ASYMPTOTIC AND INTEGRAL CLOSURE OF ELEMENTS IN MULTIPLICATIVE LATTICES

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1. INTRODUCTION

The cancellation property of ideals (if AB = AC and A ≠ 0, then B = C) in a Dedekind domain does not generalize to arbitrary commutative rings. P. Samuel [7] showed how to restore the cancellation property in Noetherian rings by replacing the properties of the ideals themselves with asymptotic properties of their high powers. For this replacement, the notion of asymptotic closure of an ideal is fundamental. In this paper we give a lattice-theoretic characterization of asymptotic closure, we generalize the cancellation law to (commutative) multiplicative lattices that satisfy the ascending chain condition (Theorem 1), and we investigate some properties of this closure operation. In particular, we give a lattice-theoretic characterization of the notion of integral closure of an ideal (in a Noetherian ring), and by means of E. W. Johnson's A-transforms of a Noether lattice [3] we show that the asymptotic closure of an element in a Noether lattice coincides with its integral closure (Theorem 3). Since not all Noether lattices are lattices of ideals of a Noetherian ring [1], Theorem 3 extends M. Nagata's result that the asymptotic and integral closure operations coincide in Noetherian rings [4].

Finally, we establish some relations between the asymptotic closure operation in a Noether lattice and its A-transforms.

2. ASYMPTOTIC CLOSURE IN MULTIPLICATIVE LATTICES

A *multiplicative lattice* is a complete lattice provided with a commutative, associative, join-distributive multiplication for which the greatest element, denoted by I, is also the multiplicative identity (0 denotes the null element). In this section, L denotes a multiplicative lattice that satisfies the ascending chain condition. We shall now generalize the asymptotic closure of ideals (as characterized by D. Rees in [6]) to multiplicative lattices. The development toward the cancellation law is patterned after [5].

Let R denote the ordered additive group of real numbers, together with an element ∞ that satisfies the relations $\alpha + \infty = \infty$, $\infty + \infty = \infty$, $\infty > \alpha$ (here α denotes a real number). A mapping v: L → R is a *pseudovaluation* on L if

a) $v(0) = \infty$,

b) $v(I) = 0$,

c) $v(AC) \geq v(A) + v(C)$ $\quad (A, C \in L)$, and

d) $v(A \vee C) \geq \min(v(A), v(C))$ $\quad (A, C \in L)$.

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A pseudovaluation is *homogeneous* if for all positive integers n it satisfies the further condition

e) $v(A^n) = nv(A)$.

For each element B different from I, we define a function v_B on L by defining

$$v_B(A) = \begin{cases} \infty & \text{if } A \le B^n \text{ for all nonnegative integers } n, \\ n & \text{if } A \le B^n, \text{ but } A \not\le B^{n+1}. \end{cases}$$

(For nonpositive integers n, we define $B^n = I$.)

Since v_B is a pseudovaluation on L, $\lim_{n\to\infty} v_B(A^n)/n$ exists for all A, B in L. The proof of this statement is a straightforward generalization of an argument used in [6]; we omit the proof.

We now construct a homogeneous pseudovaluation $\bar{v}_B$ on L defined by $\bar{v}_B(A) = \lim_{n\to\infty} v_B(A^n)/n$.

LEMMA 1. *The function* $\bar{v}_B$ *is a homogeneous pseudovaluation on* L.

Proof. For a fixed element B in L, let $v = v_B$, and let $\bar{v} = \bar{v}_B$. Observe that

$$\bar{v}(AC) = \lim_{n\to\infty} \frac{v(A^n C^n)}{n} \ge \lim_{n\to\infty} \left(\frac{v(A^n)}{n} + \frac{v(C^n)}{n}\right) = \bar{v}(A) + \bar{v}(C)$$

and $\bar{v}(A^n) = \lim_{n\to\infty} nv(A^{nm})/nm = n\bar{v}(A)$. To show that $\bar{v}(A \vee C) \ge \min(\bar{v}(A), \bar{v}(C))$, let β be a real number such that $\beta < \min(\bar{v}(A), \bar{v}(C))$. There exists a positive integer n such that $v(A^m) \ge m\beta$ and $v(C^m) \ge m\beta$ for every integer $m \ge n$. Since v is a pseudovaluation, it follows that

$$v[(A \vee C)^{kn}] \ge \min\{v(A^i C^{kn-i})\}_{i=0}^{kn} \ge \min\{v(A^i) + v(C^{kn-i})\}_{i=0}^{kn} \ge (k-1)n\beta$$

for all integers $k \ge 1$. Consequently, $\bar{v}(A \vee C) \ge \beta$. Since β is arbitrary, this completes the proof of the lemma.

If B is different from I, B_s is the join of all elements A that satisfy the condition $\bar{v}_B(A) \ge 1$; otherwise, $B_s = I$. Since B_s is a join of finitely many elements A that satisfy the inequality $\bar{v}_B(A) \ge 1$, and since $\bar{v}_B$ is a pseudovaluation (Lemma 1), we conclude that $D \le B_s$ if and only if $\bar{v}_B(D) \ge 1$. The mapping $B \to B_s$ is called the AC-*operation* on L.

The following lemma is an immediate consequence of the definition of $\bar{v}_B$.

LEMMA 2. *Let* α *be a positive real number. A necessary and sufficient condition that* $\bar{v}_B(A) \ge \alpha$ *is that to each rational number* $0 < p/q < \alpha$, *there corresponds a positive integer* n *such that* $A^{qn} \le B^{pn}$.

By Lemma 2 above and by Lemma 1 in [5], we conclude that if L^* is a lattice of ideals of a Noetherian ring, then the AC-operation on L^* is precisely the asymptotic closure operation on L^*.

We now say that the AC-operation on L is the *asymptotic closure operation* on L. For each element A in L, we call A_s the *asymptotic closure* of A. If $A = A_s$, then A is *asymptotically closed.*

LEMMA 3. *If* A *and* B *are elements of* L *different from* I, *and* n *is a positive integer, then*

a) $\bar{v}_A \leq \bar{v}_B$ *whenever* $A \leq B$,

b) $A_s \leq B_s$ *whenever* $\bar{v}_A \leq \bar{v}_B$,

c) $n\bar{v}_{A^n} = \bar{v}_A$,

d) $A_s B_s \leq (AB)_s$, *hence* $(B_s)^n \leq (B^n)_s$,

e) $\bar{v}_A = \bar{v}_{A_s}$,

f) $A_s \leq B_s$ *whenever* $A \leq B_s$, *and*

g) $\bar{v}_A \leq \bar{v}_B$ *whenever* $A_s \leq B_s$.

Proof. Statements a) and b) are obvious, while both f) and g) are immediate consequences of a) and e). To prove c), let $D \in L$, and let p/q be a rational number such that $0 < p/q < \bar{v}_A(D)$. There exists a positive integer k such that $D^{nqkm} \leq A^{npkm}$ for every integer $m \geq 1$ (Lemma 2). Therefore,

$$v_{A^n}(D^{nqkm})/nqkm \geq p/nq.$$

Since D and p/q are arbitrary, we conclude that $n\bar{v}_{A^n} \geq \bar{v}_A$. The opposite inequality is proved by a similar argument. For each rational number $0 < p/q < 1$, there exist positive integers m and n such that $A_s^{qm} \leq A^{pm}$ and $B_s^{qn} \leq B^{pn}$; hence $(A_s B_s)^{qk} \leq (AB)^{pk}$ $(k = mn)$. Statement d) now follows by Lemma 2.

Finally, we prove that $\bar{v}_A = \bar{v}_{A_s}$. Let $D \in L$, and let $0 < p/q < \bar{v}_{A_s}(D)$. For some positive integer k, $D^{qk} \leq (A_s)^{pk} \leq (A^{pk})_s$. Consequently,

$$(q/p)\bar{v}_A(D) = qk\bar{v}_{A^{pk}}(D) = \bar{v}_{A^{pk}}(D^{qk}) \geq 1.$$

From this, we deduce that $\bar{v}_A(D) \geq \bar{v}_{A_s}(D)$. The opposite inequality holds by a).

A mapping x: $L \to L$ $(A \to A_x)$ is a *semiprime operation* on L if it satisfies the following conditions for all $A, B \in L$:

a) $A \leq A_x$,

b) if $A \leq B_x$, then $A_x \leq B_x$, and

c) $A_x B_x \leq (AB)_x$.

The integral closure operation and the radical operation in a Noetherian ring R are semiprime operations on the lattice of ideals of R. The asymptotic closure operation on L is a semiprime operation (Lemma 3).

LEMMA 4. *If* A *and* B *are elements in* L *different from* I, *then*

$$\frac{1}{\bar{v}_{AB}(M)} \leq \frac{1}{\bar{v}_A(M)} + \frac{1}{\bar{v}_B(M)}$$

for all $M \in L$ *with* $\bar{v}_{AB}(M) \neq 0$.

Proof. If $0 < n/m < \bar{v}_A(M)$ and $0 < p/q < \bar{v}_B(M)$, then there exist positive integers j and k such that $M^{mj} \leq A^{nj}$ and $M^{qk} \leq B^{pk}$. From this we deduce that

$$\frac{v_{AB}(M^{(mp+nq)kjt})}{(mp+nq)kjt} \geq \frac{np}{mp+nq}$$

for every positive integer t. Therefore, $1/\bar{v}_{AB}(M) \leq m/n + q/p$. To complete the proof, let n/m approach $\bar{v}_A(M)$ and let p/q approach $\bar{v}_B(M)$.

LEMMA 5. *Let* A *and* B *be elements of* L *different from* I. *Then* $\lim_{n\to\infty} n\bar{v}_{A^n B}(M)$ *exists for all* $M \in L$. *Moreover,*

a) $\lim_{n\to\infty} n\bar{v}_{A^n B}(M) = \bar{v}_A(M)$, *if there exists a positive integer* m *such that* $M^m \leq B$;

b) $\lim_{n\to\infty} n\bar{v}_{A^n B}(M) = 0$, *if* $M^m \not\leq B$ *for all integers* m; *and*

c) $\lim_{n\to\infty} n\bar{v}_{A^n B}(M) = \bar{v}_A(M)$ *for all* $M \in L$, *if there exists a positive integer* m *such that* $A^m \leq B$.

Proof. If $M^m \not\leq B$ for all integers m, then $\bar{v}_B(M) = 0$. Since $\bar{v}_{A^n B} \leq \bar{v}_B$ for all n, statement b) holds. Now assume that there exists a positive integer m such that $M^m \leq B$. If there exists a positive integer n such that $\bar{v}_{A^n B}(M) = 0$, then $\bar{v}_A(M) = 0$, and $\lim_{n\to\infty} n\bar{v}_{A^n B}(M)$ exists and is equal to zero. Assume that $\bar{v}_{A^n B}(M) > 0$ for every positive integer n. Then, by Lemmas 3 and 4,

$$\frac{1}{n\bar{v}_{A^n B}(M)} \leq \frac{1}{\bar{v}_A(M)} + \frac{1}{n\bar{v}_B(M)}.$$

From this we deduce that

$$\limsup_{n\to\infty} \frac{1}{n\bar{v}_{A^n B}(M)} \leq \limsup_{n\to\infty} \left[\frac{1}{\bar{v}_A(M)} + \frac{1}{n\bar{v}_B(M)}\right] = \frac{1}{\bar{v}_A(M)}.$$

Consequently,

$$\bar{v}_A(M) \leq \liminf_{n\to\infty} n\bar{v}_{A^n B}(M) \leq \limsup_{n\to\infty} n\bar{v}_{A^n B}(M) \leq \bar{v}_A(M).$$

Statement c) follows from a) and b).

THEOREM 1 (Cancellation Law). *Let* A, B, *and* C *be elements in* L. *If* $(AC)_s \leq (BC)_s$ *and* $A^m \leq C$ *for some integer* m, *then* $A_s \leq B_s$.

Proof. The relation $(AC)_s \leq (BC)_s$ implies that $(A^nC)_s \leq (B^nC)_s \leq (B^n)_s$ for each integer $n \geq 1$. Consequently, $\bar{v}_{A^n C} \leq \bar{v}_{B^n}$ for each integer $n \geq 1$ (Lemma 3). Therefore,

$$\bar{v}_A(M) = \lim_{n\to\infty} n\bar{v}_{A^n C}(M) \leq \bar{v}_B(M)$$

for all $M \in L$ (Lemma 5); hence $A_s \leq B_s$.

In Lemma 2, the integer n depends upon the rational number p/q. We shall now show that n depends only on the element B, and we shall use this result to give a simpler characterization of the asymptotic closure.

LEMMA 6. *Let* B *be an element of* L. *There is a positive integer* n *such that* $(B_s)^{nq} \leq B^{np}$ *for each rational number* p/q $(0 < p/q < 1)$. *Furthermore,* $M \leq B_s$ *if and only if there exist integers* n *and* k *such that* $(M^n)^{k+i} \leq (B^n)^i$ *for every integer* $i \geq 1$.

Proof. For each integer $m \geq 1$, let

$$B_m = \{M \in L \mid M^{m(k+1)} \leq B^{mk} \text{ for every integer } k \geq 1\},$$

and let

$$(B_m)' = \{M \in L \mid \text{there exists an integer } i \leq m \text{ such that } M^{i(k+1)} \leq B^{ik} \text{ for every integer } k \geq 1\}.$$

Then $(B_m)' \subseteq (B_{m+1})'$ for each m. By the ascending chain condition, there exists an integer n^* such that

$$B^* = \bigvee\left(\bigcup_{m=1}^{\infty} (B_m)'\right) = \bigvee (B_{n^*})'.$$

Since $B_p \subseteq B_q$ if p divides q, and since $B_m \subseteq (B_m)' \subseteq B_{\Pi\{i \mid 0<i\leq m\}}$, it follows that

$$B^* = \bigvee B_{\Pi\{i \mid 0<i\leq n^*\}} \geq \bigvee B_m \quad \text{for all } m.$$

Next we prove that $B^* \in B_n$ for some integer n. Let C and D be elements in B_m. Then

$$(C \vee D)^{m(k+1)} = \bigvee_{i=0}^{m(k+1)} C^i D^{mk+m-i} \leq B^{m(k-2)}$$

for all integers $k \geq 1$. Therefore, $(C \vee D)^{3m+mk} \leq B^{mk}$ for all integers $k \geq 1$. From this, we deduce that $C \vee D \in B_{3m}$. Since $B_m \subseteq B_{3m}$, and since $\bigvee B_m$ is a join of finitely many elements of B_m, it follows that $B^* \in B_n$ for some integer n.

For each integer $k \geq 1$, there exists a positive integer h such that $(B_s)^{h(k+1)} \leq B^{hk}$; hence $B_s \leq \bigvee\left(\bigcup_{m=1}^{\infty} (B_m)'\right) = B^*$. Since $B^* \in B_n$ for some integer n, it follows that $B_s \in B_n$. Consequently, for any rational number p/q $(0 < p/q < 1)$, we have the relations $(B_s)^{nq} \leq (B_s)^{n(p+1)} \leq B^{np}$. This completes the proof of the first statement in the lemma.

If $M \leq B_s$, then there exists an integer n such that $M^{nq} \leq B^{np}$ for every rational number p/q $(0 < p/q < 1)$. In particular, $(M^n)^{k+i} \leq (B^n)^i$ for all integers $i \geq 1$ and $k \geq 1$. Conversely, if there exist integers n and k such that $(M^n)^{k+i} \leq (B^n)^i$ for all integers $i \geq 1$, then $v_B(M^{nk+ni})/n(k+i) \geq i/(k+i)$; hence $M \leq B_s$.

For $B \in L$, let

$$B_c = \bigvee \{M \in L \mid \text{there is a positive integer } n \text{ such that } M^{n+i} \leq B^i \text{ for all integers } i \geq 0\}.$$

THEOREM 2. $B_c = B_s$ *for every element* B *in* L.

Proof. There exists an integer n such that $B_c^{n+i} \leq B^i$ for every integer $i \geq 0$. Then $v_B(B_c^{n+i})/(n+i) \geq i/(n+i)$; hence $B_c \leq B_s$. Now let $M = B_s$. By Lemma 6, there exists an integer n such that $M^{nq} \leq B^{np}$ for every rational number p/q $(0 < p/q < 1)$. In particular, $(M^n)^{i+1} \leq B^{ni}$ for every integer $i \geq 0$. For each integer $i \geq 0$, we can write $i = nq + r$ $(0 \leq r < n)$. Then

$$M^{2n+i} = M^{n(2+q)+r} \leq M^{n(2+q)} \leq (B^n)^{(q+1)} \leq B^{qn+r} = B^i.$$

Therefore, $B_s \leq B_c$.

3. INTEGRAL CLOSURE IN MULTIPLICATIVE LATTICES

Throughout this section, L denotes a multiplicative lattice that satisfies the ascending chain condition.

An element M is a-*dependent* on B if there exists a positive integer n such that $M^{n+1} \leq B(M \vee B)^n$. Since $(B \vee M)^k = \bigvee_{i=0}^k M^i B^{k-i}$ for every integer $k \geq 0$, it follows that M is a-dependent on B if and only if $(B \vee M)^{n+1} = B(B \vee M)^n$ for some integer $n \geq 1$. Let B_a denote the join of all elements that are a-dependent on B. By the ascending chain condition, B_a is a join of finitely many such elements. We call the mapping $B \to B_a$ the IC-*operation* on L.

LEMMA 7. *If* M *and* N *are* a-*dependent on* B, *then so is* $M \vee N$. *Therefore* B_a *is* a-*dependent on* B.

Proof. The identities $(B \vee M)^{m+1} = B(B \vee M)^m$ and $(B \vee N)^{n+1} = B(B \vee N)^n$ imply that

$$(B \vee M \vee N)^{m+n+1} = \bigvee_{i=0}^{m+n+1} (B \vee M)^i (B \vee N)^{m+n+1-i}$$

$$\leq \bigvee_{i=0}^{m+n} B(B \vee M)^i (B \vee N)^{m+n-i} = B(B \vee M \vee N)^{m+n}.$$

Therefore, if M and N are a-dependent on B, then so is $M \vee N$. Since B_a is a join of finitely many elements that are a-dependent on B, it follows that B_a is a-dependent on B.

We shall now prove that the IC-operation on L is a semiprime operation. If $A \leq B$ and M is a-dependent on A, then M is a-dependent on B; hence $A_a \leq B_a$. Now let M be a-dependent on B_a, so that $(B_a \vee M)^{n+1} = B_a(B_a \vee M)^n$ for some positive integer n. By Lemma 7, B_a is a-dependent on B; therefore $B_a^{k+1} = BB_a^k$ for some positive integer k. Consequently,

$$(B \vee (B_a \vee M))^{n+k+1} = (B_a \vee M)^{n+k+1} = B_a^{k+1}(B_a \vee M)^n$$

$$= BB_a^k(B_a \vee M)^n = B(B_a \vee M)^{n+k} = B(B \vee (B_a \vee M))^{n+k};$$

hence $M \leq B_a$. In particular, we conclude that $B_{aa} = B_a$ and that $A \leq B_a$ implies $A_a \leq B_{aa} = B_a$. Finally, the relation $A_a B_a \leq (AB)_a$ follows from the fact that if N is a-dependent on B, then MN is a-dependent on B (for any M).

If L^* is a lattice of ideals of a Noetherian ring R, then the IC-operation on L^* is precisely the integral closure operation on L^*. Let $B \in L^*$, and let B_f denote the integral closure of B in R. Since B_a is a-dependent on B (Lemma 7), there exists a positive integer k such that $B_a^{k+1} = BB_a^k$. This implies that B and B_a have the same integral closure in R [4]; hence $B_a \subseteq B_f$. Conversely, if $b \in B_f$, then there exist elements $b_i \in B^i$ such that $b^{n+1} + b_1 b^n + \cdots + b_n b + b_{n+1} = 0$. Consequently, $(bR)^{n+1} \subseteq B(bR + B)^n$ and $bR \subseteq B_a$. Therefore $B_a = B_f$.

We now say that the IC-operation on L is the *integral closure operation* on L. We call B_a the *integral closure* of B, and we say that B is *integrally closed* if $B = B_a$.

4. ASYMPTOTIC AND INTEGRAL CLOSURE IN NOETHER LATTICES

Throughout this section, L denotes a Noether lattice [2]. In order to show that the asymptotic closure of an element in a Noether lattice coincides with its integral closure, we introduce the notion of an A-transform of L.

Let $A \neq I$ be a fixed element of L, and consider the collection R(L, A) of all formal sums $\sum_{i=-\infty}^{\infty} B_i$ of elements B_i of L for which the relations $A^i \geq B_i \geq B_{i+1} \geq AB_i$ hold for all integers i. For elements B, C $\in$ R(L, A), we define

$$B \vee C = \sum (B_i \vee C_i), \qquad B \wedge C = \sum (B_i \wedge C_i), \qquad B \cdot C = \sum_i \Big(\bigvee_{r+t=i} B_r C_t \Big),$$

and we say that $B \leq C$ if and only if $B_i \leq C_i$ for all i.

The collection R(L, A), together with the operations $\vee$, $\wedge$, $\cdot$ and the relation $\leq$, is called the A-*transform* of L. If $C \in L$ and $C \leq A^n$, then $C^{[n]}$ is the least element D of R(L, A) for which $D_n \geq C$. In the following lemma, we summarize some results from [3].

LEMMA 8. a) *The* A-*transform of* L *is a Noether lattice.*

b) *If* $C \leq A^n$, *then* $C^{[n]} = \sum CA^{i-n}$.

c) *If* $C, D \leq A^n$, *then* $C^{[n]} \vee D^{[n]} = (C \vee D)^{[n]}$.

d) *If* $B \in R(L, A)$ *and* $C \leq A^n$ *in* L, *then* $BC^{[n]} = \sum B_{i-n}C$.

e) *If* C *and* D *are elements of* L *such that* $C \leq A^n$ *and* $D \leq A^m$, *then* $C^{[n]}D^{[m]} = (CD)^{[n+m]}$.

f) *If* B *is a principal element of* L *such that* $B \leq A^n$, *then* $B^{[n]}$ *is a principal element in* R(L, A).

Let a' and s' denote the integral closure and the asymptotic closure operations in R(L, A), respectively.

THEOREM 3. *For every element* $A \in L$, *the equality* $A_a = A_s$ *holds, and* $M \leq A_s$ *implies that* M *is* a-*dependent on* A.

Proof. We show first that if B is a principal element and $M \leq B_s$, then M is a-dependent on B. Since $B_s = B_c$ (Theorem 2), there exists a positive integer n such that $M^{n+1} \leq B^i$ for every integer $i \geq 0$. Hence

$$M^{n+1} \leq B^i \wedge (M \vee B)^{n+i} = B^i((M \vee B)^{n+i} : B^i)$$

for every integer $i \geq 0$ (the equality follows since B^i is a principal element [2, p. 485]). By the ascending chain condition in L, there exists a positive integer r such that

$$(M \vee B)^{n+r+i} : B^{r+i} = (M \vee B)^{n+r} : B^r$$

for every integer $i \geq 0$. Therefore

$$M^{n+r+1} \leq B[B^r((M \vee B)^{n+r} : B^r)] \leq B(M \vee B)^{n+r};$$

hence M is a-dependent on B.

Now let A $(A \neq I)$ be an element of L, and let $M \leq A_s$. There exists an $n > 0$ such that $M^{n+i} \leq A^i$ for every $i \geq 0$; therefore

$$I^{[-i]}(M^{n+i})^{[i]} \leq I^{[-i]}(A^i)^{[i]}.$$

Consequently, $M^{[0]} \leq I^{[-1]}_{s'}$. Since $I^{[-1]}$ is a principal element in R(L, A) (Lemma 8), $M^{[0]}$ is a-dependent on $I^{[-1]}$. Therefore, there exists an $n > 0$ such that

$$(*) \qquad (I^{[-1]} \vee M^{[0]})^{n+1} = I^{[-1]}(I^{[-1]} \vee M^{[0]})^n.$$

The 0^{th} component of the left side of (*) is $(A \vee M)^{n+1}$, while the 0^{th} component of the right side of (*) is $A(A \vee M)^n$. Therefore $(A \vee M)^{n+1} = A(A \vee M)^n$, and M is a-dependent on A. In particular, $A_s \leq A_s$ implies that $A_s \leq A_a$. Since A_a is a-dependent on A (Lemma 7), there exists an integer $k \geq 1$ such that $A_a^{k+1} \leq AA_a^k$. From this, we conclude that $(A_a)_s \leq A_s$ (Theorem 1). Therefore $A_a = A_s$.

COROLLARY. *Let* A *and* B *be elements of* L. *Then* $A_s = B_s$ *if and only if there exists an element* $C \in L$ *such that* $AC = BC$ *and* $(A \vee B)^n \leq C$ *for some positive integer* n.

Proof. Let $A_s = B_s$. By Theorem 3, $(B \vee A)^{n+1} = B(B \vee A)^n$ and $(B \vee A)^{k+1} = A(B \vee A)^k$ for some positive integers n and k. Consequently, $AC = BC$ for $C = (B \vee A)^{k+n}$. The converse follows from Theorem 1.

Let x denote a semiprime operation on L. For each B in R(L, A), the formal sum $B_{x^*} = \sum [(B_n)_x \wedge A^n]$ is also in R(L, A). Furthermore, the mapping x^*: R(L, A) $\rightarrow$ R(L, A) is a semiprime operation on R(L, A). By means of this induced semiprime operation, the following theorem relates the asymptotic operation in L to the asymptotic operation in the A-transforms of L.

THEOREM 4. *For each* $B \in R(L, A)$, *the relation* $B_{s^*} \leq B_{s'}$ *holds. Furthermore, if* $P \leq A^n$, *then* $P^{[n]}_{s^*} = P^{[n]}_{s'}$.

Proof. The relation $M \leq (B_{s^*})_n$ implies that $M^{k+1} \leq B_n(M \vee B_n)^k$ for some positive integer k. Consequently, $(M^{k+1})^{[nk+n]} \leq (B_n(M \vee B_n)^k)^{[nk+n]}$. By Lemma 8,

$$\begin{aligned}(M^{[n]})^{k+1} &= (M^{k+1})^{[nk+n]} \leq (B_n(M \vee B_n)^k)^{[nk+n]} \\ &= B_n^{[n]}((M \vee B_n)^{[n]})^k = B_n^{[n]}(M^{[n]} \vee B_n^{[n]})^k \leq B(M^{[n]} \vee B)^k;\end{aligned}$$

hence $M^{[n]} \leq B_{s'}$. This completes the proof of the first statement.

Now let $P \leq A^n$. By the first statement, it is sufficient to prove that $P_{s'}^{[n]} \leq P_{s*}^{[n]}$. The relation $M \leq (P_{s'}^{[n]})_k$ implies that $(M^{[k]})^{m+1} \leq P^{[n]}(M^{[k]} \vee P^{[n]})^m$ for some positive integer m. By Lemma 8, $(M^{[k]})^{m+1} = (M^{m+1})^{[km+m]}$. Consequently,

$$M^{m+1} \leq (P^{[n]}(M^{[k]} \vee P^{[n]})^m)_{km+k} = \left(\bigvee_{i=0}^{m} (M^{[k]})^i (P^{[n]})^{m+1-i}\right)_{km+k}$$

$$= \left(\bigvee_{i=0}^{m} (M^i)^{[ik]} (P^{m+1-i})^{[nm+n-in]}\right)_{km+k}$$

$$= \bigvee_{i=0}^{m} ((M^i P^{m+1-i})^{[nm+n-in+ik]})_{km+k} = \bigvee_{i=0}^{m} M^i P^{m+1-i} A^{(k-m)(m+1-i)}$$

$$= \bigvee_{i=0}^{m} M^i (PA^{k-n})^{m+1-i} \qquad (\text{since } m+1-i \geq 0)$$

$$= (PA^{k-n})(M \vee (PA^{k-n}))^m.$$

Therefore $M \leq (PA^{k-n})_s \wedge A^k = (P_{s*}^{[n]})_k$.

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