COMPACT, ACYCLIC SUBSETS OF THREE-MANIFOLDS

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1. INTRODUCTION

Let G be a nontrivial abelian group. If X is a compact absolute neighborhood retract (abbreviated: CANR), then X is said to be G-*acyclic* if it is connected and the homology groups $H_i(X; G)$ vanish for each $i > 0$. We shall be concerned with the cases $G = Z$ (the additive group of integers) and $G = Z_2$ (the integers modulo two). We present here some theorems that we believe will frequently be useful in proving that a Z_2-acyclic CANR X embedded in a 3-manifold M^3 is a compact absolute retract (CAR). This turns out to be the case, for example, if M^3 is Euclidean 3-space E^3, and a question of Borsuk [3, p. 216] is thus answered in the affirmative. In fact, it follows from Corollary 4.1 that a Z_2-acyclic CANR X in M^3 is a CAR provided $H_1(M^3; Z)$ is a free abelian group, and provided that every Z-acyclic finite polyhedron in M^3 is simply connected.

A G-acyclic CANR X in M^3 actually possesses a property that we call *strongly* G-*acyclic* (see Section 3), and many of our proofs use this alternate hypothesis. This permits applications to other problems. A *compact decomposition* of M^3 is a decomposition whose elements consist of the components of a compact set $S \subset M^3$, plus the individual points of $M^3 - S$. Such a decomposition is upper-semicontinuous (see [8]). A corollary of Theorem 5 is that if G is a compact decomposition of the 3-sphere S^3 and the decomposition space S^3/G is a 3-manifold, then each element of G is cellular. In fact, it follows from our results and from those of R. J. Bean in [2] that an equivalence between S^3 and S^3/G can be demonstrated by means of a pseudo-isotopy.

Some of our results are valid with either of the coefficient groups Z or Z_2. In this case, terms such as Z_*-*acyclic* or Z_*-*homology* will be used, with the understanding that the reader may interpret Z_* consistently as either Z or Z_2 in a given proof or discussion.

We adopt the convention that manifolds are connected. A *closed* manifold is compact and without boundary. We use the terms "surface" and "closed 2-manifold" interchangeably. "Mapping" means "continuous mapping"; S^n denotes the n-sphere. If $f: X \times [0, 1] \to Y$ is a mapping and $t \in [0, 1]$, we let $f_t: X \to Y$ denote the mapping defined by $f_t(x) = f(x, t)$, and we say that $f_t: X \to Y$ is a *homotopy*. A *loop in* Y is a mapping $f: S^1 \to Y$. If loops f_0 and f_1 in Y are homotopic as mappings of S^1 into Y, we call them *freely homotopic*, as opposed to "base-point preserving" homotopic.

Finally, the algebraic topology used has a strongly geometric orientation. A good reference is [11]. For information on CANR's, see [3] or [5].

2. SIMPLE MOVES IN THREE-MANIFOLDS

Throughout this section, M^3 will denote an orientable, nonclosed, piecewise-linear 3-manifold, and Z^3 will denote a compact polyhedron in Int M^3 such that

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each component of Z^3 is a 3-manifold with nonempty boundary. Lemmas A, B, C, and D given here will be cited later. Their proofs are either routine or well-known.

Suppose there is a polyhedral i-cell $B^i \subset \text{Int } M^3$ (i = 1, 2, or 3) such that $\partial B^i \subset \partial Z^3$ and $Z^3 \cap \text{Int } B^i = \emptyset$. Then, if $N(B^i)$ is a "nice" regular neighborhood of B^i in $M^3 - \text{Int } Z^3$, we say that $Z^3 \cup N(B^i)$ is obtained from Z^3 by *adding an i-handle in* M^3. In this case, we shall say that $Z^3 \cup N(B^i)$ is obtained from Z^3 by a *simple annexation in* M^3 *of type* i if

(a) i = 3, or

(b) i = 2 and ∂B^2 fails to bound a 2-cell in ∂Z^3, or

(c) i = 1 and ∂B^1 intersects at least one 3-cell component of Z^3 in exactly one point.

Suppose there is a polyhedral 2-cell $B^2 \subset Z^3$ such that $B^2 \cap \partial Z^3 = \partial B^2$ and ∂B^2 fails to bound a 2-cell in ∂Z^3. Then, if $N(B^2)$ is a "nice" regular neighborhood of B^2 in Z^3, the closure of $Z^3 - N(B^2)$ is said to be obtained from Z^3 by a *simple reduction.* A *simple move* in M^3 means either a simple annexation in M^3 or a simple reduction. We associate with Z^3 an integer $c(Z^3)$ defined by

$$c(Z^3) = \sum_{n=0}^{\infty} (n+1)^2 g(n),$$

where g(n) is the number of surfaces of genus n in ∂Z^3.

Suppose, for the rest of this section, that Z_0^3 is obtained from Z^3 by simple moves in M^3. Then the following propositions hold.

LEMMA A. $1 \leq c(Z_0^3) < c(Z^3)$.

LEMMA B. *Each loop in* Z_0^3 *is freely homotopic in* M^3 *to a loop in* Z^3.

It is clear from Lemma A that by simple moves in M^3 we can always obtain from Z^3 a Z_0^3 such that no further simple moves can be applied to Z_0^3 in M^3. Such a Z_0^3 is said to be *simple in* M^3.

If S is a polyhedral, two-sided (and hence orientable) surface in Int M^3, then S is called *incompressible in* M^3 provided either S fails to be a 2-sphere and for each polyhedral 2-cell $B^2 \subset \text{Int } M^3$ such that $B^2 \cap S = \partial B^2$, it follows that ∂B^2 bounds a 2-cell in S, or else S is a 2-sphere that bounds no 3-cell in M^3. It follows from [12] that a surface S other than a 2-sphere is incompressible in M^3 if and only if the inclusion of S into M^3 induces a monomorphism on fundamental groups.

LEMMA C. *If* Z^3 *is simple in* M^3, *then the inclusion of each component of* Z^3 *into* M^3 *induces a monomorphism on fundamental groups.*

LEMMA D. *If* Z^3 *is simple in* M^3 *and* Z^3 *is not a 3-cell, then each component of* ∂Z^3 *is incompressible in* M^3.

Retaining the notation at the beginning of this section, we shall now obtain the following result.

THEOREM 1. *Suppose* Z_0^3 *is obtained from* Z^3 *by simple moves in* M^3. *Let* $H_0^3 \subset \text{Int } M^3$ *be a compact polyhedral 3-manifold-with-boundary such that* $Z_0^3 \subset \text{Int } H_0^3$. *Then there exist a compact polyhedral 3-manifold* H^3 *with boundary* ($H^3 \subset \text{Int } M^3$) *and a piecewise-linear homeomorphism* h *of* M^3 *onto* M^3 *such that*

i) *we can obtain* H^3 *from* H_0^3 *by adding* 1-*handles in* M^3,

ii) h *is isotopic to the identity, through an isotopy that reduces to the identity in the complement of some compact subset of* Int M^3, *and*

iii) $Z^3 \subset \text{Int } h(H^3)$.

Proof. It suffices to consider the case where Z_0^3 is obtained from Z^3 by one simple move in M^3. In case the simple move is an annexation, there is nothing to prove. In case the simple move is a reduction, we complete the proof by the technique described at the end of Section 2 of [7].

3. NEIGHBORHOODS OF STRONGLY ACYCLIC SUBSETS

A compact set X in Int M^3 is *strongly* G-*acyclic* if it is connected and each open set $U \subset M^3$ containing X contains an open set V such that $X \subset V$ and such that for $i > 0$, the image of $H_i(V; G)$ in $H_i(U; G)$ is zero under the inclusion-induced homomorphism. If X is a CANR, then arbitrarily close pairs U, V can be chosen so that $X \subset V \subset U$ and so that X is a strong deformation retract of V in U (see [5, proof of Theorem 1.1, p. 111]). Hence a CANR X in M^3 is G-acyclic if and only if it is strongly G-acyclic. (For a G-acyclic X in M^3 to be strongly G-acyclic, it suffices that X should be homologically locally connected in dimensions 0 and 1.) It can be shown that if $X \subset M^3$ is compact and strongly G-acyclic, then all embeddings of X in a 3-manifold are strongly G-acyclic. Further, strong Z-acyclicity implies strong Z_2-acyclicity. The proof of the following lemma is straightforward.

LEMMA 1. *Let* X *be a compact subset of* Int M^3, *where* M^3 *is a compact, piecewise-linear* 3-*manifold with nonempty boundary, and each component of* X *is strongly* Z_*-*acyclic. Then there exists a compact, polyhedral, orientable* 3-*manifold* N^3 *with connected boundary such that*

$$X \subset \text{Int } N^3 \subset N^3 \subset \text{Int } M^3$$

and such that, for $i > 0$, *each* i-*cycle in* N^3 Z_*-*bounds in* M^3.

ADDENDUM 1. *Each polyhedral* 2-*sphere in* Int N^3 *bounds a* Z_*-*homology* 3-*cell in* N^3.

ADDENDUM 2. *If* X *is connected, locally connected, and strongly* Z_2-*acyclic and if* $H_1(M^3; Z)$ *is torsion-free, then* N^3 *can be chosen so that each loop in* N^3 Z-*bounds in* M^3, *and so that each polyhedral* 2-*sphere in* N^3 *bounds a* Z-*homology* 3-*cell in* N^3.

Proof. We choose N^3 according to Lemma 1; because X is connected and locally connected, we may take N^3 so close to X that each loop in N^3 is freely homotopic in M^3 to a loop in X. Since each loop in X Z_2-bounds in N^3, the Z-homology class of each loop in N^3 is divisible by an arbitrarily large power of 2 in $H_1(M^3; Z)$. Hence, this class is zero.

Addendum 2 implies that the entire conjugate class of elements in $\pi_1(M^3)$ determined by a loop in N^3 is contained in the commutator subgroup of $\pi_1(M^3)$. An extension of this line of reasoning yields, in the terminology of [6, p. 293] a further result:

ADDENDUM 3. *If* X *is connected, locally connected, and strongly* Z_2-*acyclic, and if* X *has arbitrarily close, compact polyhedral neighborhoods* W *in* M^3 *for* *which* $H_1(W; Z)$ *is torsion-free, then* N^3 *can be chosen so that each loop in* N^3 *belongs to every group of the derived series for* $\pi_1(M^3)$.

A *homotopy* (Z_**-homology*) *3-cell* is a compact, contractible (Z_*-acyclic) 3-manifold-with-boundary. A (Z_**-homology*) *cube-with-handles* is obtained by addition of orientable 1-handles to the boundary of a (Z_*-homology) 3-cell. Similarly, we define a *homotopy cube-with-handles.*

THEOREM 2. *Let* X *be a compact subset of* Int M^3, *where* M^3 *is a piecewise-linear* 3-*manifold and each component of* X *is strongly* Z_**-acyclic. Then* $X = \bigcap_{i=1}^{\infty} H_i$, *where each component of* H_i *is a polyhedral,* Z_**-homology cube-with-handles in* M^3 *and* $H_{i+1} \subset \text{Int } H_i$.

Proof. Because of Lemma 1, we may assume without loss of generality that ∂M^3 is nonempty and connected, and that M^3 is compact, orientable, and separated by each polyhedral surface in its interior. Further, by Addendum 1 to Lemma 1, we may suppose that each polyhedral 2-sphere in Int M^3 bounds a Z_*-homology 3-cell in M^3. It suffices to show some neighborhood in M^3 of X is a Z_*-homology cube-with-handles.

According to the Finiteness Theorem of W. Haken, [4, p. 48], there exists a positive integer H such that Int M^3 does not contain H or more disjoint, incompressible polyhedral surfaces no two of which are topologically parallel (see below for the definition).

To make our argument more concise, we adopt several conventions for the remainder of this proof. A cycle will *bound* if it is Z_*-homologous to zero. "Incompressible" will mean "incompressible with respect to M^3". The phrase "for each i" will mean "for i = 1, 2, ⋯, H," with possibly additional qualifications. Z_i will always denote a compact polyhedron in Int M^3 each of whose components is a 3-manifold.

We shall consider nested, ordered H-tuples

$$\Sigma = \{Z_1, Z_2, \cdots, Z_H\}$$

of such Z_i's, with the property that, for each i, each 1-cycle in Z_i bounds in Int Z_{i-1} (we put $Z_0 = M^3$). Observe that there is such a sequence,

$$\Sigma^0 = \{Z_1^0, Z_2^0, \cdots, Z_H^0\},$$

for which in addition $X \subset \text{Int } Z_H^0$. This follows from Lemma 1. We extend the definition of c in Section 2 by putting

$$c(\Sigma) = \sum_{i=1}^{H} c(Z_i).$$

We shall also write $\partial\Sigma = \bigcup_{i=1}^{H} \partial Z_i$.

Suppose there is a polyhedral 2-cell $B^2 \subset \text{Int } M^3$ such that

$$B^2 \cap \partial\Sigma = \partial B^2 \subset \partial Z_i \qquad (i > 0)$$

and such that ∂B^2 bounds no 2-cell in ∂Z_i. Then we can apply a simple annexation of type 2 or a simple reduction to Z_i in Z_{i-1} without disturbing any Z_j ($j \neq i$). If this simple move transforms Z_i into Z_i', we say that

$$\Sigma' = \{Z_1, \cdots, Z_{i-1}, Z_i', Z_{i+1}, \cdots, Z_H\}$$

is obtained by *simplifying* Σ.

Note that Σ' possesses the required properties (see Lemma B for the case of a simple annexation). Since by Lemma A, $1 \leq c(\Sigma') < c(\Sigma)$, there exists a

$$\Sigma^* = \{Z_1^*, Z_2^*, \cdots, Z_H^*\},$$

obtained from Σ^0 by a finite number of simplifying operations, such that Σ^* cannot be simplified further. It follows that no 2-cell B^2 of the type described in the previous paragraph can exist for Σ^*. Hence, by a routine "trading disks" argument, each component of $\partial\Sigma^*$ that is not a 2-sphere is incompressible.

We claim that some ∂Z_i^* consists entirely of 2-spheres. Suppose not. Then our choice of H implies that for $i < k$ there exist *topologically parallel* surfaces $S_i \subset \partial Z_i^*$ and $S_k \subset \partial Z_k^*$ in M^3 that are not 2-spheres. (That is, there exists a compact polyhedron $A \subset \text{Int } M^3$ such that some piecewise-linear homeomorphism of $S_i \times [0, 1]$ onto A carries $S_i \times \{0\}$ onto S_i and $S_i \times \{1\}$ onto S_k.) We may assume that no surface in $(\text{Int } A) \cap \partial\Sigma^*$ is parallel to S_i in A. According to [13, Corollary 3.2], each incompressible surface in Int A is parallel to S_i in A. Hence $(\text{Int } A) \cap \partial\Sigma^*$ consists entirely of 2-spheres. Since A is irreducible, it follows that A minus the interiors of a finite disjoint collection of polyhedral 3-cells is contained in Z_i^*. But each loop in Z_k^* bounds in Z_i^*, so that each loop in S_i bounds in Z_i^*. This yields a contradiction, namely that S_i is a 2-sphere (see [10, Lemma 1]: the proof given there for the integral case extends to the Z_2-case). Thus, ∂Z_m^* (say) contains only 2-spheres.

Since each component of ∂Z_m^* bounds a Z_*-homology 3-cell in M^3, there exists a Z_*-homology 3-cell H_0^3 such that

$$Z_m^* \subset \text{Int } H_0^3 \subset H_0^3 \subset \text{Int } M^3 .$$

Applying Theorem 1, we obtain a Z_*-homology cube-with-handles H^3 and a piecewise-linear homeomorphism $h\colon M^3 \to M^3$ such that $Z_m^0 \subset \text{Int } h(H^3)$. Since $X \subset \text{Int } Z_m^0$, the proof is complete.

Following S. Armentrout [1], we say that $X \subset M$ has *property* n-UV *in* M if each open set $U \subset M$ containing X contains an open set $V \subset M$ such that $X \subset V$ and such that each singular n-sphere in V is contractible in U. We say that X has *property* UV^n provided it has property i-UV for each $i \leq n$. *Property* UV^∞ means that arbitrarily close pairs U, V can be chosen as above so that V is contractible to a point in U.

THEOREM 3. *Let* X *be a compact subset of* Int M^3, *where* M^3 *is a piecewise-linear* 3-*manifold and each component of* X *is strongly* Z_2-*acyclic and has property* 2-UV *in* M^3. *Then* $X = \bigcap_{i=1}^{\infty} H_i$, *where each component of* H_i *is a polyhedral homotopy cube-with-handles in* M^3 *and* $H_{i+1} \subset \text{Int } H_i$.

Proof. By Theorem 2, there exist polyhedra $H_i \subset M^3$ $(i \geq 0)$ such that each component of H_i is a Z_2-homology cube-with-handles, such that $X = \bigcap_{i=0}^{\infty} H_i$, and such that $H_{i+1} \subset \text{Int } H_i$. Using property 2-UV, we may choose the polyhedra so that each singular 2-sphere in H_i $(i \geq 1)$ is contractible in H_{i-1}. Hence, for $i \geq 1$, each (nonsingular) polyhedral 2-sphere in H_i bounds a homotopy 3-cell in H_{i-1} (see the first paragraph of the proof of Theorem 2 of [9]). Since the boundary of each component of H_i is connected, this homotopy 3-cell lies in H_i. Thus, for $i \geq 1$, each component of H_i is a homotopy cube-with-handles.

The next corollary follows from Theorem 3, Addendum 3 to Lemma 1, and the fact that for a free group G (we take $G = \pi_1(H_1)$ below) with derived groups $G^{(1)}, G^{(2)}, \cdots$, we have $\bigcap_{n=1}^{\infty} G^{(n)} = \{1\}$ (see [6, pp. 311 and 312]).

COROLLARY 3.1. *Let* X *be a compact, locally connected subset of* Int M^3, *where* M^3 *is a piecewise-linear* 3-*manifold. Suppose also that* X *is strongly* Z_2-*acyclic and has property* 2-UV *in* M^3. *Then* $X = \bigcap_{i=1}^{\infty} H_i$, *where* H_i *is a homotopy cube-with-handles in* M^3, $H_{i+1} \subset \text{Int } H_i$, *and* H_{i+1} *is contractible to a point in* H_i. *In particular,* X *has property* UV^{∞}.

COROLLARY 3.2. *Let* X *be a compact subset of* Int M^3, *where* M^3 *is a piecewise-linear* 3-*manifold. Suppose that* X *is strongly* Z_2-*acyclic, that* X *has property* 2-UV *in* M^3, *and that each open set* U *in* M^3 *containing* X, *contains an open set* V *such that* $X \subset V$ *and each loop in* V - X *is contractible in* U - X. *Then* X *has arbitrarily close compact, polyhedral neighborhoods* F^3 *in* M^3 *such that* F^3 *is a homotopy* 3-*cell and* F^3 - X *is topologically* $S^2 \times [0, 1)$.

Proof. This follows from Theorem 3 and the proof of [8, Theorem 1] (see the last three paragraphs of the proof cited).

4. SOME APPLICATIONS

We first give a characterization of CAR's in 3-manifolds.

THEOREM 4. *Let* X *be a compact,* Z_2-*acyclic, locally contractible subset of* M^3, *where* M^3 *is a piecewise-linear* 3-*manifold. Then* X *is an absolute retract if and only if* $\pi_2(X) = 0$, *or, equivalently, if* X *has property* 2-UV *in* M^3.

Proof. The "only-if" part is clear. Let us consider the converse. We may assume $X \subset \text{Int } M^3$. Since X is compact, finite-dimensional, and locally contractible, it is a (connected) CANR (see [3, Corollary 10.4, p. 122]). Let U be a neighborhood of X in M^3 that retracts onto X. Since X has property 2-UV and is strongly Z_2-acyclic, there exists (by Corollary 3.1) a neighborhood V of X such that $X \subset V \subset U$ and V is contractible to a point in U. It follows that X is contractible in itself and hence (see [3, Theorem 9.1, p. 96]) is a CAR.

Using the same method of proof, together with Theorem 2 and Corollary 3.1, we have the following result.

COROLLARY 4.1. *Let* X *be a compact,* Z_*-*acyclic, locally contractible subset of* M^3, *where* M^3 *is a piecewise-linear* 3-*manifold containing no* Z_*-*homology* 3-*cells that fail to be simply connected. Then* X *is an absolute retract.*

We indicate next how to use our results to study compact decompositions of 3-manifolds that yield 3-manifolds.

THEOREM 5. *Let* M^3 *and* N^3 *be closed, piecewise-linear* 3-*manifolds, and let* f *be a monotone, continuous mapping of* M^3 *onto* N^3. *Put*

$$S_f = \{x \in M^3 : f^{-1}f(x) \text{ is nondegenerate}\}.$$

Suppose that for each component C *of* S_f *there exists a homotopy* 3-*cell in* M^3 *containing* C *in its interior, and that the closure* X *of* $f(S_f)$ *is* 0-*dimensional. Then each component of* S_f *has property* UV^{∞}.

Proof. Note that the components of S_f are exactly the nondegenerate point-inverses of f. (The requirement that f be monotone is actually redundant in the presence of the other hypotheses, but here we do not prove this.) The argument to follow has two main steps. Since these steps are essentially the same, we treat them together.

Suppose that we have X expressed as $\bigcap_{i=1}^{\infty} H_i$, where each component of H_i is a tamely embedded homotopy cube-with-handles in N^3, and that $H_{i+1} \subset \operatorname{Int} H_i$, H_{i+1} is contractible in H_i, and each component of $f^{-1}(H_1)$ is interior to a homotopy 3-cell in M^3. Such sequences exist, by Theorem 2 (in fact, each component of H_i can be chosen to be a polyhedral cube-with-handles). Let K_{i+1} and K_i be components of H_{i+1} and H_i, respectively, such that $K_{i+1} \subset \operatorname{Int} K_i$, and consider the following consistent diagram of compact, connected sets:

$$\begin{array}{ccc} f^{-1}(K_{i+1}) & \xrightarrow{\ j\ } & f^{-1}(K_i) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ K_{i+1} & \xrightarrow{\ k\ } & K_i \end{array},$$

where j and k are inclusions and the vertical arrows are restrictions of f.

Note that we obtain corresponding induced algebraic diagrams of (a) first Z-homology groups and (b) fundamental groups, and that in each case k_* is trivial. Also, in each case it is true that the vertical arrows represent epimorphisms, since, for example, the restriction of f to $\partial(f^{-1}(K_i))$ is a homeomorphism onto ∂K_i. In the diagram of case (a), the vertical arrows also represent isomorphisms, since each of $f^{-1}(K_{i+1})$ and $f^{-1}(K_i)$ is interior to a homotopy 3-cell in M^3 and hence has the same first Z-homology as K_{i+1} and K_i, respectively. Hence, in case (a), $j_* = 0$, and it follows that each component of $f^{-1}(X)$ is strongly Z-acyclic.

Hence, by Theorem 2, we can write $f^{-1}(X) = \bigcap_{i=1}^{\infty} J_i$, where J_i is a compact polyhedron in M^3 each of whose components is a polyhedral homotopy cube-with-handles, and the sequence $H_i = f(J_i)$ satisfies the conditions of the second paragraph of this proof. Now consider the diagram in case (b) for this choice of H_i and with K_{i+1} and K_i as before. The vertical arrows represent epimorphisms of free groups of the same finite rank, and hence they represent isomorphisms, by [6, Theorem 2.13, p. 109]. Again we find that j_* is trivial, and hence each component of $f^{-1}(X)$ has property UV^{∞}.

From [8, Theorem 2], we obtain two further results:

ADDENDUM 1. *Each component* C *of* S_f *has arbitrarily close, compact polyhedral neighborhoods* F^3 *in* M^3 *such that* F^3 *is a homotopy* 3-*cell and* $F^3 - C$ *is topologically* $S^2 \times [0, 1)$. *Hence, if a component* C *of* S_f *lies in the interior of a* 3-*cell in* M^3, *then* C *is cellular in* M^3.

ADDENDUM 2. *If for each component* C *of* S_f *there exists a* 3-*cell in* M^3 *containing* C *in its interior, then some pseudo-isotopy of* M^3 *onto* M^3 *shrinks the collection of components of* $\overline{S}_f$ *to points. In particular,* M^3 *and* N^3 *are homeomorphic.*

Proof. Theorem 2 yields the relation $\overline{S}_f = \bigcap_{i=1}^{\infty} H_i$, where each component of H_i is a polyhedral cube-with-handles in M^3 and $H_{i+1} \subset \operatorname{Int} H_i$. We can now construct the pseudo-isotopy as in [2].

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