PIECEWISE-LINEAR CLASSIFICATION OF SOME FREE $\mathbb{Z}_p$-ACTIONS ON S^{4k+3}

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Let S^{2n+1} denote the unit $(2n+1)$-sphere. Represent each of its points by a sequence $(c_0, \cdots, c_n)$ of complex numbers with $\sum |c_i|^2 = 1$. Let (S^1, S^{2n+2}) denote the S^1-action on S^{2n+1} given by the formula

$$c \cdot (c_0, \cdots, c_n) = (cc_0, \cdots, cc_n).$$

Let p be an odd prime, and let $\mathbb{Z}_p$ be the subgroup of S^1 generated by $\exp(2\pi i/p)$. Then (S^1, S^{2n+1}) induces a $\mathbb{Z}_p$-action on S^{2n+1}. Its orbit space

$$L^n(p) = S^{2n+1}/\mathbb{Z}_p$$

is the $(2n+1)$-dimensional lens space. The purpose of this note is to study the piecewise-linear classification of all free $\mathbb{Z}_p$-actions on S^{4k+3} $(4k+3 \geq 7)$ for which the orbit space is of the same simple homotopy type as $L^{2k+1}(p)$. Our main results follow.

THEOREM I. *Let* $\mathcal{S}ht(L^{2k+1}(p))$ *denote the set of equivalence classes of simple homotopy triangulations of* $L^{2k+1}(p)$. *If* $4k+3 \geq 7$, *there exists an exact sequence of pointed sets*

$$0 \to L_{4k+4}(\mathbb{Z}_p)^{\sim} \to \mathcal{S}ht(L^{2k+1}(p)) \to [L^{2k+2}(p); G/PL] \to 0,$$

where $[L^{2k+1}(p); G/PL]$ *is the subgroup of* G/PL-*bundles on* $L^{2k+1}(p)$ *and* $L_{4k+4}(\mathbb{Z}_p)^{\sim}$ *is the reduced surgery obstruction group of C. T. C. Wall.*

THEOREM II. *There exists a one-to-one correspondence between the set*

$$\mathcal{S}ht(L^{2k+1}(p)) \times \left\{0, 1, \cdots, \frac{p-1}{2}\right\}$$

and the set of equivalence classes of free piecewise-linear $\mathbb{Z}_p$-*actions on* S^{4k+3} *whose orbit space has the same simple homotopy type as* $L^{2k+1}(p)$.

In the first section, we recapitulate some generalities about nonsimply-connected surgery, part of which has become folklore. We then carry out, in the second section, an elementary computation of the group $[L^{2k+1}(p); G/PL]$. Results of Sullivan are used in the proof of (2.1). Section 3 completes the proof of Theorem I. Section 4 is mainly a study of the homotopy classes of piecewise-linear homeomorphisms of a homotopy lens space. In the last section, we complete the proof of Theorem II.

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1. NONSIMPLY-CONNECTED SURGERY

In this section, we give a brief outline of the Browder-Novikov theorem from the point of view of nonsimply-connected surgery.

For a closed, piecewise-linear manifold M, Sullivan defines a simple homotopy triangulation of M to be a simple homotopy equivalence $f: L \to M$, where L is another closed, piecewise-linear manifold. Two homotopy triangulations (L_0, f_0) and (L_1, f_1) are equivalent if there exists a piecewise-linear homeomorphism $h: L_0 \xrightarrow{\simeq} L_1$ such that the diagram

$$\begin{array}{ccc} L_0 & & \\ {\scriptstyle h}\downarrow\simeq & \overset{f_0}{\searrow} & \\ & & M \\ & \underset{f_1}{\nearrow} & \\ L_1 & & \end{array}$$

commutes, up to homotopy. We denote the set of equivalence classes of simple homotopy triangulations of M by $\mathcal{S}ht(M)$.

Let G/PL be the fibre of the natural map $B_{PL} \to B_G$. A map $g: X \to G/PL$ consists of a piecewise-linear n-disk bundle $E \to X$ and an F-trivialization t. That is, we have the diagram

$$\begin{array}{ccc} E & \xrightarrow{\quad t \quad} & D^N \\ \Big\downarrow \pi & & \\ X & & \end{array}$$

where t induces a homotopy equivalence of each fibre pair $(p^{-1}(x), \partial p^{-1}(x))$ $(x \in X)$ with $(D^N, \partial D^N)$. Two G/PL-bundles (ξ_0, t_0) and (ξ_1, t_1) are equivalent if there exist a G/PL-bundle (ξ, t) over $X \times I$ and bundle isomorphisms $f_j: \xi_j \simeq \xi \mid X \times j$ $(j = 0, 1)$ such that $tf_j \sim t_j$.

Suppose that $X = M^k$ is a manifold. By making $t: E(\xi) \to D^N$ transverse regular to $0 \in D^N$, we may assume that there is a framed submanifold $L^k \times R^N \subset E(\xi)$ such that

$$t \mid L \times R: L^k \times R^N \to D^N = R^N$$

is a projection onto the second factor and $t(E(\xi) \setminus L \times R) = \partial D^N$, and such that the map $\pi: L^k \to M^k$, induced by the projection $\pi: E(\xi) \to M^k$, is a map of degree one. If (ξ, t) and (ξ', t') are equivalent to each other, then the framed manifolds $L \times R^N \subset E(\xi)$ and $L' \times R^N \subset E(\xi)$, given by (ξ, t) and (ξ', t'), are framed cobordant to each other. Thus for each $L^k \times R^N$ in $E(\xi)$, we can do framed surgery on $L^k \times R^N$ in $E(\xi)$ and try to make $\pi: L^k \to M^k$ a simple homotopy equivalence. According to the theory of nonsimply-connected surgery developed by Wall (see [7]), there is an obstruction in the group $L_k(\pi_1 M)$. This gives us the following maps:

$$s: [M; G/PL] \to L_k(\pi_1 M);$$

$$s': [M \times I, M \times \partial I; G/PL, *] \to L_{k+1}(\pi_1 M).$$

Next, for a prescribed simple homotopy triangulation $f\colon L^k \to M^k$, let ξ denote the piecewise-linear bundle stably equivalent to $\bar{f}^* \tau_L - \tau_M$ over M, where τ_L and τ_M are the tangent bundles and $\bar{f}$ is a homotopy inverse of f. Deform f to an embedding of L^k into the total space $E(\xi)$ of ξ. One sees by an easy computation that L^k has a trivial normal bundle in ξ. The choice of a framing $L^k \times R^N \subset E(\xi)$ determines a map $\theta(L^k, f)\colon M^k \to G/PL$ (see [6]). The notion of equivalence of G/PL-bundles guarantees that the homotopy class of $\theta(L^k, f)$ is independent of all the choices made. Thus we have constructed a sequence

$$\mathcal{S}ht(M^k) \xrightarrow{\theta} [M^k, G/PL] \xrightarrow{s} L_k(\pi_1 M),$$

which is exact as a sequence of pointed sets.

Finally, there exists an action of $L_{k+1}(\pi_1 M)$ on $\mathcal{S}ht(M^k)$ defined as follows: choose $(L^k, f) \in \mathcal{S}ht(M^k)$ and an element $x \in L_{k+1}(\pi_1 M)$; then there exists a triple (W, ϕ, F), where ϕ is a map of degree one of the triads $(W, \partial W) \to (M \times I, M \times \partial I)$, $(\partial_+ W, \phi \mid \partial_+ W) = (L, f)$, $\phi \mid \partial_- W$ is a simple homotopy equivalence, F is a stable framing for $\nu_W \oplus \phi^* \tau_M$, and the surgery obstruction satisfies the condition $\theta(W, \phi, F) = x$. Define the action of x on (L^k, f) to be $(\partial_- W, \phi \mid \partial_- W)$. Combining this with the sequence above, we have the following exact sequence of sets

$$L_{k+1}(\pi_1 M) \xrightarrow{\chi} \mathcal{S}ht(M) \xrightarrow{\theta} [M; G/PL] \xrightarrow{s} L_k(\pi_1 M)$$

$$[M \times I, M \times \partial I; G/PL, *] \xrightarrow{s'} L_{k+1}(\pi_1 M) .$$

For further information and references about nonsimply-connected surgery, see Wall [7].

2. THE GROUP $[L^n(p); G/PL]$

In this section we shall prove the following proposition.

PROPOSITION (2.1). *The map* $s\colon [L^n(p); G/PL] \to L_{2n+1}(\mathbb{Z}_p)$ *is always zero, provided that* n *is odd.*

First, consider the S^1-action (S^1, S^{2n+1}) on S^{2n+1} defined by the condition $c \cdot (c_0, \cdots, c_n) = (cc_0, \cdots, cc_n)$. Since $L^n(p)$ is the orbit space of the action of the $\mathbb{Z}_p$-subgroup in S^1, there exists an S^1-fibration

$$\pi\colon L^n(p) \to CP(n).$$

Suppose we have a homotopy triangulation (M, f) of complex projective space CP^n. Pull back this S^1-fibration by f to get a simple homotopy triangulation of $L^n(p)$. Denote this by $\pi!(M, f)$. Clearly this yields a well-defined map

$$\pi!\colon \mathcal{S}ht(CP(n)) \to \mathcal{S}ht(L^n(p)).$$

Moreover, the diagram

$$\begin{array}{ccc} \mathcal{S}ht(L^n(p)) & \xrightarrow{\theta} & [L^n(p); G/PL] \\ \uparrow \pi! & & \uparrow \pi^* \\ \mathcal{S}ht(CP(n)) & \xrightarrow{\theta} & [CP(n); G/PL] \end{array}$$

is commutative.

Next, let $L^n(p)_0$ be $L^n(p)$ with a disk D^{2n+1} removed. It is easy to see that $L^n(p)_0/L^{n-1}(p)_0$ is of the same homotopy type as $S^{2n-1} \cup_p e^{2n}$, where p denotes a map of degree p from ∂e^{2n} to S^{2n-1}. The exact sequence

$$[S^{2n-1}; G/PL] \leftarrow [S^{2n-1}; G/PL] \leftarrow [L^n(p)_0/L^{n-1}(p)_0; G/PL] \leftarrow [S^{2n}; G/PL] \leftarrow [S^{2n}; G/PL] \leftarrow \cdots$$

implies that

$$[L^n(p)_0/L^{n-1}(p)_0; G/PL] = \begin{cases} \mathbb{Z}_p & \text{if } n \equiv 0 \pmod 2, \\ 0 & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

The same sequence also shows that $[L^n(p)_0/L^{n-1}(p)_0; B_{G/PL}]$ is always zero.

LEMMA (2.2). *For every* n, $[L^n(p)_0; G/PL]$ *is a* p-*group.*

Proof. Consider the exact sequence

$$\leftarrow [L^n(p)_0/L^{n-1}(p)_0; B_{G/PL}] \leftarrow [L^{n-1}(p)_0; G/PL] \leftarrow [L^n(p)_0; G/PL] \leftarrow [L^n(p)_0/L^{n-1}(p)_0; G/PL] \leftarrow \cdots.$$

Since $[L^n(p)_0/L^{n-1}(p)_0; B_{G/PL}]$ is always zero, the group $[L^n(p)_0; G/PL]$ is either isomorphic to $[L^{n-1}(p)_0; G/PL]$ or to an extension of $[L^{n-1}(p)_0; G/PL]$ by a cyclic p-group. Lemma (2.2) follows immediately from induction on n.

LEMMA (2.3). $[L^n(p); G/PL] \cong [L^n(p)_0; G/PL]$.

Proof. There exists an exact sequence

$$\leftarrow [S^{2n+2}; G/PL] \leftarrow [L^n(p)_0; G/PL] \leftarrow [L^{n-1}(p); G/PL] \leftarrow [S^{2n+1}; G/PL] \leftarrow \cdots.$$

Since $[S^{2n+2}; G/PL]$ is p-torsion free, the homomorphism on the left-hand side of the sequence above is always zero. On the other hand, $[S^{2n+1}; G/PL] \cong 0$. Lemma (2.3) follows.

LEMMA (2.4). π^*: $[CP(n); G/PL] \to [L^n(p)_0; G/PL]$ *is a surjection.*

Proof. First, we observe that $CP(n)/CP(n-1)$ is of the homotopy type of S^{2n}. The projection map

$$\pi: L^n(p)_0/L^{n-1}(p)_0 \to CP(n)/CP(n-1)$$

induced by π: $L^n(p)_0 \to CP(n)$ is homotopic to the map that collapses S^{2n-1} in $S^{2n-1} \cup_p e^{2n}$ ($\sim L^n(p)_0/L^{n-1}(p)_0$) to a point. From this, it follows that

$$\pi^*: [CP(n)/CP(n-1); G/PL] \to [L^n(p)_0/L^{n-1}(p)_0; G/PL]$$

is always a surjective map.

Now, we assume by induction that

$$\pi^*: [CP(n-1); G/PL] \to [L^{n-1}(p)_0; G/PL]$$

is a surjective map. Chasing the following commutative diagram

$$\begin{array}{ccccccccc} \longrightarrow & [L^n(p)_0/L^{n-1}(p)_0; G/PL] & \longrightarrow & [L^n(p)_0; G/PL] & \longrightarrow & [L^{n-1}(p)_0; G/PL] & \longrightarrow \\ & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & \\ \longrightarrow & [CP(n)/CP(n-1); G/PL] & \longrightarrow & [CP(n); G/PL] & \longrightarrow & [CP(n-1); G/PL] & \longrightarrow, \end{array}$$

we see that

$$\pi^*: [CP(n); G/PL] \to [L^n(p)_0; G/PL]$$

is a surjective map. This proves Lemma (2.4).

LEMMA (2.5) (Sullivan). *For a simply-connected manifold* M *of dimension* $4k+2$, *the map*

$$s: [M^{4k+2}; G/PL] \to L_{4k+2}(1) \ (\cong \mathbb{Z}_2)$$

is a group homomorphism. In particular, for $n \equiv 1 \pmod 2$, *the map*

$$s: [CP(n); G/PL] \to L_{2n}(1)$$

is a group homomorphism.

Proof. There exists a cohomology class $K \in H^{4*+2}(G/PL; \mathbb{Z}_2)$ such that if $f: M \to G/PL$ is an element of $[M; G/PL]$, then

$$s(f) = \langle W(M) \cup f^*(K); [M] \rangle \in \mathbb{Z}_2,$$

where $W(M)$ is the total Wu class. Lemma (2.5) follows immediately from the fact that K is a primitive class (see [6]).

Now, we are in a position to prove (2.1).

Proof of (2.1). Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{S}ht(L^n(p)) & \xrightarrow[\theta]{} & [L^{2k+1}(p); G/PL] & \xrightarrow[s]{} & L_{4k+3}(\mathbb{Z}_p) \\ \uparrow{\scriptstyle \pi!} & & \uparrow{\scriptstyle \pi^*} & & \\ \mathcal{S}ht(CP(n)) & \xrightarrow[\theta]{} & [CP(n); G/PL] & \xrightarrow[s]{} & \mathbb{Z}_2 \end{array}$$

Let g be an element of $[L^{2k+1}(p); G/PL]$. Then by (2.2), (2.3), and (2.4), there exists an element g' of $[CP(n); G/PL]$ such that $\pi^*(2g') = g$. But since $s: [CP(n); G/PL] \to \mathbb{Z}_2$ is a homomorphism, we have that $s(2g') = 0$. Therefore 2g' can be represented by a simple homotopy triangulation of CP(n). It follows from the diagram above that $s(g) = 0$. The proof of (2.1) is complete.

3. PROOF OF THEOREM I

Let $L_n(1)$ denote the obstruction group of simply-connected surgery. General arguments show that $L_n(1)$ is a direct summand of $L_n(\mathbb{Z}_p)$. The reduced surgery obstruction group $L_n(\mathbb{Z}_p)^{\sim}$ in Theorem I is defined to be the kernel of the projection map $L_n(\mathbb{Z}_p) \to L_n(1)$.

PROPOSITION (3.1).

(1) $[L^n(p) \times I, L^n(p) \times \partial I; G/PL, *] \simeq \pi_{2n+2}(G/PL) \cong L_{2n+2}(1)$.

(2) *Let*

$$s': [L^n(p) \times I, L^n(p) \times \partial I; G/PL, *] \to L_{2n+2}(\mathbb{Z}_p)$$

be defined as in Section 1. *Then* $\operatorname{Im} s' = L_{2n+2}(1) \subset L_{2n+2}(\mathbb{Z}_p)$.

Proof. Let $\phi: (L \times I, L \times \partial I) \to (D^{2n+2}, \partial D)$ be a map of degree one. Since $H^i(L \times I, L \times \partial I; \pi_i(G/PL)) = 0$ for $i \neq 2n+2$, the group $[L \times I, L \times \partial I; G/PL, *]$ is generated by the image of the map

$$\phi^*: [D, \partial D; G/PL, *] \to [L \times I, L \times \partial I; G/PL, *].$$

As is well known, the group $[D^{2n+2}, \partial D; G/PL, *]$ is determined by the formula

$$[D^{2n+2}, \partial D; G/PL, *] = \pi_{2n+2}(G/PL) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 1 \pmod 2, \\ \mathbb{Z}_2 & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Represent an element $x \in [D, \partial D; G/PL, *]$ ($\simeq \mathbb{Z}$ or $\simeq \mathbb{Z}_2$) by a Kervaire-Milnor manifold (M, ϕ, F), where x is the corresponding index or Arf-invariant. Its image $\phi^*(x)$ in $[L \times I, L \times \partial I; G/PL, *]$ can be represented by the connected sum of (M, ϕ, F) with the trivial cobordism (W, ϕ, F), where $W = L \times I$, $\phi = \mathrm{id}: W \to L \times I$, and F is the natural framing of $\tau_W \oplus \phi^* \nu_{L \times I}$. Obviously,

$$s'((W, \phi, F) \# (M, \phi, F)) = s(M, \phi, F) = x.$$

Hence $[L \times I, L \times \partial I; G/PL, *]$ is isomorphic to $L_{2n+2}(1)$ under the map

$$s': [L \times I, L \times \partial I; G/PL, *] \to L_{2n+2}(1) \subset L_{2n+2}(\mathbb{Z}_p).$$

This completes the proof of (3.1).

Now Theorem I is an immediate consequence of Propositions (2.1) and (3.1).

Proof of Theorem I. Recall that we have the exact sequence of pointed sets

$$\begin{array}{ccccccc} L_{4k+4}(\mathbb{Z}_p) & \xrightarrow{\chi} & \mathcal{S}ht(L^{2k+1}(p)) & \xrightarrow{\theta} & [L^{2k+1}(p); G/PL] & \xrightarrow{s} & L_{4k+3}(\mathbb{Z}_p) \\ & \nwarrow^{s'} & & & & & \\ & & & & [L \times I, L \times \partial I; G/PL, *] & & \end{array}$$

Since (2.1) shows that the map s on the right-hand side of the sequence is always zero, and since (3.1) shows that $\operatorname{Im} s' = L_{4k+4}(1)$, we have the exact sequence

$$0 \to L_{4k+4}(\mathbb{Z}_p)^{\sim} \xrightarrow{\chi} \mathcal{S}\!\mathit{ht}(L^{2k+1}(p)) \xrightarrow{\theta} [L^{2k+1}(p); G/PL] \to 0 .$$

This proves Theorem I.

Remark (3.2). It follows from the results of D. Sullivan (see [6]) that $[L^n(p); G/PL]$ is isomorphic to the multiplicative subgroup of the virtual line bundles of $KO(L^n(p))$, modulo 2-torsions. This multiplicative structure has been completely determined by T. Kambe in [1].

Remark (3.3). To determine the group $L_{4k+4}(\mathbb{Z}_p)^{\sim}$, Wall has shown that there exist a number of invariants that can detect a nonzero element of $L_{4k+4}(\mathbb{Z}_p)^{\sim}$. Recent results of T. Petrie show that such invariants can be realized and that

$$L_{4k+4}(\mathbb{Z}_p)^{\sim} \cong \left[\frac{p-1}{2}\right]\mathbb{Z} .$$

4. HOMEOMORPHISMS

Throughout this section, a (4k + 3)-manifold is called a homotopy lens space if it has the simple homotopy type of $L^{2k+1}(p)$. In this section, we study the homotopy classes of piecewise-linear homeomorphisms of a homotopy lens space.

First, we need the following well-known facts.

LEMMA (4.1). *Let* f *and* g *be maps of* $L^{2k+1}(p)$ *into itself. If* f *and* g *induce the same map on the fundamental group and on the top homology group of* $L^{2k+1}(p)$, *in other words, if*

$$f_* = g_*: \pi_1(L^{2k+1}(p)) \to \pi_1(L^{2k+1}(p)) \quad \text{and}$$

$$f_* = g_*: H_{4k+3}(L^{2k+1}(p)) \to H_{4k+3}(L^{2k+1}(p)) ,$$

then f *is homotopic to* g.

For proof, see E. H. Spanier [5, p. 451, Theorem 10].

LEMMA (4.2). *If* $a \not\equiv \pm 1 \pmod p$, *then there exists no simple homotopy equivalence of* $L^{2k+1}(p)$ *to itself such that the map induced on the fundamental group sends* t *to* t^a.

Proof of (4.2). Denote by Δ the torsion of $L^{2k+1}(p)$, and let $N \subset Q\mathbb{Z}_p$ denote the kernel of the canonical homomorphism $Q\mathbb{Z}_p \to Q$ that carries the generator t of $\mathbb{Z}_p$ to +1. It is well known (see Milnor [3]) that

$$\Delta = (t-1)^{2k+1} \in U(N)/\mathbb{Z}_p$$

and that Δ is preserved by simple homotopy equivalence. Suppose there exists a simple homotopy equivalence $\gamma: L^{2k+1}(p) \to L^{2k+1}(p)$ such that

$$\gamma_*(t) = t^a ,$$

where $a \not\equiv \pm 1 \pmod p$. Since $\gamma_* \Delta = \Delta$, we have the relation

$$(t-1)^{2k+1} \sim (t^a - 1)^{2k+1}.$$

This, of course, contradicts the Franz Independence Lemma (see Milnor [3]). Lemma (4.2) follows.

Let c: $L^{2k+1}(p) \xrightarrow{\simeq} L^{2k+1}(p)$ be the diffeomorphism defined by conjugation of complex coordinates. Then

$$c_* = -\mathrm{id}: \pi_1(L^{2k+1}(p)) \to \pi_1(L^{2k+1}(p)) .$$

By (4.2) and (4.1), every piecewise-linear homeomorphism of $L^{2k+1}(p)$ is homotopic to either the identity or c.

Next we prove that a piecewise-linear homeomorphism similar to c: $L^{2k+1}(p) \to L^{2k+1}(p)$ always exists for a homotopy lens space.

THEOREM (4.3). *Every homotopy lens space* L *admits a piecewise-linear homeomorphism* γ: $L \xrightarrow{\simeq} L$ *such that*

(4.4)
$$\gamma_* = -\mathrm{id}: \pi_1 L \to \pi_1 L .$$

This result is essentially equivalent to the following theorem.

THEOREM (4.5). *Let* c: $L^{2k+1}(p) \xrightarrow{\simeq} L^{2k+1}(p)$ *be the piecewise-linear homeomorphism defined as before. Define* c_*: $\mathcal{S}ht(L^{2k+1}(p)) \to \mathcal{S}ht(L^{2k+1}(p))$ *by the formula*

$$c_*(L, f) = (L, c \circ f) ,$$

for every $(L, f) \in \mathcal{S}ht(L^{2k+1}(p))$. *Then* $c_* = \mathrm{id}$.

The proof of (4.5) will be based on the following lemmas.

LEMMA (4.6). *Let* θ *be defined as in Section* 1. *Then the diagram*

$$\begin{array}{ccc} \mathcal{S}ht(L^{2k+1}(p)) & \xrightarrow{\theta} & [L^{2k+1}(p); G/PL] \\ \downarrow c_* & & \downarrow c_* \\ \mathcal{S}ht(L^{2k+1}(p)) & \xrightarrow{\theta} & [L^{2k+1}(p); G/PL] \end{array}$$

commutes.

LEMMA (4.7). *Let* c: $CP(n) \xrightarrow{\simeq} CP(n)$ *be the diffeomorphism defined by the conjugation of complex coordinates. Then*

$$c^* = \mathrm{id}: [CP(n); G/PL] \to [CP(n); G/PL] .$$

The proof of (4.6) is a routine matter, and we leave it to the reader. The proof of (4.7) is easy and follows directly from the computation of the group [CP(n); G/PL] (see Sullivan [6]).

Remark (4.8). From the theory of simply-connected surgery, it is known that the map

$$\theta: \mathcal{S}ht(CP(n)) \to [CP(n); G/PL] ,$$

defined in Section 2, is an injection. Thus, by (4.7), every homotopy complex projective space HCP(n) admits a piecewise-linear homeomorphism γ such that

$$\gamma_* = \text{-id}: H^2(HCP(n)) \to H^2(HCP(n)) .$$

Since $p^*: [CP(n); G/PL] \to [L^{2k+1}(p); G/PL]$ is a surjection, we have that for each element a of $[L^{2k+1}(p); G/PL]$ there exists a simple homotopy triangulation (L, f) with $\theta(L, f) = a$, and L satisfies Theorem (4.3).

LEMMA (4.9). *Let* (L, f) *be a simple homotopy triangulation of* $L^{2k+1}(p)$ *with* $c_*(L, f) = (L, f)$. *Let*

$$\chi: L_{4k+4}(\mathbb{Z}_p)^\sim \to \mathcal{S}ht(L^{2k+1}(p))$$

be the injection defined by the action of the group $L_{4k+4}(\mathbb{Z}_p)^\sim$ *on* (L, f), *as in Section* 1. *Then the diagram*

$$\begin{array}{ccc} L_{4k+4}(\mathbb{Z}_p)^\sim & \xrightarrow{\chi} & \mathcal{S}ht(L^{2k+1}(p)) \\ \downarrow{\scriptstyle (-id)_*} & & \downarrow{\scriptstyle c_*} \\ L_{4k+4}(\mathbb{Z}_p)^\sim & \xrightarrow{\chi} & \mathcal{S}ht(L^{2k+1}(p)) \end{array}$$

is commutative.

Proof of (4.9). Let a be an element of $L_{4k+4}(\mathbb{Z}_p)$. Represent $\chi(a)$ by a simple homotopy triangulation $(\partial_+ W, \phi \mid \partial_+ W)$, as in Section 1. More precisely, we have the cobordism (W, ϕ, F), where

$$\phi: (W; \partial_+ W, \partial_- W) \to L^{2k+1}(p) \times (I; 0, 1)$$

is a map of degree one, and F is a stable framing of the bundle $\tau_W \oplus \phi^* \nu$ such that the following conditions are satisfied:

(i) $(\partial W, \phi \mid \partial_- W) = (L, f)$ in $\mathcal{S}ht(L^{2k+1}(p))$;

(ii) $\phi \mid \partial_+ W$ is a simple homotopy equivalence;

(iii) $\theta(W, \phi, F) = a$.

By definition, $(\partial_+ W, c \circ \phi \mid \partial_+ W)$ is a simple homotopy triangulation that represents the element $c_*(a)$. Note that

(i) $(\partial_- W, c \circ \phi \mid \partial_- W) = (L, c \circ f) = (L, f)$;

(ii) $(\partial_- W, c \circ \phi \mid \partial_- W)$ is connected with $(\partial_+ W, c \circ \phi \mid \partial_+ W)$ by the cobordism $(W, c \circ \phi, F)$.

Because $c_* = \text{-id}: \pi_1(L^{2k+1}(p)) \to \pi_1(L^{2k+1}(p))$, we have the relations

$$\theta(W, c \circ \phi, F) = (c_*)_* \theta(W, \phi, F) = (\text{-id})_*(a) .$$

This shows that $c_* \circ \chi(a) = \chi \circ (\text{-id})_*(a)$, and the proof of Lemma (4.9) is complete.

LEMMA (4.10). *Let* -id *denote the automorphism of* $\mathbb{Z}_p$ *that sends every element* t *of* $\mathbb{Z}_p$ *to* t^{-1}. *Then the induced map*

$$(\text{-id})_*: L_{4k+4}(\mathbb{Z}_p)^\sim \to L_{4k+4}(\mathbb{Z}_p)^\sim$$

is the identity.

The proof of (4.10) requires a different technique, and we shall omit it. For full details, we refer the reader to a forthcoming paper by Petrie (see [4]).

Proof of (4.3) *and* (4.5). Let ξ be an element of $\mathcal{S}ht(L^{2k+1}(p))$. By (4.8), there exists a simple homotopy triangulation η in $\mathcal{S}ht(L^{2k+1}(p))$ with the property that

$$c_*(\eta) = \eta \quad \text{and} \quad \theta(\eta) = \theta(\xi).$$

Thus we can think of ξ as $\chi(a)$, where a is some element of $L_{4k+4}(\mathbb{Z}_p)^\sim$ and χ is defined as in (4.9). Now Lemmas (4.9) and (4.10) imply that $c_*(\xi) = \xi$. This completes the proof of (4.5).

Next suppose we have a prescribed homotopy lens space L. Let $f: L \to L^{2k+1}(p)$ be a simple homotopy equivalence. Since $(L, f) = (L, c \circ f)$, there exists a piecewise-linear homeomorphism $\gamma: L \to L$ such that $f \circ \gamma = c \circ f$. Clearly, we have that

$$\gamma_* = -\text{id}: \pi_1(L) \to \pi_1(L).$$

This completes the proof of (4.3).

5. FREE $\mathbb{Z}_p$-ACTIONS ON S^{4k+3}

We are now in a position to prove Theorem II.

PROPOSITION (5.1). *There exists a one-to-one correspondence between* $\mathcal{S}ht(L^{2k+1}(p))$ *and the set of piecewise-linearly distinct homotopy lens spaces.*

Proof of (5.1). Every simple homotopy triangulation of $L^{2k+1}(p)$ yields a homotopy lens space. Suppose (L, f) and (L, f') yield the same lens space L. By (4.1) and (4.2), the simple homotopy equivalence $f \circ f'^{-1}: L \to L$ is either homotopic to the identity or to γ in (4.3). In either case, $f \circ f'^{-1}$ can be deformed to a piecewise-linear homeomorphism, and thus (L, f) = (L, f'). This completes the proof of (5.1).

Proof of Theorem II. Let $A(S^{4k+3}, \mathbb{Z}_p)$ denote the set of equivalence classes of free $\mathbb{Z}_p$-actions on S^{4k+3} whose orbit spaces have the same simple homotopy type as $L^{2k+1}(p)$. First observe that there exists a natural map

$$\Phi: \mathcal{S}ht(L^{2k+1}(p)) \to A(S^{4k+3}; \mathbb{Z}_p)$$

defined as follows. Let $f: M \to L^{2k+1}(p)$ be a simple homotopy triangulation in $L^{2k+1}(p)$. Pull back under f the $\mathbb{Z}_p$-bundle $S^{4k+3} \to L^{2k+1}(p)$ to get a free $\mathbb{Z}_p$-action on the universal covering space $M^\sim$ ($\cong S^{4k+3}$) of M. Clearly, $\Phi(M, f) = (\mathbb{Z}_p, M)$ is well-defined, since $(\mathbb{Z}_p, M)$ depends only on the equivalence class of (M, f) in $\mathcal{S}ht(L^{2k+1}(p))$.

Next let $T: \mathbb{Z}_p \times S^{4k+3} \to S^{4k+3}$ be a free $\mathbb{Z}_p$-action on S^{4k+3}, and let q be an integer between 0 and p - 1. Define another action $T_q: \mathbb{Z}_p \times S^{4k+3} \to S^{4k+3}$ by the relation

$$T_q(t^i, x) = T(t^{i+q}, x),$$

where t is the generator of $\mathbb{Z}_p$ and $x \in S^{4k+3}$. The $\mathbb{Z}_p$-action (T_q, S^{4k+3}) is called the qth power of (T, S^{2k+3}). Clearly, (T_q, S^{4k+3}) and (T, S^{4k+3}) have the same orbit space.

Define Φ: $\mathcal{S}ht(L^{4k+3}(p)) \times \mathbb{Z}_p \to A(S^{4k+3}, \mathbb{Z}_p)$ by the formula

$$\Phi(\xi \times q) = \text{the qth power of } \Phi(\xi),$$

where $\xi \in \mathcal{S}ht(L^{4k+3}(p))$ and $q \in \mathbb{Z}_p$. It is easy to see that Φ is onto, and by (4.3), that $\Phi(\xi, q) = \Phi(\xi, p - q)$. It remains to show that $\Phi(\xi, q) = \Phi(\xi', q')$ only if $\xi = \xi'$ and either $q = q'$ or $q = p - q'$. By (5.1), we can easily rule out the case $\xi \neq \xi'$. Suppose $q \neq q'$ or $q \neq p - q'$. Then one can find a simple homotopy equivalence of $L^{2k+1}(p)$ into itself that is not homotopic to the identity or the map c. This contradicts (4.2). The proof of Theorem II is complete.

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