SUBDIRECT DECOMPOSITIONS OF EXTENSION RINGS

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Let A and S be two rings. We say that E is an *extension of* A by S if A is an ideal of E and E/A is isomorphic to S. The purpose of this note is to give a simple counterexample to a theorem of C. W. Kohls [1], and to prove a similar result. The result is strong enough to obtain several of the corollaries in [1].

The result of Kohls [1, p. 401] states that if A is isomorphic to a subdirect sum of simple primitive rings $\{A_\alpha\}$ and if S is isomorphic to a subdirect sum of primitive rings $\{B_\beta\}$, then every extension E of A by S is isomorphic to a subdirect sum of the rings $\{A_\alpha\} \cup \{B_\beta\}$. The counterexample is as follows: Let E be the ring of all linear transformations of a vector space of countably infinite dimension into itself. Let A be the ideal consisting of all elements of E of finite rank. It is well known that the only ideals of E are the zero ideal A and E. Also A is a simple primitive ring. E/A is a simple ring with unity, and hence it is primitive. The result of Kohls would imply that E is isomorphic to a subdirect sum of A and E/A. This is clearly impossible, since E is subdirectly irreducible.

We recall that a ring is *semiprime* if it contains no nonzero nilpotent ideals.

THEOREM. *If* A *is isomorphic to a subdirect sum of semiprime rings with unity* $\{A_\alpha\}$, *and if* S *is isomorphic to a subdirect sum of rings* $\{B_\beta\}$, *then each extension* E *of* A *by* S *is isomorphic to a subdirect sum of the rings* $\{A_\alpha\} \cup \{B_\beta\}$.

Proof. For each α and each β, let

$$M_\alpha = \{a \in A: a_\alpha = 0\} \quad \text{and} \quad P_\beta = \{s \in S: s_\beta = 0\},$$

where a_α and s_β denote components in the given subdirect sums. Then $\bigcap_\alpha M_\alpha = 0$ and $\bigcap_\beta P_\beta = 0$. Also, A/M_α is isomorphic to A_α and S/P_β is isomorphic to B_β. Now there exist ideals P'_β of E containing A such that $P_\beta = P'_\beta/A$ and $\bigcap P'_\beta = A$. Let $M'_\alpha = \{x \in E: Ax \subseteq M_\alpha\}$. M'_α is clearly an ideal of E. Let b be in $A \cap M'_\alpha$. Then $Ab \subseteq M_\alpha$, which implies $A\langle b \rangle \subseteq M_\alpha$, where $\langle b \rangle$ denotes the right ideal of A generated by b. Therefore, $\langle b \rangle^2 \subseteq M_\alpha$. Now M_α is a semiprime ideal of A; therefore, $\langle b \rangle \subseteq M_\alpha$. Hence b is in M_α. Clearly, $M_\alpha \subseteq A \cap M'_\alpha$. Hence $M_\alpha = A \cap M'_\alpha$. Let $a + M_\alpha$ be the identity of A/M_α. If x is in E and y is in A, then

$$y(x - ax) = yx - yax \equiv yx - yx \pmod{M_\alpha},$$

since y is in A. Since the last quantity is zero, $y(x - ax)$ is in M_α. Hence, $x - ax$ is in M'_α for all x in E.

This implies that $E = M'_\alpha + A$; for if x is in E, then $x = (x - ax) + ax$. Therefore,

$$E/M'_\alpha = (M'_\alpha + A)/M'_\alpha \cong A/(M'_\alpha \cap A) = A/M_\alpha \cong A_\alpha.$$

Received May 16, 1969.
This work was done while the author was on an NSF Graduate Fellowship.

Also, $E/P'_\beta \cong (E/A)/(P'_\beta/A) = S/P_\beta \cong B$. Now,

$$\left(\bigcap_\alpha M'_\alpha\right) \cap \left(\bigcap_\beta P'_\beta\right) = \left(\bigcap_\alpha M'_\alpha\right) \cap A = \bigcap_\alpha (M'_\alpha \cap A) = \bigcap_\alpha M_\alpha = 0.$$

Hence E is a subdirect sum of $\{A_\alpha\} \cup \{B_\beta\}$.

The existence of at least a left unity seems to be necessary. Kohls attempted to prove the theorem for simple primitive rings by using the maximality of M_α to show that the ideal M'_α is maximal in E, so that $E = M'_\alpha + A$. His error lies in the assumption that if N is any proper ideal containing M'_α, then $N \cap A = M_\alpha$. Our example shows that this may be false.

COROLLARY. *If* A *is a semiprime ring with unity and* S *is any ring, then every extension* E *of* A *by* S *is isomorphic to the direct sum* $A \oplus S$.

Proof. The proof of our theorem shows that E can be embedded in $(E/(0)') \oplus (E/A)$ by the map $x \to (x + (0)', x + A)$. If a is the identity of A and x is in E, then $x = (x - ax) + ax$. Hence

$$x + (0)' = ((x - ax) + (0)') + (ax + (0)') = ax + (0)'.$$

If y is in A, then $(y + (0)', y + A) = (y + (0)', 0)$, which is in the image of E. Hence $(0, x + A)$ is in the image of E, since

$$(x + (0)', x + A) - (ax + (0)', 0) = (0, x + A).$$

Therefore the image of E is $(E/(0)') \oplus (E/A)$, and this is isomorphic to $A \oplus S$.

Our theorem is still strong enough to extablish Corollaries 1 and 2 in [1]. By the obvious change of hypothesis, we can obtain modified forms of Theorem 5 and Corollaries 3 and 4 [1, pp. 402-403].

REFERENCE

1. C. W. Kohls, *Properties inherited by ring extensions.* Michigan Math. J. 12 (1965), 399-404.

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