A POWER-BOUNDED OPERATOR THAT IS NOT POLYNOMIALLY BOUNDED

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Foguel has constructed an operator with uniformly bounded powers that is not similar to a contraction [1]. This counterexample answers a question asked by B. Sz.-Nagy [6]. Halmos has given a less computational version of Foguel's arguments in [2]. The purpose of this note is to reexamine Foguel's operator and to show that it is not polynomially bounded; from this it follows that Foguel's operator is not a counterexample to the conjecture that each polynomially bounded operator is similar to a contraction.

An operator T on a Hilbert space is said to be *polynomially bounded* if there exists a constant K such that

$$(*) \qquad \|\mathscr{P}(T)\| \leq K \sup\{|\mathscr{P}(z)|: |z| \leq 1\}$$

for every polynomial $\mathscr{P}$. Another way of describing this condition is to say that the unit disc is a K-spectral set for T. It is a well-known result, due to von Neumann, that the unit disc is a 1-spectral set for each contraction. The elegant proofs of this result proceed by reduction to the case of a unitary operator [4], [5]. Now, if T is similar to a contraction C (that is, if $T = S^{-1}CS$ with $\|C\| \leq 1$), then $\mathscr{P}(T) = S^{-1}\mathscr{P}(C)S$ for each polynomial $\mathscr{P}$. Thus it follows from von Neumann's theorem that T is polynomially bounded, with $K = \|S^{-1}\| \cdot \|S\|$. Therefore an operator that is not polynomially bounded is not similar to a contraction.

An operator T is said to be a *moment operator* if for each pair of vectors x and y there exists a complex-valued function g on $[0, 2\pi]$, of finite variation, such that

$$\langle T^n x, y \rangle = \int_0^{2\pi} e^{int} dg(t) \qquad \text{for } n = 0, 1, 2, \cdots.$$

LEMMA. *An operator is polynomially bounded if and only if it is a moment operator.*

Proof. If T is polynomially bounded, consider $\langle \mathscr{P}(T)x, y \rangle$ as a function of $\mathscr{P}$, and apply the Schwarz inequality, (*), and the maximum-modulus, Hahn-Banach, and Riesz representation theorems to show the existence of a g such that

$$\langle \mathscr{P}(T)x, y \rangle = \int \mathscr{P} dg.$$

(The argument has been used in another context [3].)

To prove the converse, note that if T is a moment operator and $|\mathscr{P}(z)| \leq 1$ for $|z| \leq 1$, then

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$$|\langle \mathscr{P}(T)x, y \rangle| = \left| \int \mathscr{P} dg \right| \leq \operatorname{var} g.$$

It follows from two applications of the uniform-boundedness principle that there exists a constant K such that $\|\mathscr{P}(T)\| \leq K$ for all polynomials $\mathscr{P}$ that are bounded by 1 in the unit disc. Thus T is polynomially bounded.

THEOREM. *There exists a power-bounded operator that is not polynomially bounded.*

Proof. Halmos represents Foguel's operator as the matrix

$$A = \begin{pmatrix} S^* & Q \\ O & S \end{pmatrix},$$

where S is the unilateral shift on a Hilbert space H with orthonormal basis $\{e_0, e_1, e_2, \cdots\}$, and where Q is the projection of H onto the span of $\{e_j\}$, j being a power of 3. Halmos observes that

$$A^n = \begin{pmatrix} S^{*n} & Q_n \\ O & S^n \end{pmatrix},$$

where $Q_{n+1} = \sum_0^n S^{*n-i} Q S^i$; moreover, Q_{n+1} is a partial isometry, so that A is power-bounded. Since S and S^* are contractions, hence moment operators, it is easily seen that A is a moment operator if and only if for each pair of vectors x and y in H there exists a g of finite variation such that

$$(1) \qquad \langle Q_n x, y \rangle = \int_0^{2\pi} e^{int} dg(t) \qquad (n = 1, 2, \cdots).$$

We consider the case $x = y = e_0$; here

$$\langle Q_{n+1} e_0, e_0 \rangle = \left\langle \sum_0^n S^{*n-i} Q S^i e_0, e_0 \right\rangle = \sum_0^n \langle Q e_i, e_{n-i} \rangle.$$

Each term of the last sum is 0, unless $i = 3^k$ and $n - i = 3^k$. Thus $\langle Q_{n+1} e_0, e_0 \rangle$ is 1 or 0, according as $n = 2 \cdot 3^k$ or not. We shall show that this 0-1 condition makes it impossible to satisfy (1) with $x = y = e_0$. Suppose the contrary, and let

$$f(z) = \int \frac{dg(t)}{1 - e^{it} z} = \sum_0^\infty \langle Q_n e_0, e_0 \rangle z^n.$$

The function f is analytic in the open unit disc. Fubini's theorem and an elementary estimate show that f belongs to the Hardy class H^p, for $0 < p < 1$. By a theorem of Riesz (extended by Zygmund to the case $p < 1$), f has radial limits almost everywhere on the unit circle [7, p. 276]. Thus the Fourier series

$$(2) \qquad \sum \langle Q_n e_0, e_0 \rangle e^{int}$$

is Abel-summable almost everywhere, and by a theorem of Zygmund on lacunary series [7, p. 203], the coefficients of (2) are square-summable. This contradiction completes the proof.

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