LIGHT OPEN MAPPINGS ON A TORUS WITH A DISK REMOVED

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1. INTRODUCTION

Suppose δ is a continuous mapping of a Jordan curve J into Euclidean 2-space R^2. We shall consider J as the boundary of T, a 2-dimensional torus with a disk removed, and also as the boundary of a 2-cell D, and we shall study the relation between the case where δ has a light, open, continuous extension to T and the case where δ has a similar extension to D.

Generally, definitions and notation not given in the paper will be as in [3]. All light open mappings will be assumed to be sense-preserving, unless it is otherwise specified.

2. MAIN RESULTS

Definition 1. Suppose J is a Jordan curve on a 2-dimensional torus that bounds a disk; let T be the other component of the complement of J. Suppose δ is a continuous mapping of J into R^2. We say that δ is a t-*boundary* if there exists a properly interior mapping $f: \overline{T} \to R^2$ such that $f \mid J = \delta$. (We use the term *properly interior* in the sense of [7], not [3].)

We shall consider J as embedded in R^2 and oriented as in Figure 1.

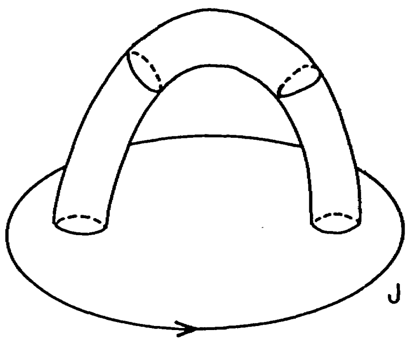

Figure 1.

Definition 2. Suppose $I = [a, b]$ is a closed interval of real numbers, A is some closed arc, and $\delta: A \to R^2$. We extend the definition of normality as in [6, p. 1084] and say that δ is *topologically normal* (briefly, t-*normal*) if there exist homeomorphisms $h: I \to A$ and $k: R^2 \to R^2$ such that $k \circ \delta \circ h$ is normal in the sense of [6]. Also, if M is an oriented Jordan curve and δ maps M into R^2, then δ is t-*normal* if there exists a mapping $\psi: I \to M$ such that $\psi(a) = \psi(b)$, ψ is one-to-one on (a, b), $\psi(x) \neq \psi(a)$ for $x \in (a, b)$, and $\delta \circ \psi$ is t-normal as defined in the previous sentence. Let δ map the closed arc A_1 into R^2; let η map the closed arc A_2 into R^2. We say that δ and η *intersect* t-*normally* if there exist homeomorphisms $h_1: I \to A_1$, $h_2: I \to A_2$, and $k: R^2 \to R^2$ such that $k \circ \delta \circ h_1$ and $k \circ \eta \circ h_2$ intersect normally in the sense of [3, p. 50].

LEMMA 1. *Let* U *be an open connected subset of a metrizable 2-dimensional manifold, and suppose* $f: U \to R^2$ *is light and open. For any two points* p *and* q *in* U, *there exists an arc* A *in* U *with end points* p *and* q *such that* $f \mid A$ *is* t-*normal. Also, if* A_1 *is any arc and* $g: A_1 \to R^2$ *is* t-*normal, then* A *can be chosen so that* $f \mid A$ *and* $g \mid A_1$ *intersect* t-*normally. Finally, if* U *is bounded by a finite number of Jordan curves and* f *is a local homeomorphism at points of* $\overline{U} - U$, *then the*

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conclusion still holds even if one or both points p *and* q *are in* $\overline{U} - U$. (*We can choose* A *so that* $A \cap (\overline{U} - U) = \{p, q\}$.)

Proof. First suppose p and q are in U. By Stoïlow's theorem [10, Theorem 1.1, p. 103], there exist a Riemann surface X, a homeomorphism $\psi: X \to U$, and a complex analytic function $\alpha: X \to R^2$ such that $\alpha = f \circ \psi$. By the definition of normality, it suffices to find an arc B of X joining $\psi^{-1}(p)$ and $\psi^{-1}(q)$ such that $\alpha \mid B$ is t-normal and intersects $g \mid A_1$ t-normally. Then $\psi^{-1}(B)$ will be the desired arc A.

There exists a chain of open sets U_i $(1 \leq i \leq n)$ from $\psi^{-1}(p)$ to $\psi^{-1}(q)$ such that $\alpha \mid U_i$ is an analytic homeomorphism $(1 \leq i \leq n)$. Pick a sequence of points x_i $(0 \leq i \leq n)$, where $x_0 = \psi^{-1}(p)$ and $x_n = \psi^{-1}(q)$, and where $x_i \in U_i \cap U_{i+1}$ for $1 \leq i \leq n - 1$. If we now prove the theorem for $p = x_i$, $q = x_{i+1}$, and $U = U_i$, then we will be done.

Now $\alpha(U_i)$ is an open subset of R^2, and it contains the points $\alpha(x_i)$ and $\alpha(x_{i+1})$. Certainly we can find an analytic arc C in $\alpha(U_i)$ that joins $\alpha(x_i)$ and $\alpha(x_{i+1})$ and intersects $g(A_i)$ normally. Since α is an analytic homeomorphism on U_i, the set $\alpha^{-1}(C) \cap U_i$ is an analytic arc in U_i with the desired properties.

If p, say, is in $\overline{U} - U$, use Church's extension of Stoïlow's theorem [1, p. 86] locally at p to get away from $\overline{U} - U$ and into U. The proof then proceeds as before.

LEMMA 2. *Let* J *and* δ *satisfy the conditions in Definition* 1. *Suppose* K *is a Jordan curve in* Ins J. *Let* K_1, K_2, K_3, K_4 *be four closed arcs of* K, *intersecting only at end points, numbered consecutively in the counterclockwise direction, oriented in the counterclockwise direction, and such that* $K = K_1 \cup K_2 \cup K_3 \cup K_4$. *Suppose* $\eta: K \to R^2$ *is continuous and* $\eta \mid K_i = -\eta \mid K_{i+2}$ $(i = 1, 2)$. *Then* δ *is a* t-*boundary if and only if there exists an* η *as above such that* (δ, η) *is an* a-*boundary. Moreover,* η *can be selected to be topologically normal.*

Proof. Suppose δ has a light open extension f to $\overline{T}$. Lemma 1 implies the existence of Jordan curves M, a meridian of T, and L, a longitude of T, such that $f \mid M$ and $f \mid L$ are t-normal and $f \mid M$ and $f \mid L$ intersect t-normally. Cutting T along $M \cup L$, we get an annulus, and then we obtain η by restricting f to the cut.

Suppose (δ, η) has a light, open extension f to the annulus bounded by J and K. Identifying the arcs K_1 and K_3 and K_2 and K_4, we obtain a surface T consisting of a torus from which a disk has been removed. Because of the hypothesis on η, f induces a natural map f^* on $\overline{T}$. That f^* is light and open on the identified points follows immediately from [9, Theorem 9, p. 336].

THEOREM 1. *If* δ *is a* t-*boundary, then* $w(\delta, p) \geq 0$ *for every* $p \in R^2 - [\delta]$ *and there is some* $p \in R^2$ *such that* $w(\delta, p) \geq 2$ ($w(\delta, p)$ denotes the winding number of δ at p).

Proof. There exists an η as in Lemma 2 such that (δ, η) is an a-boundary. By [4, Theorem 20.2, p. 72], $w(\delta, p) - w(\eta, p)$ is equal to the number of pre-images of p under any properly interior extension f of (δ, η); thus, $w(\delta, p) - w(\eta, p) \geq 0$. However, $\eta \mid K_i = -\eta \mid K_{i+2}$ $(i = 1, 2)$ implies that $w(\eta, p) = 0$; therefore $w(\delta, p) \geq 0$.

Also, some point p must have two or more pre-image points under f. Otherwise, the torus with a disk removed would be homeomorphic to a subset of the plane, which is impossible. Thus, for that p, $w(\delta, p) \geq 2$.

Not all representations of nonnegative circulation are t-boundaries, as we shall see in Section 3.

THEOREM 2. *Suppose* K *is a Jordan curve in* R^2 *and* δ *maps* K *into* R^2. *If* δ *is a topologically normal interior boundary, then there exist a homeomorphism* $h: K \to S^1$, *a homeomorphism* $k: R^2 \to R^2$, *and a complex polynomial* P *such that* $\delta = k \circ P \circ h$.

Proof. Since δ is topologically normal, there exist a homeomorphism $h_1: S^1 \to K$ and a homeomorphism k_1 on R^2 such that $\delta_1 = k_1 \circ \delta \circ h_1$ is normal. Since δ is an interior boundary, there exists a properly interior mapping $f: K \cup \text{Ins } K \to R^2$ such that $f \mid K = \delta$. By [5, Lemma 5.2, p. 193], there exists a local homeomorphism g, defined on $\{z \mid 1 \le |z| \le r\}$ for some $r > 1$, such that $g \mid S^1 = \delta_1$. Define f^* on $S = \{z \mid 0 \le |z| \le r\}$ by $f^* = k_1 \circ f \circ h_1$ for $|z| \le 1$ and $f^* = g$ for $1 \le |z| < r$; then f^* is light and open [9, Theorem 9, p. 336].

There exists a homeomorphism h_2 of S onto itself such that $f^* \circ h_2 = F$ is analytic [10, p. 103]. There exists a sequence $\{Q_n\}$ of polynomials that approaches F' uniformly on compact subsets. Each Q_n has an appropriate antiderivative P_n such that $\{P_n\}$ approaches F uniformly on compact subsets. Hence, $\{P_n\}$ tends to F in the C^1-norm on the differentiable Jordan curve $h_2(S^1)$. By [6, Lemma 1, p. 1084], there exists an integer m so large that $P_m \mid h_2(S^1)$ and $F \mid h_2(S^1)$ have the same intersection sequence. Thus, there are homeomorphisms h_3 and k_3 of R^2 onto itself such that $k_3 \circ P_m \circ h_3 = F$ on $h_2(S^1)$ [7, Theorem 3, p. 49]. If we set $h = h_3 \circ h_2^{-1} \circ h_1^{-1}$ and $k = k_1^{-1} \circ k_3$, then $\delta = k \circ P_m \circ h$, and the theorem is proved.

THEOREM 3. *If* $\delta: J \to R^2$ *is a* t-*boundary, then there exists a properly interior mapping* $f: J \cup \text{Ins } J \to S^2$ *such that* $f \mid J = \delta$ *and* $f^{-1}(\infty)$ *is empty or contains one element.*

Proof. By Lemma 2, there exists a t-normal η such that (δ, η) is an a-boundary. Let J, K, and K_i $(1 \le i \le 4)$ be as in Lemma 2. By the definition of topological normality, there exist a homeomorphism $h: S^1 \to K$ and a homeomorphism $k: R^2 \to R^2$ such that $k \circ \eta \circ h$ is normal. There exist an open annulus U in R^2 with $S^1 \subset U$ and a local homeomorphism $g: \overline{U} \to R^2$ such that $g \mid S^1 = k \circ \eta \circ h$. By the Schoenflies theorem, we can extend h to R^2; let $V = h(U) \cap \text{Ins } K$. Applying Lemma 1 to $g \circ h^{-1}$ on $\overline{V}$, we obtain an arc B in $\overline{V}$ whose end points p and q are the first end point of K_1 and the last end point of K_2 (first and last refer to orientation of K). Also, B intersects $\overline{V} - V$ only in p and q. Finally, $g \circ h^{-1}$ is topologically normal on B, and $g \circ h^{-1} \mid B$ intersects $\eta \mid (K_1 \cup K_2)$ normally. Let M denote the Jordan curve $K_1 \cup K_2 \cup B$, and define ψ on M by $\psi \mid B = g \circ h^{-1}$ and $\psi \mid (K_1 \cup K_2) = \eta$. Then ψ is t-normal on M.

By Theorem 2, there exist a homeomorphism $h_1: M \to S^1$, a homeomorphism $k_1: R^2 \to R^2$, and a complex polynomial P such that $\psi = k_1 \circ P \circ h_1$. If P is of degree n, there exists a Jordan curve L such that $S^1 \subset \text{Ins } L$ and P is topologically equivalent to the power mapping z^n on L. Thus, $(z^n, P \mid S^1)$ is an a-boundary, which implies that (z^n, ψ) is an a-boundary. By [3, Lemma 5.3, p. 54], $(-\psi, z^{-n})$ is an interior boundary. This implies that $-\psi$ has a properly interior extension $v: M \cup \text{Ins } M \to S^2$ such that $v^{-1}(\infty)$ contains one element. Let $N = B \cup K_3 \cup K_4$, and give N the positive orientation in the plane. Define $\psi_1: N \to R^2$ by

$$\psi_1 \mid B = g \circ h^{-1} \quad \text{and} \quad \psi_1 \mid (K_3 \cup K_4) = \eta .$$

Since $\eta \mid K_i = -\eta \mid K_{i+2}$ $(i = 1, 2)$, the mapping ψ_1 is topologically equivalent to $-\psi$. Thus, ψ_1 has a properly interior extension $v_1: N \cup \text{Ins } N \to R^2$ such that $v_1^{-1}(\infty)$ contains one element.

Suppose f^* is a properly interior extension of (δ, η) to the annulus A bounded by J and K. Define a mapping f on $J \cup \text{Ins } J$ by

$$f \,|\, A = f^*, \qquad f \,|\, (M \cup \text{Ins } M) = g \circ h^{-1}, \qquad f \,|\, (N \cup \text{Ins } N) = v_1.$$

By [9, Theorem 9, p. 336], f is light and open and constitutes the desired map.

COROLLARY. *Suppose* $f: \overline{T} \rightarrow R^2$ *is a properly interior mapping and that* f *is a local homeomorphism at each point of* $J = \overline{T} - T$. *Then there exists a homeomorphism* $h: J \cup \text{Ins } J \rightarrow R^2$ *such that* $f \circ h^{-1}$ *is analytic in* Ins h(J) *or meromorphic in* Ins h(J) *with exactly one pole.*

This corollary follows immediately from Theorem 3 and the theorem and remark in [1, p. 86 and p. 88]. The next theorem is a partial converse to Theorem 3.

THEOREM 4. *Suppose* $\delta: J \rightarrow R^2$ *is an interior boundary and* $w(\delta, p) \geq 2$ *for some* $p \in R^2 - [\delta]$. *Then* δ *is a* t-*boundary.*

Proof. Let f be a properly interior extension of δ to D = Ins J. Since the branch points of f are isolated, we may assume p is not the image of a branch point of f. There are two points x_1 and x_2 in D such that $f(x_1) = f(x_2) = p$ [4, Theorem 20.2, p. 72]. Since x_1 and x_2 are not branch points, there exist Jordan curves K and L in D such that $(K \cup \text{Ins } K) \cap (L \cup \text{Ins } L) = \phi$, $x_1 \in \text{Ins } K$, $x_2 \in \text{Ins } L$, $f \,|\, K$ and $f \,|\, L$ describe positively oriented Jordan curves, and $f(L) \subset \text{Ins } f(K)$. Without loss of generality, we can take

$$K = \{z \,|\, |z - a| = r\} \qquad \text{and} \qquad L = \{z \,|\, |z - b| = s\},$$

for some appropriate complex numbers a and b and positive real numbers r and s.

It follows from [3, Theorem 3, p. 55] that there exists a properly interior mapping g, defined on the annulus $X = \{z \,|\, 1 \leq |z| \leq 2\}$, such that

$$g(2e^{i\theta}) = f(a + re^{i\theta}) \qquad \text{and} \qquad g(e^{i\theta}) = f(b + se^{-i\theta}).$$

Let $Y = \overline{D} - \text{Ins } K - \text{Ins } L$. Define h: Boundary $X \rightarrow Y$ by

$$h(2e^{i\theta}) = a + re^{i\theta} \qquad \text{and} \qquad h(e^{i\theta}) = b + se^{-i\theta}.$$

Attach X to Y by h, forming $Z = X \cup_h Y$ (see [2, pp. 127-129]); Z is a torus with an open disk removed. Since $g(x) = f(h(x))$ for all $x \in$ Boundary X, the mappings g and f define a continuous function on Z, and this function is properly interior [9, Theorem 9, p. 336] and extends δ.

COROLLARY. *If* δ *is a normal interior boundary and* δ *does not represent a Jordan curve, then* δ *is a* t-*boundary.*

Proof. In view of the theorem, it is only necessary to show that $w(\delta, p) \geq 2$ for some point p. If δ does not represent a Jordan curve, then δ has a vertex v. Let p, q, r be as in Figure 2. Now

$$w(\delta, r) = w(\delta, q) - 1 \qquad \text{and}$$

$$w(\delta, p) = w(\delta, q) + 1 = w(\delta, r) + 2$$

[6, Lemma 2, p. 1085]. Since δ has nonnegative circulation, $w(\delta, p) \geq 2$.

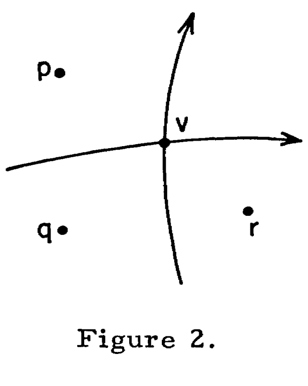

Figure 2.

3. EXAMPLES

In this paragraph we produce examples to show that the converses of Theorems 3 and 4 are false.

Example 1. Let δ represent the curve in Figure 3. Figure 4 shows that δ has a light open extension g mapping into S^2 such that $g^{-1}(\infty)$ contains one point.

Note that δ is of nonnegative circulation and that there exist points p such that $w(\delta, p) \geq 2$ (the tangential winding number of δ is negative; but the example can be modified to make this positive); however, δ is not a t-boundary.

Suppose, if possible, that δ is a t-boundary. Let f: $\overline{T} \rightarrow R^2$ be a properly interior extension of δ: $J \rightarrow R^2$. If such an f exists, we can also find one that is a local homeomorphism on a neighborhood of J in $\overline{T}$; we therefore assume that f has this property. Let δ_0, δ_1, δ_2 be as in Figure 3; suppose

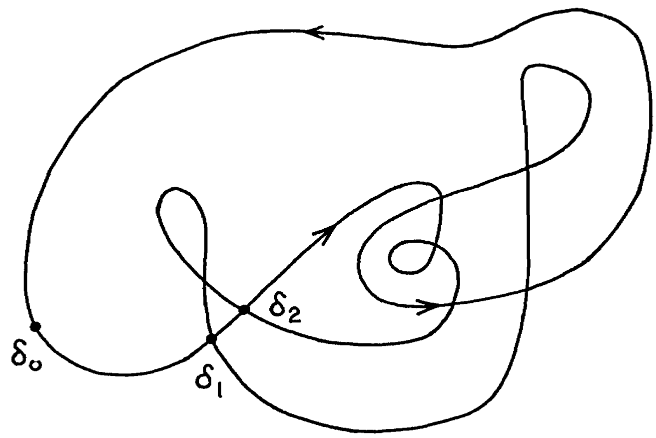

Figure 3.

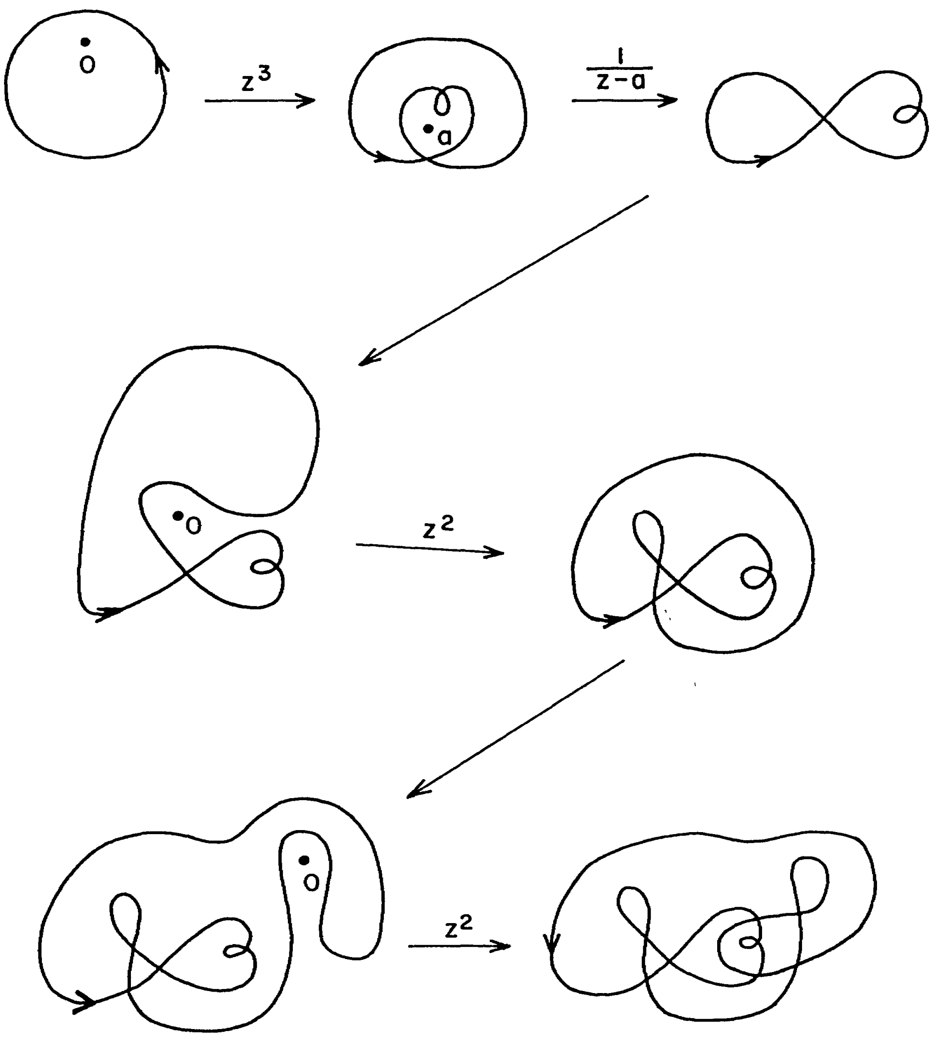

Figure 4.

$f^{-1}(\delta_i) = \{d_i, d_i^*\}$ $(i = 1, 2)$ and $f^{-1}(\delta_0) = \{d_0\}$. We encounter these points in the order $d_0, d_1, d_2, d_2^*, d_1^*$ as we traverse J in the positive orientation. There exists an arc γ in $\overline{T}$ that intersects J only in its end points, such that $f(\gamma)$ is the arc of $[\delta]$ from δ_1 to δ_2, and such that one end point of γ is d_2^* [3, Theorem 1, p. 49]. Since f is a local homeomorphism at each point of J, the other end point of γ must be d_1^*. Now $\overline{T} - J - \gamma = D_1 \cup D_2$, where D_1 and D_2 are open and either

(1) $D_1 = \phi$ and D_2 is an open annulus or

(2) D_1 is a disk.

In case (1), $(\delta^{**}, -\delta^*)$ would be an a-boundary (δ^* and δ^{**} are the Titus cuts of δ [7]; see Figure 5). But (δ^*, δ^{**}) is not an a-boundary, by (1) of Theorem 2 [3, p. 50]. In case (2), either δ^* or δ^{**} is an interior boundary. However, δ^* is not an interior boundary, because it has points of negative circulation; δ^{**} is not an interior boundary by [7, Theorem 8, p. 56].

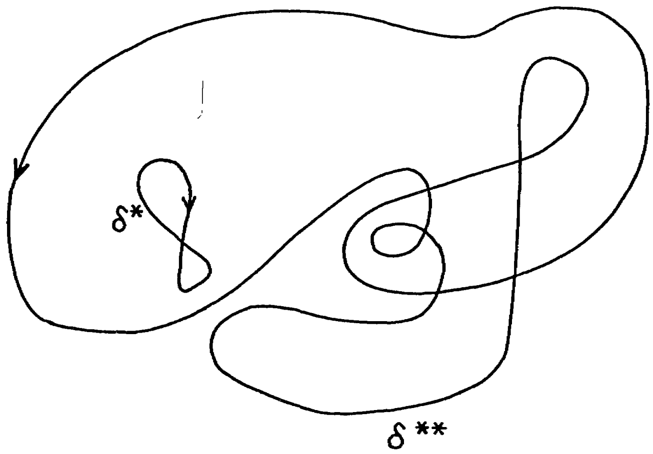

Figure 5.

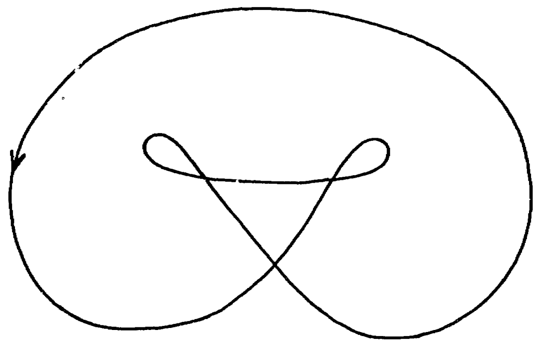

Figure 6.

Example 2. The curve of Figure 6 is not an interior boundary [8, p. 203]. It is a t-boundary; this can be seen as follows. The pair of curves in Figure 7 is an a-boundary [3], and therefore the pair of curves (δ, η) in Figure 8 is an a-boundary. Note that the arc γ is traced by both curves. Suppose f is a properly interior extension of (δ, η) to an annulus A in the plane bounded by Jordan curves J and K. There are arcs $B_1 \subset J$ and $B_2 \subset K$ such that $f(B_1) = f(B_2) = \gamma$. The decomposition space M of A whose nondegenerate elements have the form $f^{-1}(x) \cap J \cap K$ for all $x \in \gamma$ is a torus with an open disk removed. If ψ is the natural map of A onto M,

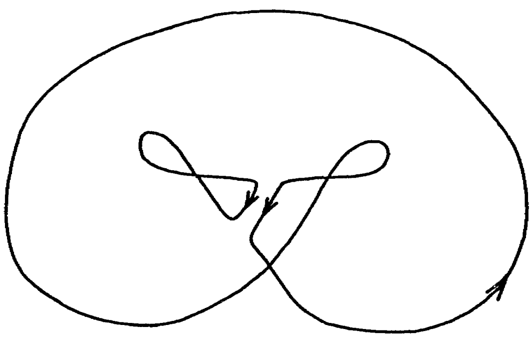

Figure 7.

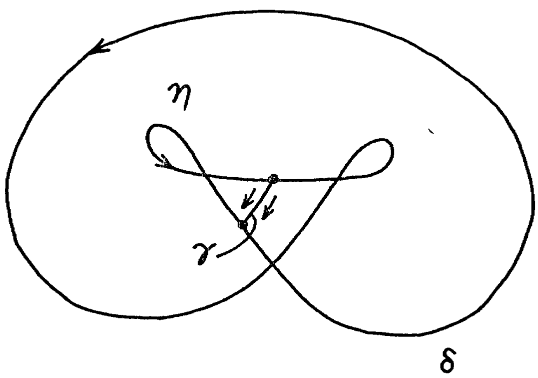

Figure 8.

then $f \circ \psi^{-1}$ is a properly interior extension of M such that on the boundary of M the curve of Figure 6 is described.

4. NORMAL t-BOUNDARIES

The following theorem, used in conjunction with [3] and [7], yields a finite algorithm for determining whether a prescribed normal representation is a t-boundary. We do not prove the theorem, because all the necessary techniques are in [3].

THEOREM 5. *Suppose* δ *is a normal representation and* δ *is a* t-*boundary.*

(1) *If* δ *has a cut of Type II, then either*

(a) δ^* *is a* t-*boundary and* δ^{**} *is an interior boundary,*

(b) δ^{**} *is a* t-*boundary and* δ^* *is an interior boundary, or*

(c) $(\delta^*, -\delta^{**})$ *is an* a-*boundary.*

(2) *If* δ *has a cut of Type I at* δ_k, *then either*

(a) δ^{**} *is a* t-*boundary,*

(b) *for some integer* n, $(\psi, -\eta)$ *is an* a-*boundary, where* ψ *is topologically equivalent to the power mapping* z^n *and*

$$\eta = \delta(0)\,\delta(d_k)\,(\delta) + \sum_{i=1}^{n} \delta(d_k)\,\delta(d_k^*)\,(-\delta) + \delta(d_k^*)\,\delta(2\pi)\,(\delta)\,.$$

Conversely, if any of the conditions (1a), (1b), (1c), (2a), or (2b) *holds, then* δ *is a* t-*boundary.*

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