MULTIPLIERS AND SETS OF UNIQUENESS OF L^p

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Let G be a nondiscrete, locally compact Abelian (LCA) group and Γ its character group. In this paper, we establish the existence of subsets of G of positive Haar measure that are sets of uniqueness for Fourier transforms of functions in $L^p(\Gamma)$ $(1 \leq p < 2)$. We draw from this result several consequences concerning multipliers of $L^p(\Gamma)$, and we establish related results for other function spaces. Specifically, in Section 1, we prove that there exist subsets $E \subset G$ of positive Haar measure with the property that no nonzero function whose Fourier transform belongs to

$$\bigcup_{1 \leq p < 2} L^p(\Gamma)$$

can be carried by E. This is an extension of a theorem of Y. Katznelson: Katznelson proved this result for the case where $G = T$ (the circle group) and $\Gamma = Z$ (the integers) (see [10], [11, p. 101]), and he has indicated to one of us that the proof of his theorem can be modified to give the same result for the real line. The proof presented here is in some sense independent of the algebraic structure of the group and, in fact, yields also an analogous result for orthogonal systems on a nonatomic measure space (see Remark 1.3 below). In Section 2, we use the results of Section 1 to extend to general nondiscrete LCA groups G several theorems concerning multipliers and Fourier transforms of functions in $L^p(\Gamma)$. These theorems were proved for special classes of groups by the authors (working independently) [3], [6], by R. E. Edwards, and by L. Hörmander [9].

In Section 3, we characterize the multipliers of the space $L^1_p(\Gamma)$ consisting of those functions in $L^1(\Gamma)$ whose Fourier transforms are in $L^p(G)$. Our result has been stated previously in [12]; the proof of this result given in [12] is, however, incorrect.

1. SETS OF UNIQUENESS FOR L^p

If G is a nondiscrete LCA group and Γ is its character group, we define a set of uniqueness for $L^p(\Gamma)$ $(1 \leq p < 2)$ as follows.

Definition. Let E be a measurable subset of G; then E is called a *set of uniqueness* for $L^p(\Gamma)$ if no nonzero integrable function f, carried by E, satisfies the condition $\hat{f} \in L^p(\Gamma)$.

(Here, as in the sequel, $\hat{f}$ denotes the Fourier transform of f.)

We can now state the main result of this section.

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THEOREM 1.1. *For each measurable set* $M \subset G$ *of finite positive Haar measure, there exists a measurable set* $E \subset M$ *such that the measure of* E *is arbitrarily close to that of* M *and such that* E *is a set of uniqueness for* $L^p(\Gamma)$, *for every* p *in the range* $[1, 2)$.

The proof can easily be obtained from the following lemma.

LEMMA 1.2. *Suppose that* M *is a measurable subset of* G *of finite positive measure and that* m *denotes the Haar measure on* G. *For each* p $(1 \leq p < 2)$ *and each* $\varepsilon > 0$, *there exists a measurable set* $E_{\varepsilon,p} \subset M$ *such that*

(i) $m(E_{\varepsilon,p}) \geq (1 - \varepsilon) m(M)$, *and*

(ii) *for each function* $f \in L^1(G)$ *carried by* $E_{\varepsilon,p}$, *the inequality*

$$\|\hat{f}\|_{L^\infty(\Gamma)} \leq \varepsilon \|\hat{f}\|_{L^p(\Gamma)}$$

holds.

Proof. We need the following result: for each measurable set $M \subset G$ of finite positive measure, there exists a measurable subset $M' \subset M$ such that

$$m(M') = m(M)/2.$$

This follows [8, p. 174] from the fact that G is nondiscrete and is therefore a nonatomic measure space with respect to m. By applying this result repeatedly, we can define a sequence $\{\pi_n\}$ of partitions of M into measurable sets as follows: the partition π_1 consists of two subsets of M of equal measure, and the partition π_n is obtained by dividing each set of π_{n-1} into two sets of equal measure. For each n, we define on M the Rademacher function r_n relative to the partition π_n: we let $r_0 = 1$, and we let r_n be constant and equal to $+1$ or -1 on each set of π_n in such a way that the mean value of r_n over each set of the partition π_{n-1} is zero. Of course, the Rademacher functions are not uniquely determined; for example, at each step, we could choose $-r_n$ instead of r_n; however, we choose a fixed sequence $\{r_n\}$ of Rademacher functions relative to the sequence $\{\pi_n\}$ of partitions. By forming all possible finite products of Rademacher functions, we obtain a system $W = \{w_i\}$ of orthogonal functions whose L^2-norm is $m(M)^{1/2}$. We call the functions w_i the Walsh functions, and we refer to W as the Walsh system on M.

Suppose now that N is a (large) positive integer. We shall define N functions $\phi_1, \cdots, \phi_N$ that vanish on the complement of M, are orthogonal, and are, on M, real linear combinations of the Walsh functions; the functions $\phi_1, \cdots, \phi_N$ will have the following properties.

$$\int_M |\phi_j| \, dx \leq 2m(M); \qquad \int_M \phi_j^2 \, dx \leq 2^{k+1} m(M). \tag{1}$$

$$m(\{x \in M: \phi_j(x) = 1\}) \geq (1 - \varepsilon/N) m(M). \tag{2}$$

$$|\hat{\phi}_1(\gamma) + \cdots + \hat{\phi}_N(\gamma)| \leq 4m(M) \qquad (\gamma \in \Gamma). \tag{3}$$

Here k is a positive integer, depending on N, that will be chosen later. The precise way to define k will appear naturally as the proof progresses.

To construct these functions, we let $\phi_1 = \chi_M$, the characteristic function of M, and we suppose that n orthogonal functions $\phi_1, \cdots, \phi_N$ $(n < N)$ have been defined that satisfy the conditions listed above, with (3) replaced by the condition

(3') $$|\hat{\phi}_1(\gamma) + \cdots + \hat{\phi}_n(\gamma)| \leq (2 + 2n/N)\,m(M).$$

Since the ϕ_j are integrable functions, the Riemann-Lebesgue Lemma permits us to choose a compact set of characters $K \subset \Gamma$ such that $|\hat{\phi}_j(\gamma)| \leq m(M)/N$, for $1 \leq j \leq n$ and all $\gamma \notin K$. Since the family $\{w_i\}$ is orthogonal, Bessel's inequality implies that, for each $\gamma \in K$,

$$\sum |(\gamma, w_i)|^2 \leq \|\gamma\chi_M\|^2_{L^2(G)}\, m(M).$$

The last inequality shows that $\sum |(\gamma, w_i)|^2$ is continuous in γ. Dini's theorem, applied to the partial sums of $\sum |(\gamma, w_i)|^2$, now implies that $\sum |(\gamma, w_i)|^2$ converges uniformly on the compact set K. It is therefore possible to choose a finite set $F \subset W$ such that

$$\sum_{w_i \notin F} |(\gamma, w_i)|^2$$

is uniformly (and arbitrarily) small on K; we shall assume further that F contains all those Walsh functions that appear with nonzero coefficients in the (Walsh) expansions of the functions $\phi_1, \cdots, \phi_n$. This last aspect of the choice of F will ensure that if $w \in W \setminus F$, then w is orthogonal to ϕ_j $(1 \leq j \leq n)$. For any such choice of F, we can find a partition π_m of M such that the elements of F and the functions ϕ_j are constant on the sets of π_m. Let $E_1, \cdots, E_{2^m}$ denote the sets of π_m; for each j $(1 \leq j \leq 2^m)$, we consider the partition of E_j into 2^k subsets that is induced by the partition π_{m+k} of M. We write $E_j = E'_{1j} \cup \cdots \cup E'_{2^k j}$, and we define ϕ_{n+1} as follows:

$$\phi_{n+1}(x) = \begin{cases} 0 & (x \notin M), \\ (\eta - 1)/\eta & (x \in E'_{1j}), \\ 1 & (x \in E_j \setminus E'_{1j}), \end{cases}$$

where, for convenience, we have written $\eta = 1/2^k$. Since $\int_{E_j} \phi_{n+1}\,dx = 0$, and since every $w \in F$ is constant on each E_j, it follows that

$$\int_M \phi_{n+1} w\,dx = 0 \qquad (w \in F).$$

In particular, ϕ_{n+1} is orthogonal to each ϕ_j $(1 \leq j \leq n)$. Therefore

(4) $$\phi_{n+1} = \chi_M \sum_{w_j \notin F} a_j w_j,$$

where the a_j are real numbers and the sum is finite. It is easy to check that condition (1) holds and that

$$m(\{x \in G: \phi_{n+1}(x) = 1\}) = m(M)\{1 - 1/2^k\}.$$

At this juncture, we restrict k by the requirement that $1/2^k < \varepsilon/N$; hence (2) holds.

If $\gamma \notin K$, we have the inequality $|\hat{\phi}_j(\gamma)| \leq m(M)/N$ $(1 \leq j \leq n)$; therefore

$$|\hat{\phi}_1(\gamma) + \cdots + \hat{\phi}_{n+1}(\gamma)| \leq n\, m(M)/N + |\hat{\phi}_{n+1}(\gamma)| \leq m(M)(2 + n/N).$$

On the other hand, if $\gamma \in K$, we have that

$$|\hat{\phi}_{n+1}(\gamma)| \leq \sum_{w_j \notin F} \left| a_j \int_M w_j(x)\gamma(-x)\,dx \right| \leq \sum_{w_j \notin F} |a_j|\, |(w_j, \gamma)|$$

$$\leq \left(\sum |a_j|^2\right)^{1/2} \left(\sum_{w_j \notin F} |(w_j, \gamma)|^2\right)^{1/2}$$

$$\leq (1/m(M)^{1/2})\, \|\phi_{n+1}\|_{L^2(G)} \left(\sum_{w_j \notin F} |(w_j, \gamma)|^2\right)^{1/2}$$

But by choosing F large enough, we can make the right-hand side of this inequality arbitrarily small. Therefore we may suppose F to have been chosen so that $|\hat{\phi}_{n+1}(\gamma)| \leq 2m(M)/N$. Hence the existence of functions $\phi_1, \cdots, \phi_N$ satisfying (1), (2), and (3) is now assured.

To complete the proof of the lemma, write

$$\Phi = (\phi_1 + \cdots + \phi_N)/N.$$

By construction, the ϕ_j are mutually orthogonal, and it follows that

$$\|\Phi\|_{L^2(G)} = \frac{1}{N}\left(\sum \|\phi_j\|^2_{L^2(G)}\right)^{1/2} < \{2m(M)/\eta N\}^{1/2}.$$

The integer k has already been restricted to satisfy the condition $\eta = 1/2^k < \varepsilon/N$; we now impose the further condition that $\varepsilon/N \leq 1/2^{k-1} = 2\eta$. Then $\eta N > \varepsilon/2$, so that

$$\|\Phi\|_2 \leq C,$$

where the constant C is independent of N. On the other hand,

$$\|\hat{\Phi}\|_{L^\infty(\Gamma)} \leq \left\|\sum \hat{\phi}_j\right\|_{L^\infty(\Gamma)}/N \leq \rho/N,$$

where the constant ρ is independent of N, by (3). Finally, if $2 < q \leq \infty$ and $1/p + 1/q = 1$, the relation

$$\|\hat{\Phi}\|_{L^q(\Gamma)} \leq \|\hat{\Phi}\|^{2/q}_{L^2(\Gamma)} \|\hat{\Phi}\|^{1-2/q}_{L^\infty(\Gamma)} = O(N^{-1+2/q})$$

holds. If N is large enough, this inequality implies that $\|\hat{\Phi}\|_{L^q(\Gamma)} \leq \varepsilon$.

Write $E_{\varepsilon,p} = \{x \in G: \Phi(x) = 1\}$; then

$$E_{\varepsilon,p} \supset \bigcap_{j=1}^{N} \{x: \phi_j(x) = 1\},$$

and therefore $m(E_{\varepsilon,p}) \geq (1-\varepsilon)m(M)$. Let (μ_i) be an approximate identity in $L^1(G)$ consisting of functions $\mu_i \in L^2(G)$ with $\|\mu_i\|_1 = 1$. Then if f is integrable and is carried by $E_{\varepsilon,p}$, we have the relations

$$\begin{aligned} |\hat{f}(\gamma)| &= \lim_i \left| \int_G f * \mu_i(x)\gamma(-x)\,dx \right| \\ &= \lim_i \left| \int_G f * \mu_i(x)\Phi(x)\gamma(-x)\,dx \right| \\ &= \lim_i \left| \int_\Gamma \hat{f}(\gamma' + \gamma)\hat{\mu}_i(\gamma' + \gamma)\hat{\Phi}(\gamma')\,d\gamma' \right| \\ &\leq \limsup_i \|\hat{\Phi}\|_{L^q(\Gamma)} \|\hat{f}\hat{\mu}_i\|_{L^p(\Gamma)} \\ &\leq \varepsilon \|\hat{f}\|_{L^p(\Gamma)} \qquad (\gamma \in \Gamma). \end{aligned}$$

The proof of the lemma is complete.

Proof of Theorem 1.1. Let M be a set of positive measure, and write $\varepsilon_n = 3^{-n-n_0}$ and $p_n = 2 - \varepsilon_n$, where n_0 is a fixed positive integer. Construct sets E_{ε_n,p_n} as in the lemma, and let

$$E = \bigcap_{n=1}^{\infty} E_{\varepsilon_n,p_n}.$$

Then whatever the value of n_0, the inequality $m(E) \geq (1/2)m(M)$ holds, and by choosing n_0 large enough, we can make $m(E)$ arbitrarily close to $m(M)$. If f is carried by E and $\|\hat{f}\|_p < \infty$ for some $p < 2$, then, by the Hausdorff-Young Theorem, we may assume that $f \in L^2(G)$, so that $\hat{f} \in L^2(\Gamma)$ and $\hat{f} \in L^r(\Gamma)$ $(p \leq r \leq 2)$. On the other hand, for all sufficiently large n, we have that $p_n > p$, f is carried by E_{ε_n,p_n}, and hence

$$\|\hat{f}\|_\infty \leq \varepsilon_n \|\hat{f}\|_{p_n}.$$

Since $p < p_n < 2$ and $\|\hat{f}\|_2$ and $\|\hat{f}\|_p$ are both fixed, Hölder's inequality shows that $\|\hat{f}\|_{p_n}$ is bounded as $n \to \infty$. Since $\varepsilon_n \to 0$, it follows that $\|\hat{f}\|_\infty = 0$, and therefore $f = 0$ a.e. on E.

Remark 1.3. The core of the argument in the proofs of Lemma 1.2 and Theorem 1.3 consists in showing (roughly speaking) that if M is an arbitrary set of positive measure, then there exists a function Φ on G with the properties that $\Phi = 1$ on almost the whole of M, $\Phi = 0$ on $G \setminus M$ and $|\hat{\Phi}|$ is small. The proof we have presented of this fact does not depend on the algebraic structure of G. We can therefore formulate an analogous result for arbitrary complete orthonormal systems as follows. (The proof is an obvious modification of that of Lemma 1.2.)

Let (X, m) *be a finite nonatomic measure space, and suppose that* (u_i) *is a bounded and complete orthonormal system in* $L^2(X, m)$. *Then if* $\varepsilon > 0$ *is arbitrary and* $2 < q \leq \infty$, *there exist a measurable subset* E *of* X *and a function* $\Phi \in L^\infty(X)$ *having the properties that*

(i) $\Phi = 1$ *on* $X \setminus E$,

(ii) $m(X \setminus E) \leq \varepsilon$, *and*

(iii) $\sum_i |(\Phi, u_i)|^q \leq \varepsilon$.

There is of course a corresponding form of Theorem 1.1, which we leave to the reader to formulate. We remark that, in formulating this analogue, it is necessary to impose some further conditions on the bounded, complete, orthonormal system (u_i) in order to be able to carry through the final step of the proof of Lemma 1.2. For example, it would suffice to assume that there exists a constant K, independent of i and j, with the property that

$$|(\bar{u}_i \Phi, u_j)| \leq K |(\Phi, u_j)| \qquad (\Phi \in L^\infty(X)).$$

2. MULTIPLIERS OF L^p

We recall the definition of the space of multipliers of $L^p(\Gamma)$.

Definition. A bounded linear operator T on $L^p(\Gamma)$ with values in $L^p(\Gamma)$ $(1 \leq p \leq \infty)$ is called a multiplier if it commutes with translations; in the case $p = \infty$, we require in addition that T be continuous in the weak* topology of $L^\infty(\Gamma)$. We denote the space of multipliers of $L^p(\Gamma)$ by $M_p(\Gamma)$.

It is known (and easy to prove) that $M_p(\Gamma)$ can be identified by duality with $M_{p'}(\Gamma)$ $(1/p + 1/p' = 1)$ and that $M_p(\Gamma) \subset M_r(\Gamma) \subset M_2(\Gamma)$, provided

$$|1/p - 1/2| \geq |1/r - 1/2|.$$

One of the principal goals of this section is to prove that if Γ is infinite and the inequalities $|1/p - 1/2| > |1/r - 1/2| > 0$ hold, then the inclusion relations above are strict. We shall achieve this by using Theorem 1.1 and the following convexity theorem.

If $0 \leq t \leq 1$ *and* $1/q = t/q_1 + (1 - t)/q_2$ $(1 \leq q_i \leq \infty)$ *and* $T \in M_{q_1} \cap M_{q_2}(\Gamma)$, *then* $T \in M_q(\Gamma)$ *and*

$$\|T\|_{M_q} \leq \|T\|_{M_{q_1}}^t \, \|T\|_{M_{q_2}}^{1-t}. \tag{5}$$

It is also known that $M_2(\Gamma)$ can be identified isometrically with $L^\infty(G)$ in the following sense: $T \in M_2(\Gamma)$ if and only if there exists $\phi \in L^\infty(G)$ such that $(Tf)^\wedge = \phi\hat{f}$ for all $f \in L^2(\Gamma)$; in this case $\|T\| = \|\phi\|_\infty$. With this result in mind, we define the Fourier transform $\hat{T}$ of $T \in M_2(\Gamma)$ by writing $\hat{T} = \phi$. This correspondence between elements of $M_2(\Gamma)$ and elements of $L^\infty(G)$, together with the fact that $M_p(\Gamma) \subset M_2(\Gamma)$, allows us to associate in a one-to-one fashion to each $T \in M_p(\Gamma)$ an element $\hat{T} \in L^\infty(G)$. As in the case of the space $L^2(\Gamma)$, $\hat{T}$ is characterized by the condition that $(Tf)^\wedge = \hat{T}\hat{f}$ for $f \in L^p \cap L^2$; and if $1 \leq p \leq 2$, this formula extends to the whole space $L^p(\Gamma)$. Another well-known fact is that $M_1(\Gamma)$ is isometrically isomorphic to the space $M(\Gamma)$ of bounded regular Borel measures on

Γ, in the sense that if $T \in M_1(\Gamma)$, then there exists $\mu \in M(\Gamma)$ such that $Tf = \mu * f$; of course $\hat{T} = \hat{\mu}$, where $\hat{\mu}$ denotes the Fourier-Stieltjes transform of μ. We refer the reader to [2, Vol. II, Chapter 16], [4], [6], [9], and [17] for proofs of these remarks and for other results on multipliers of L^p.

It is an immediate consequence of Theorem 1.1 that if Γ is a noncompact group and $p \neq 2$, then $M_p(\Gamma)$ is different from $M_2(\Gamma)$. For if E is a subset of G satisfying the conclusions of Theorem 1.1, then χ_E is the transform of a nonzero element of $M_2(\Gamma)$. On the other hand, since E is a set of uniqueness, the only bounded measurable function carried by E that is the transform of a multiplier of $L^p(\Gamma)$ is the zero function. Observe further that for some $g \in L^1(G)$, the convolution $g * \chi_E$ is not the Fourier transform of an element of $M_p(\Gamma)$. For otherwise, an easy application of the closed-graph theorem would yield the inequality

$$\|(g * \chi_E)^\wedge\|_{M_p(\Gamma)} \leq D\,\|g\|_{L^1(G)},$$

for some constant D independent of $g \in L^1(G)$, where $(g * \chi_E)^\wedge$ is that element of $M_p(\Gamma)$ of which $g * \chi_E$ is the Fourier transform. But it is a simple matter to deduce from this inequality that χ_E is itself the transform of an element of $M_p(\Gamma)$. For if the function g ranges over the elements of an approximate identity (g_i) in $L^1(G)$, the corresponding multipliers $(g_i * \chi_E)^\wedge$ are bounded in norm. However, $M_p(\Gamma)$ is isometrically isomorphic to the dual of the Banach space $A_p(\Gamma)$ [4], and hence the net $\{(g_i * \chi_E)^\wedge\}$ has a weak* limit point in $M_p(\Gamma)$. If one uses the definition of the space $A_p(\Gamma)$, it is a routine calculation to verify that this weak* limit point is unique, and that it is the operator defined by multiplication of Fourier transforms by the function χ_E. But we have already noted that χ_E is not the Fourier transform of an element of $M_p(\Gamma)$; thus we have arrived at a contradiction. Finally, let $C_0(G)$ denote the space of continuous functions vanishing at infinity on G. Then $g * \chi_E \in C_0(G)$. Hence we conclude that if Γ is a noncompact group and $p \neq 2$, then $M_p(\Gamma)^\wedge \cap C_0(G) \neq C_0(G)$.

The same conclusions can be reached, but by different methods, when Γ is an infinite compact group. In this case, well-known results on random Fourier series can be applied to prove that $M_p(\Gamma) \neq M_2(\Gamma)$ if $p \neq 2$ and that

$$M_p(\Gamma)^\wedge \cap C_0(G) \neq C_0(G).$$

See [2, Vol. II, Chapters 14, 16].

We now utilize the remarks above to prove that the inclusions

$$M_p(\Gamma) \subset M_r(\Gamma) \subset M_2(\Gamma)$$

are proper whenever Γ is infinite and $|1/p - 1/2| > |1/r - 1/2| > 0$.

THEOREM 2.1. *If* $|1/p_1 - 1/2| \neq |1/p_2 - 1/2|$ $(1 \leq p_1, p_2 \leq \infty)$ *and* Γ *is infinite, then*

$$M_{p_1}(\Gamma)^\wedge \cap C_0(G) \neq M_{p_2}(\Gamma)^\wedge \cap C_0(G).$$

Proof. By the duality result mentioned earlier, we may suppose without loss of generality that $1 \leq p_1 < p_2 \leq 2$. The proof distinguishes two cases.

(i) $1 \leq p_1 < p_2 = 2$. In this case, the remarks preceding the theorem yield the desired result.

(ii) $1 \leq p_1 < p_2 < 2$. We use case (i) and a process of interpolation. Suppose that $M_{p_1}(\Gamma)^\wedge \cap C_0(G) = M_{p_2}(\Gamma)^\wedge \cap C_0(G)$. Each space is a Banach space when given its appropriate M_p-norm. (The M_p-norm is at least as strong as the L^∞-norm on $M_p(\Gamma)^\wedge$, as can be seen by taking $q_1 = p$, $q_2 = p'$, $q = 2$ in (5).) Therefore equality of the spaces implies equivalence of their norms; in other words, there exists a constant B such that the inequality

$$(6) \qquad \|T\|_{M_{p_2}(\Gamma)} \leq \|T\|_{M_{p_1}(\Gamma)} \leq B\,\|T\|_{M_{p_2}(\Gamma)}$$

holds for all $T \in M_{p_1} \cap M_{p_2}(\Gamma)$ with $\hat{T} \in C_0(G)$.

At the same time, we know that $M_{p_1}(\Gamma)^\wedge \cap C_0(G) \neq C_0(G)$ and that $L^1(\Gamma)^\wedge$ is contained in each space and is dense in the second. Therefore $M_{p_1}(\Gamma)$ and $M_2(\Gamma)$ induce different topologies on $L^1(\Gamma)$, and there exists a sequence (f_n) of elements of $L^1(\Gamma)$ satisfying the conditions $\|f_n\|_{M_{p_1}} = 1$ and $\|\hat{f}_n\|_\infty \to 0$. However, since $\|f_n\|_{M_{p_1}}$ is bounded, (5) (applied to the case $q = p_2$, $q_1 = p_1$, $q_2 = 2$) shows that $\|f_n\|_{M_{p_2}} \to 0$ because $t < 1$ in this case. On the other hand, (6) shows that $\|f_n\|_{M_{p_2}}$ is bounded away from zero. This is a contradiction, and the proof is complete.

Theorem 2.1 has some interesting corollaries. In order to state two of them, we recall the definition of the space $A_p(\Gamma)$ of [4].

Definition. Let $A_p(\Gamma)$ $(1 \leq p \leq \infty)$ denote the subspace of $C_0(\Gamma)$ consisting of those functions g that can be written in the form

$$(7) \qquad g = \sum_{i=1}^{\infty} h_i * k_i,$$

where $h_i, k_i \in C_c(\Gamma)$ (the space of continuous functions with compact supports) and

$$(8) \qquad \sum \|h_i\|_p \|k_i\|_{p'} < \infty.$$

The space A_p becomes a Banach space if we define the norm of g to be the infimum of all sums (8) over all possible representations (7).

It is a consequence of the Riesz convexity theorem that the inclusion $A_{p_2}(\Gamma) \subset A_{p_1}(\Gamma)$ holds if $1 \leq p_1 \leq p_2 \leq 2$. Theorem 2.1 shows that this inclusion is in general strict.

COROLLARY 2.2. *If* $1 \leq p_1 < p_2 \leq 2$, *then* $A_{p_1}(\Gamma) \neq A_{p_2}(\Gamma)$, *unless* Γ *is finite.*

The corollary follows easily from the main result of [4], namely that M_p is the dual space of A_p.

COROLLARY 2.3. *If* $1 < p < \infty$, *then* $L^p * L^{p'}(\Gamma) = C_0(\Gamma)$ *if and only if* Γ *is finite.*

Proof. Since $1 < p < \infty$, we have that $L^p * L^{p'}(\Gamma) \subset A_p(\Gamma)$; now the factorization theorem [1, Theorem 2.2] implies that $C_0(\Gamma) = A_1(\Gamma)$.

Remarks. (a) Corollary 2.3 is an extension of a known result of I. E. Segal [16] that the space $A(\Gamma) = A_2(\Gamma)$ of Fourier transforms of integrable function on G is equal to $C_0(\Gamma)$ if and only if Γ is finite.

(b) Theorem 2.1 has been proved independently, and in a slightly more general form, by J. F. Price [13]. His proof also applies to the compact, noncommutative situation.

We now present another application of Theorem 1.1, namely a result concerning multiplication of transforms of L^p-functions by functions in $C_0(G)$. For functions defined on compact groups, there are well-known results concerning random changes of the signs of their Fourier coefficients. One might expect analogous results for functions defined on noncompact groups. Our result shows that such an analogue fails dramatically.

The following theorem was proved by R. E. Edwards for the case where Γ contains an infinite discrete subgroup [6, Theorem 2.7], and by one of the authors of the present paper for $\Gamma = R^n$ [3]. (See also [15, Lemma 3.3].)

THEOREM 2.4. *Let* Γ *be a noncompact* LCA *group, and let* G *be its character group; let* p *be a fixed number* $(1 \leq p < 2)$. *Suppose that* F *is a function on* G *with the property that* $\phi F \in L^p(\Gamma)^\wedge$ *for each* $\phi \in C_0(G)$. *Then* $F = 0$ l.a.e..

Proof. If F is not locally negligible, we can find a function $\psi \in C_c(G)$ such that ψF has the same property as F and ψF is not negligible. Hence we may assume that F itself has a compact support. In this case, the Hausdorff-Young Theorem shows that $F \in L^2(G)$; we may therefore assume that $p > 1$.

We now assert that if E is any compact subset of G, then $\chi_E F$ is also in $L^p(\Gamma)^\wedge$. For it is easy to see that the mapping $\phi \to (\phi F)^\wedge$, which (by hypothesis) carries $C_0(G)$ into $L^p(\Gamma)$, is continuous. (This is a simple application of the closed-graph theorem.) Hence there exists a constant C such that

$$\|(\phi F)^\wedge\|_{L^p(\Gamma)} \leq C\,\|\phi\|_\infty \qquad (\phi \in C_0(G)). \tag{9}$$

Suppose (ϕ_i) is a net of functions in $C_c(G)$ whose supports shrink to E, and suppose each ϕ_i takes its values in [0, 1] and equals 1 on E. Then if we substitute ϕ_i for ϕ in (9) and take limits, it is easy to deduce (recall that $p > 1$) that

$$(\chi_E F)^\wedge \in L^p(\Gamma).$$

But by Theorem 1.1, we can choose E so that it is a set of uniqueness and so that, at the same time, $\chi_E F$ is not negligible. This is a contradiction.

Remark. By applying Baire's category theorem to the space $C_0(G)$, one can prove that the conclusion of Theorem 2.4 holds with the apparently weaker hypothesis that

$$\phi F \in \bigcup_{1 \leq p < 2} L^p(\Gamma)^\wedge$$

for every $\phi \in C_0(G)$. This is the approach of R. E. Edwards in [6, Theorem 2.7].

Theorems 2.1 and 2.4 have a number of other important consequences concerning multipliers. To quote these results would take us too far afield and would in any case necessitate the use of the fairly elaborate language (developed in [5] and [6]) for Fourier transforms of entities more general than bounded measures or elements of $L^p(\Gamma)$ $(1 \leq p \leq 2)$. We quote just one result of this kind, referring the reader to [6] for a detailed discussion of this and other results. The result we quote is the most fundamental of all the results of [6].

THEOREM 2.5. *Let* Γ *be a noncompact group, and let* $p > 2$. *Then there exists a function* $f \in L^p(\Gamma)$ *such that* $\hat{f}$ *is not a measure.*

Remarks. We mention in conclusion that the results of [6], which are proved there for the case where Γ contains an infinite discrete subgroup, can be extended to the general situation by use of the principal structure theorem for LCA groups. This approach is developed in detail in [7].

3. MULTIPLIERS OF L_p^1

The space $L_p^1(\Gamma)$ $(1 \leq p < \infty)$ is the following dense subspace of $L^1(\Gamma)$:

$$L_p^1(\Gamma) = \{f \in L^1(\Gamma): \hat{f} \in L^p(G)\}.$$

The space $L_p^1(\Gamma)$ becomes a Banach space if we define $\|f\| = \|f\|_1 + \|\hat{f}\|_p$. In this section, we characterize the multipliers of $L_p^1(\Gamma)$, where Γ is noncompact, as the space $M(\Gamma)$: the identification is isometric, each multiplier being defined by convolution with an element μ of $M(\Gamma)$. It is easy to see that it suffices to show that the space of multipliers of $L_p^1(\Gamma)$ into $L^1(\Gamma)$ is isometrically isomorphic to $M(\Gamma)$. As usual, a multiplier is a continuous linear operator commuting with translations.

The space $L_p^1(\Gamma)$ was studied previously in [12], where Theorem 3.1 was actually stated. However, the proof given in [12] is incorrect; more precisely, it is not always possible to find a function $f \in L_1^1(\Gamma)$ that satisfies the conditions (i), (ii), and (iii) specified at the bottom of page 375 of [12].

THEOREM 3.1. *Let* Γ *be a noncompact* LCA *group with character group* G. *Then the space of multipliers from* $L_p^1(\Gamma)$ *into* $L^1(\Gamma)$ *is isometrically isomorphic to* $M(\Gamma)$, *each multiplier being defined by convolution with an element* μ *of* $M(\Gamma)$, *and conversely.*

Proof. One half of the theorem is trivial. For the converse, suppose that T is a multiplier of $L_p^1(\Gamma)$ into $L^1(\Gamma)$. Then

$$\|Tf\|_1 \leq \|T\|(\|f\|_1 + \|\hat{f}\|_p) \qquad (f \in L_p^1(\Gamma)). \tag{10}$$

We want to prove that

$$\|Tf\|_1 \leq \|T\| \cdot \|f\|_{L^1}. \tag{11}$$

Once (11) is established, the theorem will follow from the facts that $L_p^1(\Gamma)$ is dense in $L^1(\Gamma)$ and that the space $M_1(\Gamma)$ is isometrically isomorphic to $M(\Gamma)$ (see [13]).

In proving (11), we may assume that f is continuous and has a compact support. Fix $\varepsilon > 0$ and choose a compact set K in Γ containing the support of f and having the property that

$$\int_{\Gamma\setminus K} |Tf|\, d\gamma < \varepsilon .$$

Since Γ is not compact, we can find inductively elements $\gamma_1, \cdots, \gamma_n, \cdots$ in Γ such that $(K+\gamma_i) \cap (K+\gamma_j) = \emptyset$ for $i \neq j$. We define

$$f_n = (f_{\gamma_1} + \cdots + f_{\gamma_n})/n ,$$

where f_{γ_i} denotes the translate of f by the amount γ_i. The choice of K and the fact that T commutes with translations imply that

$$\|Tf_n\|_1 \geq \|Tf\|_1 - 2\varepsilon .$$

Also, since K contains the support of f, we have that $\|f_n\|_\infty \to 0$ as $n \to \infty$. But $\|f_n\|_1$ is bounded; hence Hölder's inequality shows that $\|f_n\|_2 = \|\hat{f}_n\|_2 \to 0$. Since $\hat{f}_n \to 0$ in $L^2(G)$, we may, by passing to a subsequence if necessary, assume that $\hat{f}_n \to 0$ a.e.. But $|\hat{f}_n| = |\phi_n \hat{f}|$, where ϕ_n is the sum of n characters of G divided by n, so that $|\phi_n| \leq 1$. This implies that $|\hat{f}_n| \leq |\hat{f}|$; hence we may apply Lebesgue's dominated-convergence theorem and conclude that $\|\hat{f}_n\|_p \to 0$. Writing f_n in place of f in (10), we find that

$$\|Tf\|_1 - 2\varepsilon \leq \|Tf_n\|_1 \leq \|T\|\,(\|f_n\|_1 + \|\hat{f}_n\|_p) \leq \|T\|\,(\|f\|_1 + \|\hat{f}_n\|_p).$$

Letting $n \to \infty$, we obtain the inequality

$$\|Tf\|_1 - \varepsilon \leq \|T\|\;\|f\|_1 .$$

Since ε was arbitrary, (11) follows, and the proof is complete.

Using Theorem 3.1, one can show that if Γ is noncompact, the multipliers of $L^1(\Gamma) \cap L^p(\Gamma)$ $(1 \leq p < \infty)$ (respectively, $L^1(\Gamma) \cap C_0(\Gamma)$) into $L^1(\Gamma)$ are precisely the operators defined by convolution with elements of $M(\Gamma)$. This result can, however, be established more simply by a direct argument, which we now present. The proof is due to Frank Forelli. We give the space $L^1(\Gamma) \cap L^p(\Gamma)$ (respectively, $L^1(\Gamma) \cap C_0(\Gamma)$) the natural norm $\|f\| = \|f\|_1 + \|f\|_p$.

THEOREM 3.2. *Let* Γ *be a noncompact* LCA *group. The space of multipliers of* $L^1(\Gamma) \cap L^p(\Gamma)$ $(1 < p < \infty)$ *(respectively,* $L^1(\Gamma) \cap C_0(\Gamma)$*) into* $L^1(\Gamma)$ *is isometrically isomorphic to the space of bounded measures on* Γ.

Proof. It is obvious that each $\mu \in M(\Gamma)$ defines a multiplier T_μ and that $\|\mu\|_{M(\Gamma)} \geq \|T_\mu\|$.

Conversely, let T be a multiplier, so that the inequality

$$\|Tf\|_1 \leq \|T\|\,(\|f\|_1 + \|f\|_p) \tag{12}$$

holds for all $f \in L^1(\Gamma) \cap L^p(\Gamma)$ (respectively, $L^1(\Gamma) \cap C_0(\Gamma)$). Replace f in (12) by $f + f_a$. Then

$$\|Tf + (Tf)_a\|_1 \leq \|T\|\,(\|f + f_a\|_1 + \|f + f_a\|_p). \tag{13}$$

Letting $a \to \infty$ in (13), we get the inequality

$$2\|Tf\|_1 \leq \|T\| (2\|f\|_1 + 2^{1/p}\|f\|_p),$$

or, equivalently,

$$\|Tf\|_1 \leq \|T\| (\|f\|_1 + 2^{p^{-1}-1}\|f\|_p).$$

Carrying out this shifting process n times, we find that

$$\|Tf\|_1 \leq \|T\| (\|f\|_1 + 2^{n(p^{-1}-1)}\|f\|_p). \tag{14}$$

The exponent $p^{-1} - 1$ is negative; letting n tend to infinity in (14), we get the relation

$$\|Tf\|_1 \leq \|T\| \; \|f\|_1 .$$

As in Theorem 3.1, this inequality allows us to complete the proof.

Remarks. (a) The situation where Γ is compact is altogether different in each of the cases above. We leave it to the reader to observe that when Γ is compact, the corresponding multiplier problems are either trivial (and have quite different solutions from those enunciated in Theorems 3.1 and 3.2) or impossible.

(b) [12] contains another correct result with an incorrect proof. It is Theorem 5, part (ii). A correct proof is readily obtained by using the fact that $L_p^1(\Gamma)$ contains every integrable function whose transform has compact support. See [14], where a much more general class of algebras is studied.

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