A RESULT ON GRAPH-COLORING

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The following problem was proposed by Erdös [4]:

Let $m > 1$ and $k \geq 1$ be integers, and let G be a graph with km nodes, each of the nodes having valence at least $(m - 1)k$. Does G contain k disjoint subgraphs each of which is a complete graph with m nodes?

In a number of cases an affirmative answer to Erdös' problem has been established (or follows from more general theorems). The author believes that the following list is complete:

(i) $m = 2$ (Dirac [3]);

(ii) $m = 3$ (Corrádi and Hajnal [2]);

(iii) $k \leq 3$ (Zelinka [5]).

It is the aim of the present note to enlarge the list by establishing an affirmative answer in the case $k = 4$.

Considering the graph complementary to the one of Erdös' problem (that is, reformulating Erdös' problem as a coloring problem) and making a mild generalization, one is led to the following conjecture (which reduces to Erdös' problem in case $n/k = m$ is an integer).

Conjecture. If a graph G has n nodes, and the valence of each node is less than k, then it is possible to color the nodes of G with k colors in such a manner that each color is assigned to at least $[n/k]$ nodes and to at most $[n/k] + 1$ nodes.

Zelinka's paper [5] adopts essentially the coloring point of view, and a trivial modification of his proof establishes the conjecture in case $k \leq 3$. The contribution of the present note is the proof of the conjecture in case $k = 4$.

THEOREM. *If* G *is a graph with* n *nodes and with maximal valence* 3, *it is possible to color the nodes of* G *by four colors in such a manner that each color is assigned either to* $[n/4]$ *or to* $[n/4] + 1$ *nodes.*

Proof. We use induction on the number of nodes of the graph G, noting that each node of G has valence at most 3. If every connected component of G has at most 3 nodes, the assertion is obviously true. If some connected component of G is the triod T (Figure 1) the assertion follows if we assign the four colors to the four nodes of T and use the inductive assumption to color the rest of G.

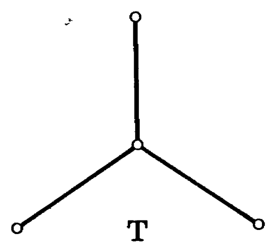

Figure 1

For the remaining part of the proof, we need the notion of a *nice* 3-*path*. We shall say that the nodes A_1, A_2, A_3, A_4 of a graph C with maximal valence 3 form a *nice* 3-*path* provided

(1) the nodes A_1, A_2, A_3, A_4 are all distinct;

(2) C contains the edges (A_1, A_2), (A_2, A_3), and (A_3, A_4) (and possibly additional edges between the A_i);

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(3) if C contains the edges (B, A_2) and (B, A_3) for some node B, then $B = A_1$ or $B = A_4$.

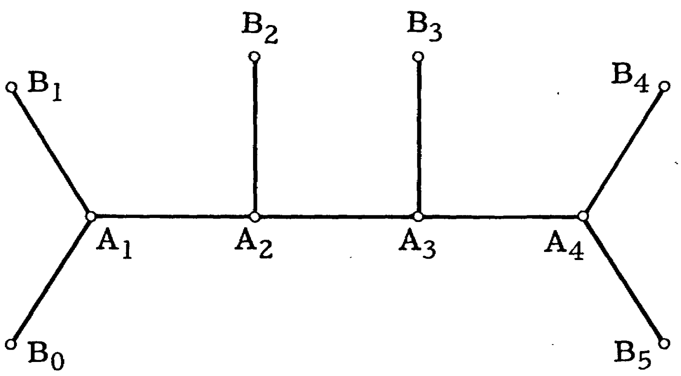

Figure 2

In other words, a nice 3-path may be represented by the diagram in Figure 2, with the possibility that some of the B_i coincide or are missing, or are the same as some of the A_j, provided only that $B_2 \neq B_3$ if both B_2 and B_3 are present.

The following lemma is easily established, and we omit its proof.

LEMMA. *If* C *is a connected graph with maximal valence at most* 3 *and with at least four nodes, then either* C *is the triod* T, *or else* C *contains a nice* 3-*path.*

Resuming the proof of the theorem, we note that we may now restrict our attention to graphs G that contain nice 3-paths. Let A_1, A_2, A_3, A_4 be the nodes of a nice 3-path in G; we construct a new graph G^* by deleting from G the nodes A_1, A_2, A_3, A_4 (and the edges incident with them) and adjoining the edges (B_0, B_1), (B_2, B_3), (B_4, B_5), if possible (that is, if the B_i in question are nodes of G^* and do not coincide, and if the corresponding edge is not already present). Since G^* has fewer nodes than G, the inductive assumption implies that G^* is 4-colorable in the manner required by the theorem. From this 4-coloring of G^* we obtain a 4-coloring of the desired type for G by assigning to each node of G that belongs to G^* its color in G^*, while A_1, A_2, A_3, A_4 receive the colors indicated in Table 1.

This completes the proof of the theorem.

Remark 1. It seems that the results of Dirac [2] and of Corrádi and Hajnal [3] do not yield a proof of the conjecture for $k = [n/2]$ and $k = [n/3]$ unless $n = 2k$ or $n = 3k$, respectively.

Colors in the 4-coloring of G^*				Colors in the 4-coloring of G			
B_0, B_1	B_2	B_3	B_4, B_5	A_1	A_2	A_3	A_4
1, 2	1	2	3, 4	3	4	1	2
1, 2	1	3	3, 4	3	4	1	2
1, 2	3	2	3, 4	3	4	1	2
1, 2	2	1	1, 3	3	1	2	4
1, 2	2	3	1, 3	3	1	2	4
1, 2	2	4	1, 3	3	1	2	4
1, 2	1	2	1, 3	3	2	1	4
1, 2	3	2	1, 3	3	2	1	4
1, 2	1	4	1, 3	3	2	1	4
1, 2	4	2	1, 3	3	2	1	4
1, 2	1	2	1, 2	3	2	1	4
1, 2	1	3	1, 2	3	2	1	4
1, 2	3	4	1, 2	3	2	1	4

Table 1

Remark 2. The conjecture may be compared with a theorem of Brooks [1], which asserts that if the maximal valence of a graph G is less than k, then G may be (k - 1)-colored (unless $k \leq 3$, or some component of G is the complete graph with k nodes). According to our conjecture, the increase (by one) of the number of available colors allows one to insist that each color be assigned to equally many nodes as nearly as possible. It is easy to find examples showing that insistence on any other frequency-distribution of colors would invalidate the conjecture.

Added in proof (March 6, 1968): In a recent paper (*On the number of complete subgraphs and circuits in a graph*, to appear in Proc. Roy. Soc. Ser. A), D. G. Larman establishes a number of results related to the subject of the present paper. One of Larman's results concerns relations between Erdös' problem and our conjecture; from it and the result of Corrádi and Hajnal [2], Larman deduces the validity of our conjecture for $k = [n/2]$ and $k = [n/3]$.

REFERENCES

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