second part, as is easily seen, follows from (4.18), (4.17), (4.16), (4.10), and Lemma 2.III. The proof is now complete.

If $f = 0$, the conclusion of Theorem 4.I can be strengthened in the sense that on a preassigned rectangle, the derivative z_x of the solution can be made arbitrarily small by taking $\overline{u}'$, ϕ', and $\underline{u}'$ sufficiently small. Indeed, consider the homogeneous problem

(4.21) $$z_t + A(x, t, z) z_x = 0 \qquad (\text{on } \mathscr{R}(a, T)),$$

(4.22) $$z(x, 0) = \phi(x) \qquad (x \in [0, a]),$$

(4.23) $$\overline{z}(0, t) = \overline{u}(t), \quad \underline{z}(a, t) = \underline{u}(t) \qquad (t \in [0, T]).$$

THEOREM 4.II. *Fix* $m = \overline{m} + \underline{m}$, $0 < \varepsilon < b \leq \infty$, $0 < T' < \infty$, $\mathscr{R} = \mathscr{R}(b, T)$, *and* $A \in \Sigma(\mathscr{R}, m, \alpha)$. *Then there exist positive numbers* c *and* N *such that if* a ($a \in [\varepsilon, b] \cap R$), ϕ, $\overline{u}$, *and* $\underline{u}$ *satisfy* (4.5) *with* $f = 0$; *if* $0 < T < T'$; *if* $z \in C^1(\mathscr{R}(a, T))$ *satisfies* (4.21) *to* (4.23); *and if* $\max(|\phi'|, |\overline{u}'|, |\underline{u}'|) \leq c$, *then*

$$|z_x| \leq N(|\overline{u}'| + |\phi'| + |\underline{u}'|).$$

The proof of Theorem 4.II is an obvious simplification of that of Theorem 4.I, and hence we omit its details; indeed, it suffices to note that (4.13) is now replaced by the inequality

$$|\overline{v}(0, t)| \leq M'(|\overline{u}'| + |\phi'|);$$

hence one obtains the relation

$$|v|(t) \leq M(|\overline{u}'| + |\phi'| + |\underline{u}'|) + \int_0^t [(|v|(s))^2 + |v|(s)]\, ds$$

instead of (4.10); and this, by the usual reasoning, gives the conclusion of Theorem 4.II.

5. MIXED BOUNDARY PROBLEMS WITH SOLUTIONS ON A PREASSIGNED RECTANGLE

An immediate consequence of the previous estimates and the standard existence theorem for the mixed boundary problem is that the problems studied in Section 4 possess a solution on a preassigned rectangle, whenever the data are conveniently restricted. To make this precise, let a and T be positive real numbers, put

$$\tau(a, T) = \left\{(x, t) \colon 0 \leq t \leq T,\ 0 \leq x \leq a - t\frac{a}{T}\right\},$$

and consider the problem

(5.1) $$z_t + A(x, t, z) z_x = f(t, z) \qquad ((x, t) \in \tau(a, T)),$$

$$z(x, 0) = \phi(x) \quad (x \in [0, a]), \tag{5.2}$$

$$\bar{z}(x, t) = \bar{u}(t) \quad (t \in [0, T]). \tag{5.3}$$

It is known (see Lax [8, p. 107], Courant and Lax [4, p. 271], R. Courant, E. Isaacson, and M. Rees [5, p. 253], G. Prouse [11], V. Thomée [12]) that under appropriate hypotheses, (5.1) to (5.3) have a unique solution in the small. Indeed, the following result is known (for the notation, see Section 4).

THEOREM 5.I. *Fix* $m = \overline{m} + \underline{m}$ *and positive real numbers* a, T', c_0 $(c_0 < \alpha)$, *and* c. *Put* $\mathscr{R} = \mathscr{R}(a, T')$, *and suppose* $(A, f) \in \overline{\Sigma}(\mathscr{R}, m, \alpha)$. *Then, there exists a number* T $(0 < T \leq T')$ *such that if* ϕ, $\bar{u}$ *satisfy* (4.5) *and if the inequalities* $|\phi| \leq c_0$ *and* $\max(|\phi'|, |\bar{u}'|) \leq c$ *hold, there exists* $z \in C^1(\tau(a, T), R_m)$ *that satisfies* (5.1) *to* (5.3). *Furthermore, if* $T \leq T_0 = \min(T', a/|\underline{D}|)$, *where* $D = SAS^{-1}$, *then* z *is uniquely determined in* $C^1(\tau(a, T), R_m)$.

A proof of Theorem 5.I can be obtained by following the proof of Douglis [6] of the analogous result for the initial-value problem (Theorem 8, p. 149); the modifications needed are minor and are due to the fact that in Theorem 5.I some of the characteristics starting in $\tau = \tau(a, T_0)$ will reach the lateral boundary $x = 0$ of τ rather than the lower boundary $t = 0$.

We note that if it is also assumed that S, D, f are of class C^2 and the derivatives ϕ', $\underline{u}'$ are Lipschitz-continuous, then a proof of the conclusion of Theorem 5.I appears in [11]. Hence Theorem 5.I is also a consequence of the result in [11], an a priori bound implied by the inequality (4.10), and the approximation technique in [6, pages 132 to 137].

In view of the standard existence theorem for the initial-value problem (for instance [6, p. 149] or [7, p. 855]), Theorem 5.I implies an analogous result in the small for the mixed boundary problem (4.2) to (4.4); hence, by using the a priori estimates proved in Theorems 4.I and 4.II, one obtains the following continuation results. The notation is defined in Section 4 and the proofs are omitted, since they are simple and similar to the proof of Theorem 3.II.

THEOREM 5.II. *Fix* $\alpha > 0$, $T' > 0$, $m = \overline{m} + \underline{m}$, $0 < b \leq \infty$, $\mathscr{R} = \mathscr{R}(b, T')$, *and* $(A, f) \in \overline{\Sigma}(\mathscr{R}, m, \alpha)$. *Suppose* $\lim_{w \to 0} |f(t, w)|/|\overline{w}| = 0$, *for each* $t \in [0, T']$, *and let* N *and* ε *be positive reals with* $\varepsilon < b$. *Then there exist positive numbers* c_0 *and* c *such that if* a $(a \in [\varepsilon, b] \cap R)$, ϕ, $\bar{u}$, *and* $\underline{u}$ *satisfy* (4.5), *if* $|\phi| \leq c_0$ *and* $\max(|\phi'|, |\bar{u}'|, |\underline{u}'|) \leq c$, *then there exists a (unique)* $z \in C^1(\mathscr{R}(a, T'), R_m)$ *that satisfies* (4.2) *to* (4.4) *with* $T = T'$, *and, moreover,* $|z| \leq 2c_0$, $|z_x| \leq N$.

THEOREM 5.III. *Fix* $0 < c_0 < \alpha$, $T' > 0$, $m = \overline{m} + \underline{m}$, $0 < \varepsilon < b \leq \infty$, $\mathscr{R} = \mathscr{R}(b, T')$, *and* $A \in \overline{\Sigma}(\mathscr{R}, m, \alpha)$. *Then there exist positive numbers* c *and* N *such that if* a $(a \in [\varepsilon, b] \cap R)$, ϕ, $\bar{u}$, *and* $\underline{u}$ *satisfy* (4.5) *with* $f = 0$, *if* $|\phi| \leq c_0$ *and* $\max(|\phi'|, |\bar{u}'|, |\underline{u}'|) \leq c$, *then there exists a (unique)* $z \in C^1(\mathscr{R}(a, T'), R_m)$ *satisfying* (4.21) *to* (4.23) *with* $T = T'$, *and moreover,* $|z_x| \leq N(|\bar{u}'| + |\phi'| + |\underline{u}'|)$.

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