REPRESENTATIONS OF SOLVABLE LIE ALGEBRAS

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Ado's theorem [1] states that every finite-dimensional Lie algebra over a field of characteristic zero has a faithful finite-dimensional representation. The proofs of this theorem in the literature at the present time do not appear to provide a bound for the dimension of the representation space.

In a forerunner to the theorem of Ado, G. Birkhoff proved that a finite-dimensional, nilpotent Lie algebra has a faithful representation whose space has dimension not greater than $1 + n + n^2 + \cdots + n^{k+1}$, where n is the dimension of the algebra and k is the nilpotency class [2].

The main result of this paper is that every finite-dimensional, solvable Lie algebra over an algebraically closed field of characteristic zero has a faithful representation whose space has dimension not greater than $1 + n + n^n$, where n is the dimension of the algebra. In the nilpotent case, we obtain the bound $1 + n^k$, where k is the nilpotency class. This is sharper than the bound of Birkhoff.

In particular, Section 1 contains a proof of Birkhoff's theorem about nilpotent Lie algebras. We also prove that the enveloping associative algebra determined by the representation has the same nilpotency class as the original algebra. In the second part of this section, we prove that if $\mathscr{L}$ is a finite-dimensional Lie algebra that can be written as a semidirect sum $\mathscr{L} = \mathscr{L}_1 + N$ (N a nilpotent ideal), then Birkhoff's representation of N can be extended to $\mathscr{L}$.

Section 2 deals with splittable Lie algebras. Here we prove that every solvable Lie algebra $\mathscr{L}$ of finite dimension over an algebraically closed field of characteristic zero can be embedded in a solvable algebra of given dimension, determined by the dimension of $\mathscr{L}$ and the dimension of the nilradical of $\mathscr{L}$.

Finally, in Section 3 we obtain a faithful representation of an n-dimensional solvable Lie algebra $\mathscr{L}$ over an algebraically closed field of characteristic zero. This is done by embedding $\mathscr{L}$ in a solvable splittable algebra $\mathscr{L}_1$, as in Section 2, and constructing a representation of $\mathscr{L}_1$ from the results of Section 1.

1. PROPOSITION 1. *Let $\mathscr{L}$ be an n-dimensional nilpotent Lie algebra with lower central series $\mathscr{L} = \mathscr{L}^1 \supset \mathscr{L}^2 \supset \cdots \supset \mathscr{L}^k \supset \mathscr{L}^{k+1} = 0$. Then $\mathscr{L}$ is isomorphic to a Lie algebra $\mathscr{A}$ of linear transformations of a vector space M of dimension not greater than $1 + n^k$. Moreover, the product of any $k+1$ elements of $\mathscr{A}^*$ is zero.*

Proof. We first choose a basis $x_1, \cdots, x_n$ in the following way:

$$x_1, \cdots, x_{p(1)} \text{ is a basis for } \mathscr{L}^k,$$

$$x_1, \cdots, x_{p(2)} \text{ is a basis for } \mathscr{L}^{k-1},$$

$$\cdots,$$

$$x_1, \cdots, x_{p(k)} = x_n \text{ is a basis for } \mathscr{L}.$$

Received November 9, 1968.

Let $\mathscr{U}$ be the universal enveloping algebra of $\mathscr{L}$. Recall that by the Poincaré-Birkhoff-Witt theorem, the element 1 together with elements of the form $x_1^{\rho_1} x_2^{\rho_2} \cdots x_n^{\rho_n}$ (called standard monomials) form a basis for $\mathscr{U}$.

Now consider the subalgebra $\mathscr{U}'$ of $\mathscr{U}$ generated by the standard monomials. We define a function ϕ from $\mathscr{U}'$ into the natural numbers as follows:

$$\phi(x_i) = m \qquad \text{if } x_i \in \mathscr{L}^m \text{ and } x_i \notin \mathscr{L}^{m+1},$$

$$\phi(x_1^{\rho_1} x_2^{\rho_2} \cdots x_n^{\rho_n}) = \sum_{i=1}^{n} \rho_i \phi(x_i).$$

Finally, for a linear combination $x = \sum_{i=1}^n c_i y_i$ of standard monomials y_i we define $\phi(x)$ to be the minimum of the set $[\phi(y_i)]$.

Now observe that $\phi[x_i, x_j] \geq \phi(x_i) + \phi(x_j)$. This implies that

$$\phi(y_1, \cdots, y_s) \geq \phi(y_1) + \cdots + \phi(y_s) \qquad \text{for } y_1, \cdots, y_s \in \mathscr{U}'.$$

Let $J = \{y \in \mathscr{U}' \mid \phi(y) > k\}$. Then J is an ideal of $\mathscr{U}'$. It is also clear that J is an ideal in $\mathscr{U}$. Consider the quotient algebras $\mathscr{U}/J$ and $\mathscr{U}'/J$. We see that $\mathscr{U}'/J$ is an associative algebra whose Lie algebra contains $\mathscr{L}$ (since ϕ of any linear combination of the x_i is not greater than k). Moreover, $\mathscr{U}'/J$ has the property that the product of any $k+1$ of its elements is zero. Let R be the representation of $\mathscr{U}'/J$ acting in the space $\mathscr{U}/J$ induced by the regular representation of $\mathscr{U}/J$. Since $(\mathscr{U}'/J)_{\mathscr{L}}$ contains $\mathscr{L}$, R induces a representation $\overline{R}$ of $\mathscr{L}$ acting in the space $\mathscr{U}/J$. Let $M = \mathscr{U}/J$ and $\mathscr{A} = \overline{R}(\mathscr{L})$. We need only show that the dimension of M is not greater than $1 + n^k$.

The dimension of the space $M = \mathscr{U}/J$ is not greater than 1 plus the number of standard monomials of length less than or equal to k.

If n is 1 or 2, then $k = 1$, and the result is obvious; now suppose n is greater than 2. The number of standard monomials of length 1 is n. The number that have length 2 is $n(n+1)/2$, and the number of length j is $n^{j-1}(n+1)/2$. Hence, the number of standard monomials of length less than or equal to k is

$$1 + n + \frac{n(n+1)}{2} + \frac{n^2(n+1)}{2} + \cdots + \frac{n^{k-1}(n+1)}{2}$$

$$= 1 + n + n^2 + \cdots + n^{k-1} + \frac{n+n^k}{2} = \frac{n^k - 1}{n-1} + \frac{n+n^k}{2}.$$

By a straightforward argument, this number is not greater than $n^k + 1$. This completes the proof.

PROPOSITION 2. *Let $\mathscr{L}$ be a Lie algebra of dimension* n *such that* $\mathscr{L} = \mathscr{L}_1 + N$ *(semidirect sum), where* N *is a nilpotent ideal of $\mathscr{L}$ containing the center. Let* $N = N^1 \supset N^2 \supset \cdots \supset N^k \supset N^{k+1} = 0$ *be the lower central series of* N. *Then $\mathscr{L}$ has a faithful representation* R *with representation space* M *of dimension not greater than* $1 + n + n^k$.

Proof. Let $\mathscr{U}_{\mathscr{L}}$ and $\mathscr{U}_N$ be the universal enveloping algebras of $\mathscr{L}$ and N, respectively. Let J be the ideal of $\mathscr{U}_N$ of the preceding proposition. We shall first define a representation of $\mathscr{L}$ acting in the space $\mathscr{U}_N$, and then we shall show that this representation induces a representation of $\mathscr{L}$ acting in the quotient space $\mathscr{U}_N/J$. Moreover, the representation will be an extension of the representation of the preceding proposition. With each $x \in \mathscr{L}_1$, we associate a function ϕ_x mapping $\mathscr{U}_{\mathscr{L}}$ into $\mathscr{U}_{\mathscr{L}}$, defined by $\phi_x(u) = xu - ux$ (we now think of x as an element of $\mathscr{U}_{\mathscr{L}}$). Note that ϕ_x is the extension to $\mathscr{U}_{\mathscr{L}}$ of the derivation ad x of $\mathscr{L}$. With each element $y \in N$, we associate a function θ_y mapping $\mathscr{U}_{\mathscr{L}}$ into $\mathscr{U}_{\mathscr{L}}$, defined by $\theta_y(u) = yu$.

Now $\mathscr{U}_N$ is invariant under ϕ_x, for each $x \in \mathscr{L}$. For if $u = x_{\lambda(1)} x_{\lambda(s)}$ is a standard monomial in $\mathscr{U}_N$, then

$$\phi_x(u) = \sum_{i=1}^{s} x_{\lambda(1)} x_{\lambda(2)} \cdots [x_{\lambda(i)}, x] \cdots x_{\lambda(s)},$$

and the right-hand side is an element of $\mathscr{U}_N$, since N is an ideal of $\mathscr{L}$. Obviously, $\mathscr{U}_N$ is invariant under θ_y for each y is N.

Now, for elements $x, y \in \mathscr{L}_1$ and an element u of $\mathscr{U}_N$,

$$\begin{aligned}[\phi_x, \phi_y]u &= \phi_x\phi_y(u) - \phi_y\phi_x(u) = \phi_x(yu - uy) - \phi_y(xu - ux) \\ &= xyu - xuy - yux + uyx - yxu + yux + xuy - uxy \\ &= (xy - yx)u - u(xy - yx) = [x, y]u - u[x, y] = \phi_{[x,y]}(u).\end{aligned}$$

Thus the mapping defined by $x \to \phi_x$ defines a representation of $\mathscr{L}_1$ acting in the space $\mathscr{U}_N$.

Since each N_i is an ideal in $\mathscr{L}$, it follows that J is invariant under ϕ_x, for each $x \in \mathscr{L}_1$; hence, letting $\bar{\phi}_x$ be the induced operator on $\mathscr{U}_N/J$, we now see that the map $x \to \bar{\phi}_x$ defines a representation of $\mathscr{L}_1$ with representation space $\mathscr{U}_N/J$.

Similarly, we obtain a representation $y \to \bar{\theta}_y$ of N on $\mathscr{U}_N/J$. Moreover, this representation is the representation constructed in the previous proposition.

Observe that for $x \in \mathscr{L}_1$ and $y \in N$,

$$\begin{aligned}[\phi_x, \theta_y]u &= \phi_x\theta_y(u) - \theta_y\phi_x u = \phi_x(yu) - \theta_j(xu - ux) \\ &= xyu - yux - yxu + yux = (xy - yx)u = [x, y]u = \theta_{[x,y]}(u).\end{aligned}$$

Hence, the map defined by $x \to \phi_x$ if $x \in \mathscr{L}_1$, ad $y \to \theta_y$ if $y \in N$ determines a representation of $\mathscr{L}$ with representation space $\mathscr{U}_N$. It follows that the map defined by $x \to \bar{\phi}_x$ ad $y \to \bar{\theta}_y$ determines a representation R' of $\mathscr{L}$ with representation space $\mathscr{U}_N/J$. Moreover, on N the representation R' agrees with the representation of the previous proposition, and hence it is faithful on N.

Now let R be the direct sum of R' and the adjoint representation. Since the adjoint representation has as kernel the center of $\mathscr{L}$, and since R' is faithful on N and hence on the center, it follows that the representation R is faithful on $\mathscr{L}$. Moreover, it is clear that the representation space M of R has dimension not greater than $1 + n + n^k$. The proof is now complete.

2. Malčev [5] proved that every finite-dimensional, solvable Lie algebra over an algebraically closed field of characteristic zero can be embedded in a solvable, splittable Lie algebra. We prove this theorem in this section, specifying the dimension of the splittable algebra. Our proof is more constructive than that of Malčev. We shall need several definitions and results from [5].

Definition. Let $\mathscr{L}$ be a finite-dimensional Lie algebra over an algebraically closed field of characteristic zero. Then an element $x \in \mathscr{L}$ is said to be *semiregular* if the Jordan form of the matrix of $\mathrm{ad}_{\mathscr{L}} x$ canonical is diagonal.

Definition. Let $\mathscr{L}$ satisfy the conditions above. An element x of $\mathscr{L}$ is called splittable if it can be represented as a sum $x = a + y$, where a is a semiregular element of $\mathscr{L}$ and y is a nilpotent element of $\mathscr{L}$. If $[a, y] = 0$, then the decomposition $x = a + y$ is called *normal* and the element x is said to be *normally splittable.*

Definition. A finite-dimensional Lie algebra $\mathscr{L}$ over an algebraically closed field of characteristic zero is called *splittable* if each of its elements is normally splittable.

THEOREM. *Every splittable element of a solvable Lie algebra* $\mathscr{L}$ *is normally splittable.*

From the above theorem, it follows that if $\mathscr{L}$ is a solvable Lie algebra and each element of $\mathscr{L}$ is splittable, then $\mathscr{L}$ is splittable.

Suppose $\mathscr{L}$ is a finite-dimensional Lie algebra over an algebraically closed field Φ of characteristic zero, and suppose D is a derivation of $\mathscr{L}$. We recall that as a linear operator, D can be written as the commutable sum of a diagonalizable operator d and a nilpotent operator n. Moreover, the operators d and n are unique, and each is a polynomial in D. A proof of this can be found in the book [3] of K. Hoffman and R. Kunze.

PROPOSITION 3. *Let* $\mathscr{L}$, D, d, *and* n *be as above. Then* d *and* n *are derivations.*

Proof. The derivation D generates an abelian Lie algebra of linear transformations acting in the space $\mathscr{L}$. Let $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_\alpha + \mathscr{L}_\beta + \cdots + \mathscr{L}_\gamma$ be the decomposition of $\mathscr{L}$ into weight spaces relative to this Lie algebra, where $\mathscr{L}_0$ is the weight space corresponding to the weight zero.

Let $\mathscr{L}_1 = \Phi D + \mathscr{L}$ denote the semidirect sum of the Lie algebra generated by the derivation D and $\mathscr{L}$. Then

$$\mathscr{L}_1 = (\Phi D + \mathscr{L}_0) + \mathscr{L}_\alpha + \mathscr{L}_\beta + \cdots + \mathscr{L}_\gamma$$

is the decomposition of $\mathscr{L}_1$ into root spaces relative to $\mathrm{ad}_{\mathscr{L}_1} \Phi D$, where $\Phi D + \mathscr{L}_0$ is the root space corresponding to the root zero.

We shall now define a linear operator $\bar{d}$ on $\mathscr{L}_1$. Each element x of $\mathscr{L}_1$ can be written as a sum $x = x_0 + x_\alpha + x_\beta + \cdots + x_\gamma$ with $x_0 \in \Phi D + \mathscr{L}_0$, $x_\alpha \in \mathscr{L}_\alpha$, $x_\beta \in \mathscr{L}_\beta$, $\cdots$. We define $\bar{d}(x)$ to be

$$\alpha(D)x_\alpha + \beta(D)x_\beta + \cdots \gamma(D)x_\gamma .$$

Clearly, $\bar{d}$ is a linear operator on $\mathscr{L}_1$.

We now show that $\bar{d}$ is a derivation on $\mathscr{L}_1$. It is sufficient to show that $\bar{d}[x_\rho, x_n] = [\bar{d}x_\rho, x_n] + [x_\rho, \bar{d}x_n]$, where x_ρ and x_n are elements of the root spaces $\mathscr{L}_\rho$ and $\mathscr{L}_n$, respectively, in the above decomposition of $\mathscr{L}_1$.

Recall that $[\mathscr{L}_\rho, \mathscr{L}_n] \subseteq \mathscr{L}_{\rho+n}$ if $\rho + n$ is a root and that $[\mathscr{L}_\rho, \mathscr{L}_n] = 0$ if $\rho + n$ is not a root [4].

If $\rho + n$ is not a root, then $[x_\rho, x_n] = 0$ and the derivation condition is obviously satisfied. If $\rho + n$ is a root, then $[x_\rho, x_n] \in \mathscr{L}_{\rho+n}$ and

$$\bar{d}[x_\rho, x_n] = (\rho + n)(D)[x_\rho, x_n].$$

Also,

$$\begin{aligned}[dx_\rho, x_n] + [x_\rho, dx_n] &= [\rho(D)x_\rho, x_n] + [x_\rho, n(D)x_n] \\ &= \rho(D)[x_\rho, x_n] + n(D)[x_\rho, x_n] = (\rho + n)(D)[x_\rho, x_n].\end{aligned}$$

Hence $\bar{d}$ is a derivation of $\mathscr{L}_1$. Since $\mathscr{L}$ is invariant under $\bar{d}$, it follows that the restriction $\bar{d}'$ to $\mathscr{L}$ is a derivation of $\mathscr{L}$. That $\bar{d}'$ is a diagonalizable operator is obvious from the definition.

Since each weight space $\mathscr{L}_\rho$ in the decomposition of $\mathscr{L}$ is invariant under the operator D, we easily deduce that D and $\bar{d}'$ commute. Letting $\bar{n} = D - \bar{d}'$, we see that $\bar{d}'$ and $\bar{n}$ commute. It also follows from the definition of a weight that $\bar{n}$ is nilpotent on each weight space in the decomposition of $\mathscr{L}$ and is therefore nilpotent on $\mathscr{L}$.

Thus we have written D as the sum of a diagonalizable operator $\bar{d}'$ and a nilpotent operator $\bar{n}$ that commute. It follows that $d' = d$ and $\bar{n} = n$, and that the operators d and n are derivations.

PROPOSITION 4. *Let* $\mathscr{L}$ *be a solvable Lie algebra of dimension* n *over an algebraically closed field of characteristic zero, and let* k *be the dimension of* $\mathscr{L}/N$, *where* N *is the nilradical of* $\mathscr{L}$. *Then* $\mathscr{L}$ *is isomorphic to a subalgebra of a splittable solvable Lie algebra* $\mathscr{L}_1$ *of dimension* $k + n$.

Proof. Choose a basis $x_{k+1}, x_{k+2}, \cdots, x_n$ for N, and choose $x_1 \in \mathscr{L}$ so that the set $\{x_1, x_{k+1}, x_{k+2}, \cdots, x_n\}$ is linearly independent. Consider the derivation ad x_1, and let d_{x_1} be its diagonalizable part as in the preceding proposition.

Now decompose $\mathscr{L}$ into weight spaces relative to the abelian Lie algebra of linear transformations generated by d_{x_1}. Then

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_\alpha + \mathscr{L}_\beta + \cdots + \mathscr{L}_\gamma$$

(here we are letting $\mathscr{L}_0$ denote the space corresponding to the weight zero).

Recall that since $\mathscr{L}$ is a solvable Lie algebra over a field of characteristic zero, $D(\mathscr{L}) \subseteq N$ for each derivation D of $\mathscr{L}$. Let x be an element of any nonzero-weight space in the above decomposition, say $x \in \mathscr{L}_\alpha$. Then $(d_{x_1} - \alpha(d_{x_1})I)^k(x) = 0$ for some integer k. This implies that x is in the image of the derivation d_{x_1} and hence x is in the nilradical N of $\mathscr{L}$.

It follows that there exists an element $x_2 \in \mathscr{L}_0$ such that $\{x_1, x_2, x_{k+1}, \cdots, x_n\}$ is a linearly independent set. (Here we assume that k is greater than 1; for $k = 1$, we could proceed with the second part of the proof. This will be obvious in what follows.) Since x_2 belongs to the zero-weight space $\mathscr{L}_0$, there exists an integer r such that $d_{x_1}^r(x_2) = 0$. But d_{x_1} is a diagonalizable operator; therefore $d_{x_1}(x_2) = 0$. Now form the semidirect-sum algebra $\Phi d_{x_1} + \mathscr{L}$ and consider its representation $x \to \text{ad}_{\mathscr{L}} x$. The condition $d_{x_1}(x_2) = 0$ implies that

$$[d_{x_1}, \mathrm{ad}\, x_2] = d_{x_1} \mathrm{ad}\, x_2 - \mathrm{ad}\, x_2 d_{x_1} = 0.$$

Let d_{x_2} be the diagonalizable part of ad x_2 acting on $\mathscr{L}$. Since d_{x_2} is a polynomial in ad x_2 and ad x_2 commutes with d_{x_1}, we now see that d_{x_1} and d_{x_2} commute.

The derivations d_{x_1} and d_{x_2} generate an abelian Lie algebra of linear transformations acting on the vector space $\mathscr{L}$. We may now decompose $\mathscr{L}$ into weight spaces relative to this Lie algebra and choose an element x_3 in the zero-weight space to obtain a derivation d_{x_3} that commutes with d_{x_1} and d_{x_2}.

Continuing the process described above, we obtain elements $x_1, x_2, \cdots, x_k$ of $\mathscr{L}$ such that the set $[x_1, \cdots, x_k, x_{k+1}, \cdots, x_n]$ is a linearly independent set and hence a basis for $\mathscr{L}$. Moreover, there exist derivations $d_{x_1}, \cdots, d_{x_k}$ of $\mathscr{L}$ that commute with each other and have the property that the operator ad $x_i - d_{x_i}$ is nilpotent.

Let A be the Lie algebra generated by the derivations $d_{x_1}, \cdots, d_{x_k}$, and form the semidirect-sum Lie algebra $\mathscr{L}_1 = A + \mathscr{L}$. Observe that $\mathrm{ad}(x_i - d_{x_i})\mathscr{L}_1 \subseteq \mathscr{L}$ for each i and that the operator $\mathrm{ad}(x_i - d_{x_i})$ restricted to $\mathscr{L}$ is ad $x_i - d_{x_i}$. From this it follows that the elements $x_i - d_{x_i}$ $(1 \leq i \leq k)$ are nilpotent elements of $\mathscr{L}_1$.

It is clear that if we let N_1 be the subalgebra of $\mathscr{L}_1$ with basis

$$x_1 - d_{x_1}, x_2 - d_{x_2}, \cdots, x_k - d_{x_k}, x_{k+1}, \cdots, x_n,$$

then $\mathscr{L}_1 = A + N_1$, and therefore $\mathscr{L}_1$ is splittable.

Now we must show that $\mathscr{L}_1$ has dimension $n + k$. For this, it is sufficient to show that the set $\{d_{x_1}, d_{x_2}, \cdots, d_{x_k}\}$ is linearly independent. Suppose this is not the case. Then there exist elements $c_1, c_2, \cdots, c_n$ (not all zero) of the base field such that

$$c_1 d_{x_1} + c_2 d_{x_2} + \cdots + c_k d_{x_k} = 0.$$

This implies that

$$\sum_{i=1}^{k} c_i(x_i - d_{x_i}) = \sum_{i=1}^{k} c_i x_i$$

is a nilpotent element of $\mathscr{L}_1$, hence of $\mathscr{L}$ (this follows, since for each i, $x_i - d_{x_i}$ is a nilpotent element of $\mathscr{L}$, and the sum of nilpotent elements is nilpotent). But $\sum_{i=1}^{k} c_i x_i$ cannot be a nilpotent element of $\mathscr{L}$, since this would contradict our choice of bases for N and $\mathscr{L}$. Hence the set $\{d_{x_1}, \cdots, d_{x_k}\}$ is linearly independent, and the proof is complete.

Actually, N_1 is the nilradical of $\mathscr{L}_1$. To prove this, it is sufficient to show that no linear combination of the set $\{d_{x_1}, \cdots, d_{x_k}\}$ of derivations is nilpotent. Suppose this is not the case. Then there exist scalars $c_1, c_2, \cdots, c_k$ in the base field such that $c_1 d_{x_1} + c_2 d_{x_2} + \cdots + c_k d_{x_k}$ is nilpotent. The element

$$\sum_{i=1}^{k} c_i(x_i - d_{x_i})$$

is a nilpotent element of $\mathscr{L}$, since $x_i - d_{x_i}$ is a nilpotent element of $\mathscr{L}_1$. But this, along with our assumption that the element $\sum_{i=1}^{k} c_i d_{x_i}$ is nilpotent, implies that $\sum_{i=1}^{k} c_i x_i$ is nilpotent in $\mathscr{L}_1$ and hence in $\mathscr{L}$. This again, however, contradicts our choice of bases for N and $\mathscr{L}$.

3. Now we construct a faithful representation of a solvable Lie algebra $\mathscr{L}$ of dimension n over an algebraically closed field of characteristic zero. Let $\mathscr{L}_1$ be the splittable solvable Lie algebra containing $\mathscr{L}$, from Proposition 4, and let N be the nilradical of $\mathscr{L}_1$. The proof of Proposition 4 and the remarks following Proposition 4 imply that the dimension of N is n.

By the corollary following Proposition 2, $\mathscr{L}_1$ has a representation R' of degree not greater than $1 + n^k$, where k is the nilpotency class of N. Since $\mathscr{L}_1 \supset \mathscr{L}$, this representation induces a representation R on $\mathscr{L}$ of degree not greater than $1 + n^k$. Moreover, since R' is faithful on N and N contains the nilradical of $\mathscr{L}$, R is faithful on the nilradical of $\mathscr{L}$ and hence on the center of $\mathscr{L}$. Now let $\overline{R}$ be the direct sum of R and the adjoint representation of $\mathscr{L}$. Then $\overline{R}$ is faithful on $\mathscr{L}$ and has degree not greater than $1 + n + n^k$.

Since the nilpotency class k of N is at most n, we now have the following result.

THEOREM. *Let $\mathscr{L}$ be a solvable Lie algebra of dimension* n *over an algebraically closed field of characteristic zero. Then $\mathscr{L}$ has a faithful representation of degree not greater than* $1 + n + n^n$.

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