ON GENERATORS OF VON NEUMANN ALGEBRAS

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1. Recently, C. Pearcy and D. Topping [4] and the author [5] separately considered a certain class of von Neumann algebras, and they showed that each algebra of the class can be generated by various small sets of special operators, projections, unitary operators, idempotents, and so forth. The purpose of this paper is to clarify the relation among these results.

2. Throughout the paper, we assume that $\mathbb{H}$ is a separable Hilbert space, and by an *operator* we mean a bounded linear operator on a Hilbert space. A von Neumann algebra $\mathbb{M}$ acting on a Hilbert space is said to be *generated by a family* $\{A, B, \cdots\}$ of operators if $\mathbb{M}$ is the smallest von Neumann algebra containing each member of the family $\{A, B, \cdots\}$, and it is denoted by $\mathbb{M} = R(A, B, \cdots)$. We shall consider von Neumann algebras $\mathbb{M}$ on $\mathbb{H}$ with the property

$$(*) \qquad \mathbb{M} \text{ is } *\text{-isomorphic to } \mathbb{M}_2,$$

where $\mathbb{M}_2$ is the algebra of all 2×2 matrices over $\mathbb{M}$. The following is our result. It sharpens the results in [4] and [5].

THEOREM. *Let* $\mathbb{M}$ *be a von Neumann algebra with property* (*). *Then the following statements are equivalent.*

(a) $\mathbb{M}$ *has a single generator.*

(b) $\mathbb{M}$ *is generated by one partial isometry.*

(c) $\mathbb{M}$ *is generated by two operators.*

(d) $\mathbb{M}$ *is generated by two unitary operators.*

(e) $\mathbb{M}$ *is generated by three projections.*

3. The essential part of this paper is the proof of the equivalence of (a), (c), and (e).

LEMMA 1. *Suppose that a von Neumann algebra* $\mathbb{M}$ *is generated by two operators* A *and* B. *Then* $\mathbb{M}_2$ *is generated by the three operators*

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The proof of this lemma consists of easy computations, which we omit.

LEMMA 2. *Let a von Neumann algebra* $\mathbb{M}$ *be generated by two operators, and suppose that one of these generators is a normal operator. Then* $\mathbb{M}_2$ *is generated by a single operator.*

Proof. Let A and B be the generators of $\mathbb{M}$, and suppose that B is normal. We can assume that A is invertible and $\|A\| < 1$. We can also assume that B is an

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invertible normal operator. Next, let S and Z be the positive square roots of $I - A^*A$ and $I - AA^*$, respectively. Define two operators in $\mathbb{M}_2$ by

$$U = \begin{pmatrix} A & Z \\ -S & A^* \end{pmatrix}, \quad T = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}.$$

Then U is a unitary operator (see [2]), and T is a normal operator. Let $\mathbb{R} = R(U, T)$. Then, since $R(U)$ and $R(T)$ are abelian von Neumann subalgebras of $\mathbb{R}$, $\mathbb{R}$ has a single generator, by [6, Lemma 3]. Thus it suffices to prove that $\mathbb{R} = \mathbb{M}_2$. Since B is invertible,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

is contained in $\mathbb{R}$, and so

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is an element of $\mathbb{R}$. Thus all matrices $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ with $X \in R(A)$ are contained in $\mathbb{R}$. Of course, $\begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ belongs to $\mathbb{R}$. Now, we have the relations

$$U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^* = \begin{pmatrix} AA^* & -AS \\ -SA^* & I - A^*A \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I - A^*A \end{pmatrix}.$$

Since $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = U^*U - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, the operator $\begin{pmatrix} 0 & 0 \\ 0 & I - A^*A \end{pmatrix}$ is contained in $\mathbb{R}$, and thus $\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}$ is an element of $\mathbb{R}$, for every $X \in R(I - A^*A)$. In particular, $\begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix}$ is contained in $\mathbb{R}$. By the equations

$$\begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix}\begin{pmatrix} AA^* & -AS \\ -SA^* & I - A^*A \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ -A^* & 1 \end{pmatrix},$$

and

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -A^* & 1 \end{pmatrix}\right]^* = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

the operator

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

is contained in $\mathbb{R}$. Therefore, by Lemma 1, the proof of Lemma 2 is complete.

Remark. The proof of Lemma 2 is essentially the same as that of [5, Theorem 2].

COROLLARY. *Let* $\mathbb{M}$ *be a von Neumann algebra generated by three normal operators. Then* $\mathbb{M}_2$ *has a single generator.*

Proof. Let A, B, and C be normal operators such that $\mathbb{M} = R(A, B, C)$. Since R(A) and R(B) are abelian von Neumann subalgebras of R(A, B), R(A, B) has a single generator, by [6, Lemma 3]. Thus $\mathbb{M}$ is generated by two operators, one of which is a normal operator. It now follows from Lemma 2 that $\mathbb{M}_2$ is generated by a single operator.

LEMMA 3. *Suppose that a von Neumann algebra* $\mathbb{M}$ *is generated by two operators. Then* $\mathbb{M}_2$ *is generated by three unitary operators on* $\mathbb{H} \oplus \mathbb{H}$.

Proof. Let two operators A and B generate $\mathbb{M}$. We can assume that both A and B are invertible and that they are strict contractions. We define three elements of $\mathbb{M}_2$ by

$$U = \begin{pmatrix} A & \sqrt{I - AA^*} \\ -\sqrt{I - A^*A} & A^* \end{pmatrix}, \quad V = \begin{pmatrix} B & \sqrt{I - BB^*} \\ -\sqrt{I - B^*B} & B^* \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then U and V are unitary operators, and E is a projection. It suffices to show that $R(U, V, E) = \mathbb{M}_2$. For this, it is sufficient to note that

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

belong to $\mathbb{M}_2$, by Lemma 1. Computing EUE and EVE, we see that $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ belong to $\mathbb{M}_2$. To show that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is contained in $\mathbb{M}_2$, it suffices to trace the last part of the proof of Lemma 2; we omit the details.

Proof of the theorem. By the assumption, we can identify $\mathbb{M}$ with $\mathbb{M}_2$. The implications (b) $\Rightarrow$ (a) and (d) $\Rightarrow$ (c) are trivial. The implication (a) $\Rightarrow$ (b) follows from [3, Lemma 1].

The implication (a) $\Rightarrow$ (d) follows from [5, Theorem 2], but we give a direct proof of the equivalence of (a) and (d). Let T be a generator of $\mathbb{M}$, and let $T = A + iB$ be the decomposition into the real and imaginary parts of T. Then $\mathbb{M} = R(A, B)$. We can assume that A and B are self-adjoint contractions. Then

$U = A + i\sqrt{I - A^2}$ and $V = B + i\sqrt{I - B^2}$ are unitary operators that generate $\mathbb{M}$. Conversely, the implication (d) $\Rightarrow$ (a) follows from [6, Lemma 3].

To show that (c) implies (a), let $\mathbb{M}$ be generated by two operators A and B. Then, by Lemma 3, there exist three unitary operators that generate $\mathbb{M}_2$. Thus the algebra $(\mathbb{M}_2)_2$ of all 2×2 matrices over $\mathbb{M}_2$ is generated by a single operator, by the corollary to Lemma 2, and therefore $\mathbb{M}$ has a single generator, by the hypothesis (*). The equivalence of (a) and (e) follows from [5, Theorem 1] and the corollary to Lemma 2, and the proof of our theorem is complete.

Remark. From the above proof, it is easily seen that the statement (e) can be replaced by

(e)' $\mathbb{M}$ is generated by three normal operators.

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