THE DECOMPOSITION OF MATRIX-VALUED MEASURES

James B. Robertson and Milton Rosenberg

1. INTRODUCTION

By an $m \times n$ - *matrix-valued measure* on a σ-algebra $\mathscr{B}$ over Ω we shall mean a function M from $\mathscr{B}$ into the set of all $m \times n$ matrices over the complex numbers such that for every disjoint sequence of sets $A_1, A_2, \cdots$ in $\mathscr{B}$ with union A, $M(A) = \sum_{k=1}^{\infty} M(A_k)$. A *$\mathscr{B}$-measurable $\ell \times m$-matrix-valued function* on Ω will be a function Φ from Ω into the set of all $\ell \times m$ matrices such that the entries $\Phi_{ij}(\omega) = [\Phi(\omega)]_{ij}$ are $\mathscr{B}$-measurable. We shall define the integral $\int_\Omega \Phi dM$ for a suitable class of $\ell \times m$-matrix-valued functions with respect to an $m \times n$-matrix-valued measure. Such measures and integrals are important in the spectral analysis of multivariate, weakly stationary, stochastic processes (see Masani [8, Section 8]). It was our interest in this subject that led us to the present study, and in Section 7 we shall indicate how our results apply to this theory.

The primary purpose of this paper is to define and study appropriate notions of the total variation of a matricial measure and of the absolute continuity, Radon-Nikodým derivative, and singularity of one matricial measure with respect to another, and to prove matricial versions of the Hahn-Jordan decomposition, Radon-Nikodým theorem, and Lebesgue decomposition. We are able to obtain a reasonably complete theory, but only by renouncing seemingly reasonable definitions. The Hahn-Jordan decomposition of a matrix-valued measure into nonnegative hermitian matrix-valued measures is best viewed not as a finite sum of such measures, but as an integral thereof. Even for a complex-valued measure M, this seems to have been overlooked; but, of course, if M is real-valued, it yields the usual decomposition (Section 4). To get the Radon-Nikodým theorem, we have to define absolute continuity in terms of certain derivatives of the measures rather than in terms of the measures themselves (Section 5). If two matrix-valued functions are equal almost everywhere with respect to a matrix-valued measure M, and if N is absolutely continuous with respect to M, it does not necessarily follow that the functions are equal almost everywhere with respect to N. This is because matrix multiplication is not commutative. Unless we exercise great care in the definition of Radon-Nikodým derivatives, the Radon-Nikodým derivatives of two equivalent measures will not always turn out to be inverses of one another. Here the notion of generalized inverse due to Penrose [10] is very useful (Section 2). The usual notion of the carrier of a measure must be supplanted by that of a projection matrix-valued function, which is the matricial analogue of the indicator function (Section 6). There is not just a single Lebesgue decomposition of one measure with respect to another, but to each measure of a certain class there corresponds a distinct decomposition (Section 6). This has helped to clarify certain problems arising in the orthogonal decomposition of stochastic processes (Section 7).

Hardly any work seems to have been done on the problem of obtaining a Radon-Nikodým derivative and a Lebesgue decomposition of one operator-valued measure with respect to another. Even for vector-valued measures, the literature is scanty

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[3], [4], [11]. In [3, p. 545], N. Dinculeanu and C. Foiaş construct under restrictive conditions a "weak derivative" Φ of a vector-valued (not matrix-valued) measure M with respect to a real measure μ. In [11], M. M. Rao investigates the more general case of two vector measures M and N valued in two distinct Banach spaces. Under restrictive conditions, he obtains a Lebesgue decomposition

$$N(B) = \int_B \Phi\, dM + N(B \cap E),$$

where E is a set such that M is zero on every measurable subset of E. (His assumptions imply the Bochner-integral representability of M and of N.) In complete contrast to Rao's result, our absolutely continuous and singular parts of N with respect to M are not concentrated on disjoint sets [see (6.14)].

In this paper, we have restricted our results to the case of finite matricial measures on σ-algebras. (Elsewhere, we hope to extend our results to the case of σ-finite matricial measures.) Further, we have only dealt with the derivative $dN \cdot dM^{\#}$ [see (5.4)], which should properly be called a *left-hand derivative*. By an obvious dual procedure (using the adjoint measure) we can obtain a theory of *right-hand derivatives* $dM^{\#} \cdot dN$ with reference to integrals of the form $\int_B dM \cdot \Phi$.

The generalized inverse due to Penrose [10] plays an important role in our work. In Section 2 we establish several results concerning it. Section 3 is devoted to the fundamentals of matricial measures and integrals.

We thank the referee for suggesting several improvements, and especially for some results on generalized inverses.

2. GENERALIZED INVERSES

Throughout the paper, vectors and matrices will have complex entries. Matrices will be denoted by capital letters, and vector spaces by script letters. $\mathscr{R}(A) = \{y: y = xA\}$ will denote the *range* of A, and $\mathscr{N}(A) = \{x: xA = 0\}$ will denote the *null space* of A. As usual, A^* will denote the conjugate transpose of A. By a *projection* we shall mean a matrix J such that $J^2 = J$. The matrix J is said to be a projection *onto* $\mathscr{M}$ *along* $\mathscr{N}$ if J is a projection such that $\mathscr{R}(J) = \mathscr{M}$ and $\mathscr{N}(J) = \mathscr{N}$ (see Halmos [5, Section 33] for usage). If it is intended that J is an *orthogonal* projection (that is $J^* = J = J^2$), we shall specifically say so. In particular, $P_{\mathscr{M}}$ will denote the orthogonal projection onto $\mathscr{M}$. $\mathscr{M}^{\perp}$ will denote the orthogonal complement of $\mathscr{M}$. The Euclidean norm of A is defined by $\|A\|_E = \sqrt{\text{trace }(AA^*)}$.

The concept of the generalized inverse of a linear operator has been in the folklore of operator theory (see for example von Neumann [9] and Hestenes [6, Section 3]). The corresponding concept for matrices was introduced by Penrose, who proved the following theorem [10, p. 406]:

2.1 THEOREM. *Let* A *be any* $m \times n$ *matrix. Then there exists a unique* $n \times m$ *matrix* X *such that*

$$A = AXA, \qquad X = XAX,$$

$$(AX)^* = AX, \qquad (XA)^* = XA.$$

The matrix X in this theorem is called the *generalized inverse* of A, and it will be denoted by $A^{\#}$.

The generalized inverse of A may be written in the following two useful forms:

$$A^{\#} = P_{\mathscr{R}(A)} A^{-1} P_{\mathscr{N}(A)^{\perp}}, \tag{2.2}$$

where A^{-1} is the multivalued inverse of A (that is, yA^{-1} can be any x such that $xA = y$), and

$$A^{\#} = A^{*} H, \tag{2.3}$$

where H is any hermitian matrix such that $AA^{*}H = P_{\mathscr{R}(A^{*})} = P_{\mathscr{N}(A)^{\perp}}$.

The verification that $A^{\#}$, given by either (2.2) or (2.3), satisfies the conditions of Theorem 2.1 is routine, and we omit it.

The following properties follow immediately if (for example) we use (2.2) and the fact that $\mathscr{R}(A^{\#}) = \mathscr{N}(A)^{\perp}$ and that $\mathscr{N}(A^{\#}) = \mathscr{R}(A)^{\perp}$. We omit the proofs.

(2.4) $AA^{\#} = P_{\mathscr{R}(A^{*})} = P_{\mathscr{N}(A)^{\perp}}$.

(2.5) $A^{\#}A = P_{\mathscr{R}(A)} = P_{\mathscr{N}(A^{*})^{\perp}}$.

(2.6) $(A^{\#})^{\#} = A$.

(2.7) $(A^{*})^{\#} = (A^{\#})^{*}$.

(2.8) If $\mathscr{R}(A) = \mathscr{N}(B)^{\perp}$, then $(AB)^{\#} = B^{\#}A^{\#}$.

(2.9) $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ if and only if $AB^{\#}B = A$.

(2.10) Let H be hermitian and $G = AHA^{*}$. Then $\mathscr{R}(H) \subseteq \mathscr{R}(A)$ if and only if $A^{\#}G(A^{\#})^{*} = H$.

The next two propositions concern the continuity properties of the generalized inverse; 2.11 is due to Penrose [10, page 408], and 2.12 follows from (2.4), (2.5), (2.7), and 2.11.

2.11 LEMMA. *If* A *is a matrix and* $\varepsilon > 0$, *then there exists a* $\delta > 0$ *such that* $\|A^{\#} - B^{\#}\|_E < \varepsilon$ *whenever* $\|A - B\|_E < \delta$ *and* $\operatorname{rank}(A) = \operatorname{rank}(B)$.

2.12 LEMMA. *If* $A_n \to A$, *then the following conditions*

(a) $A_n^{\#} \to A^{\#}$, (b) $A_n^{*\#} \to A^{*\#}$, (c) $P_{\mathscr{R}(A_n)} \to P_{\mathscr{R}(A)}$, (d) $P_{\mathscr{N}(A_n)} \to P_{\mathscr{N}(A)}$

are equivalent.

The next two propositions concern projections. The proof of 2.13 is straightforward, and we omit it.

2.13 LEMMA. *The following conditions are equivalent:*

(a) $AB = 0$.

(b) *There exists a projection* J *along* $\mathscr{N}(B)$ *such that* $AJ = 0$.

(c) *For all projections along* $\mathcal{N}(B)$, $AJ = 0$.

2.14 LEMMA. *Let* A *and* B *be* $m \times q$ *and* $n \times q$ *matrices such that*

$$\mathscr{C}^q = \mathscr{R}(B) + \mathscr{R}(A)^\perp \quad \text{and} \quad \mathscr{R}(B) \cap (\mathscr{R}(A)^\perp) = \{0\}.$$

Then $J = A^*(BA^*)^\# B$ *is the projection onto* $\mathscr{R}(B)$ *along* $\mathscr{R}(A)^\perp$.

Proof. Using Theorem 2.1, we can easily verify that J is a projection. It also follows that

$$\mathscr{R}(J) \subseteq \mathscr{R}(B) \quad \text{and} \quad \mathcal{N}(J) \supseteq \mathcal{N}(A^*) = \mathscr{R}(A)^\perp.$$

The proof will therefore be completed if we show that $\operatorname{rank}(B) \leq \operatorname{rank}(J)$. But by Theorem 2.1, $BJA^* = BA^*$, and hence $\operatorname{rank}(BA^*) \leq \operatorname{rank}(J)$. But clearly, $\operatorname{rank}(B) = \operatorname{rank}(BA^*)$, since $\mathcal{N}(A^*) \cap \mathscr{R}(B) = \{0\}$. ∎

Nonnegative hermitian matrices play a central role in our development, and we present here some facts concerning them.

2.15 LEMMA. *Let* H *be an* $n \times n$ *nonnegative hermitian matrix. If* A *is a matrix such that* $\mathscr{R}(A) \cap \mathcal{N}(H) = \{0\}$, *then*

$$\mathscr{C}^n = \mathscr{R}(AH^\#) + \mathscr{R}(A)^\perp \quad \text{and} \quad \mathscr{R}(AH^\#) \cap (\mathscr{R}(A)^\perp) = \{0\}.$$

Proof. Since H is hermitian, $\mathcal{N}(H^\#) = \mathcal{N}(H)$. Hence, by hypothesis, $\mathscr{R}(A) \cap \mathcal{N}(H^\#) = \{0\}$, and so $\operatorname{rank}(AH^\#) = \operatorname{rank}(A)$. Hence

$$\dim \mathscr{R}(AH^\#) + \dim \mathscr{R}(A)^\perp = n,$$

and it is sufficient to show that $\mathscr{R}(AH^\#) \cap (\mathscr{R}(A)^\perp) = \{0\}$. If $x = yA$ and $xH^\# \perp x$, then, since $H^\#$ is nonnegative hermitian, x is in $\mathcal{N}(H^\#) \cap \mathscr{R}(A) = \{0\}$. Thus $\mathscr{R}(AH^\#) \cap (\mathscr{R}(A)^\perp) = \{0\}$. ∎

2.16 THEOREM. *Let* H *be an* $n \times n$ *nonnegative hermitian matrix, and let* A *and* B *be matrices such that* $\mathscr{R}(A) \subseteq \mathscr{R}(H)$, $\mathscr{R}(B) \subseteq \mathscr{R}(H)$. *Then* $AH^\# B^* = 0$ *if and only if there exist nonnegative hermitian matrices* M *and* N *such that*

(i) $\mathscr{R}(A) \subseteq \mathscr{R}(M)$, (ii) $\mathscr{R}(B) \subseteq \mathscr{R}(N)$, (iii) $\mathscr{R}(M) \cap \mathscr{R}(N) = \{0\}$, (iv) $H = M + N$.

Proof. Suppose M and N are nonnegative hermitian matrices satisfying (i) to (iv). Since $H^\# H = P_{\mathscr{R}(H)}$, (iv) yields the relations

$$M = MH^\# H = MH^\# M + MH^\# N.$$

Therefore (iii) implies that $MH^\# N = 0$. Multiplying by $M^\#$ and $N^\#$, we see that $P_{\mathscr{R}(M)} H^\# P_{\mathscr{R}(N)} = 0$. Hence, using (i) and (ii), we get the equations

$$AH^\# B^* = (AP_{\mathscr{R}(M)}) H^\# (BP_{\mathscr{R}(N)})^* = A (P_{\mathscr{R}(M)} H^\# P_{\mathscr{R}(N)}) B^* = 0.$$

Now suppose $\mathscr{R}(A), \mathscr{R}(B) \subseteq \mathscr{R}(H)$, and $AH^\# B^* = 0$. Multiplying by $A^\#$ and $B^{\#*}$, we find that $P_{\mathscr{R}(A)} H^\# P_{\mathscr{R}(B)} = 0$. Let $\mathscr{M} = \mathscr{R}(AH^\#)^\perp \cap \mathscr{R}(H)$. Then $\mathscr{R}(B) \subseteq \mathscr{M}$. Further, $P_{\mathscr{R}(A)} H^\# P_{\mathscr{M}} = 0$, and $\mathscr{R}(AH^\#) + \mathscr{M} = \mathscr{R}(H)$. Also, $\mathscr{R}(A) \cap \mathscr{M} = \{0\}$, since if x is in both, then $xH^\# \perp x$ and $x \in \mathscr{R}(H^\#) \cap \mathcal{N}(H^\#) = \{0\}$. Since $\mathscr{R}(A) \subseteq \mathscr{R}(H) = \mathscr{R}(H^\#)$, $\dim \mathscr{R}(A) = \dim \mathscr{R}(AH^\#)$. Thus

$$\dim \mathscr{R}(A) + \dim \mathscr{M} = \dim \mathscr{R}(H),$$

and hence $\mathscr{R}(A) + \mathscr{M} = \mathscr{R}(H)$. Now let J denote the projection onto $\mathscr{R}(A)$ along $\mathscr{M} + \mathscr{N}(H)$. Then $P_{\mathscr{R}(H)} - J$ is the projection onto $\mathscr{M}$ along $\mathscr{R}(A) + \mathscr{N}(H)$. We can then verify that

$$M = HJ = (HJ)H^{\#}(HJ)^* \quad \text{and} \quad N = H(P_{\mathscr{R}(H)} - J) = H(P_{\mathscr{R}(H)} - J)H^{\#}(H(P_{\mathscr{R}(H)} - J))^*$$

have the desired properties. ■

We close this section with a statement of the polar decomposition.

2.17 LEMMA. *Suppose* (i) A *is an* $m \times n$ *matrix,* (ii) $S = \sqrt{A^*A}$, *and* (iii) $W = AS^{\#}$. *Then* W *is a partial isometry, and*

$$\mathscr{R}(W) = \mathscr{R}(A), \qquad \mathscr{N}(W) = \mathscr{N}(A)$$

$$W^*W = P_{\mathscr{R}(A)}, \qquad WW^* = P_{\mathscr{N}(A)^{\perp}}, \qquad A = WS.$$

3. MATRICIAL INTEGRALS

In the sequel, $\mathscr{B}$ denotes a σ-algebra of subsets of an abstract set Ω. A $\mathscr{B}$-*measurable* $m \times n$-*matrix-valued function* on Ω is a function Φ from Ω into the set of all $m \times n$ matrices such that the entries $\Phi_{ij}(\omega) = [\Phi(\omega)]_{ij}$ are $\mathscr{B}$-measurable.

3.1 LEMMA. *If* Φ *is a* $\mathscr{B}$-*measurable* $m \times n$-*matrix-valued function of* Ω, *then* $\Phi^{\#}$ *is a* $\mathscr{B}$-*measurable* $n \times m$-*matrix-valued function on* Ω.

Proof. This follows from [4] if we note that $\Phi\Phi^*$ is nonnegative hermitian, and that we may therefore construct a $\mathscr{B}$-measurable H for (2.3).

If Φ is a $\mathscr{B}$-measurable $m \times n$-matrix-valued function on Ω, and if μ is a nonnegative σ-finite measure on $\mathscr{B}$, we define (for $B \in \mathscr{B}$)

$$\int_B \Phi \, d\mu = \left[\int_B \Phi_{ij} \, d\mu \right],$$

provided $\int_B \Phi_{ij} \, d\mu$ exists for all i and j. The set of all such Φ will be denoted by $L_1(B, \mathscr{B}, \mu)$. In discussing certain examples below, we shall need the following lemma.

3.2 LEMMA. *Let* Φ *be an* $n \times n$ *nonnegative hermitian matrix-valued function on* Ω *such that* $\int_B \Phi \, d\mu$ *exists. Then*

(a) *for almost all* ω *in* B, $\mathscr{N}\left(\int_B \Phi \, d\mu\right) \subseteq \mathscr{N}(\Phi(\omega))$;

(b) *for almost all* ω *in* B, $\mathscr{R}(\Phi(\omega)) \subseteq \mathscr{R}\left(\int_B \Phi \, d\mu\right)$;

(c) *if for almost all* ω *in* B, $\mathscr{R}(\Phi(\omega)) \subseteq \mathscr{M}$, *where* $\mathscr{M}$ *is a subspace of* $\mathscr{C}^n$, *then* $\mathscr{R}\left(\int_B \Phi \, d\mu\right) \subset \mathscr{M}$.

Proof. It is clear that for any vector x constant with respect to ω,

$$x \cdot \int_B \Phi \, d\mu \cdot x^* = \int_B (x\Phi x^*) \, d\mu .$$

Since $x\Phi x^* \geq 0$, with equality if and only if x is in $\mathscr{N}(\Phi)$, we see that $x \in \mathscr{N}\left(\int_B \Phi \, d\mu\right)$ if and only if $x \in \mathscr{N}(\Phi(\omega))$ for almost all ω in B. This implies (a). The conclusion (b) follows if we take orthogonal complements. Similarly, since $\mathscr{R}(\Phi(\omega)) \subseteq \mathscr{M}$ is the same as $\mathscr{M}^{\perp} \subseteq \mathscr{N}(\Phi(\omega))$, we obtain (c). ■

M is called an $m \times n$ - *matrix-valued measure* on $\mathscr{B}$ if M is a function from $\mathscr{B}$ into the set of all $m \times n$ matrices, and if $M(A) = \sum_{k=1}^{\infty} M(A_k)$ whenever $A_1, A_2, \cdots$ is a disjoint sequence of sets in $\mathscr{B}$ whose union is A. Obviously, $M = [M_{ij}]$ is a matrix-valued measure if and only if each of its entries M_{ij} is a complex-valued measure on $\mathscr{B}$. If μ is a nonnegative σ-finite measure on $\mathscr{B}$, we say that M is *absolutely continuous* with respect to μ ($M \ll \mu$) if each entry M_{ij} is absolutely continuous with respect to μ. In this case, M'_{μ} will denote the $\mathscr{B}$-measurable $m \times n$ - matrix-valued function whose entries are the Radon-Nikodým derivatives of the elements M_{ij} with respect to μ. Clearly, $M(B) = \int_B M'_{\mu} \, d\mu$.

In [12, 3.1], Rosenberg defined and studied $\int_B \Phi \cdot dM \cdot \Psi$ for an $n \times n$ - nonnegative hermitian matrix-valued measure M. We now extend his definition to arbitrary matrix-valued measures M. Let Φ and Ψ be $\mathscr{B}$-measurable $k \times m$ - and $n \times \ell$ - matrix-valued functions on Ω, and let M be an $m \times n$ - matrix-valued measure on $\mathscr{B}$. If μ is a nonnegative, σ-finite measure on $\mathscr{B}$, such that $M \ll \mu$, and if $\int_B (\Phi \cdot M'_{\mu} \cdot \Psi)_{ij} \, d\mu$ exists for all i and j, then we define

$$\int_B \Phi \, dM\Psi = \left[\int_B (\Phi \cdot M'_{\mu} \cdot \Psi)_{ij} \, d\mu \right].$$

If ν is another measure, then

$$\int_B (\Phi M'_{\mu} \Psi)_{ij} \, d\mu = \int_B (\Phi M'_{\mu+\nu} \Psi)_{ij} \, d(\mu + \nu) = \int_B (\Phi M'_{\nu} \Psi)_{ij} \, d\nu ,$$

by the usual chain rule for Radon-Nikodým derivatives. Thus $\int_B \Phi \, dM\Psi$ is independent of the choice of μ. In case $\Psi(\omega)$ is the identity matrix for almost all ω, we shall simply write $\int_B \Phi \, dM$; $\int_B dM\Psi$ is defined similarly.

The proof of the following result is easy, and we omit it.

3.3 THEOREM. (a) *Let* M *be an* $m \times n$ - *matrix-valued measure on* $\mathscr{B}$, *and* Φ *a* $\mathscr{B}$-*measurable* $\ell \times m$ - *matrix-valued function on* Ω *such that* $\int_{\Omega} \Phi \, dM$ *exists. Let* $N(B) = \int_B \Phi dM$ $(B \in \mathscr{B})$. *Then* N *is an* $\ell \times n$ - *matrix-valued measure.*

(b) *If* Ψ *is a* $\mathscr{B}$*-measurable* $k \times \ell$*-matrix-valued function on* Ω *and* N *is as in* (a), *then for each* $B \in \mathscr{B}$, $\int_B \Psi\, dN$ *exists if and only if* $\int_B \Psi\, \Phi dM$ *exists. If the two integrals exist, they are equal.*

4. TOTAL VARIATION AND THE HAHN-JORDAN DECOMPOSITION

There are several ways of defining the total variation of a complex-valued measure μ. The form we have chosen to generalize is as follows: Let ν be any nonnegative measure such that $\mu \ll \nu$. (For example ν may be the sum of the total variations of the real and imaginary parts of μ.) Then the total variation of μ is given by

$$|\mu|(B) = \int_B \left|\frac{d\mu}{d\nu}\right| d\nu \qquad (B \in \mathscr{B}). \tag{4.1}$$

We have not investigated what form the resulting theory will take for a different definition of $|\mu|$.

4.2 LEMMA. *Let* M *be an* $n \times m$*-matrix-valued measure on* $\mathscr{B}$, *and let* μ *be a nonnegative measure such that* $M \ll \mu$. *Then*

(a) $\sqrt{M_\mu'^* M_\mu'}$ *is in* $L_1(\Omega, \mathscr{B}, \mu)$, *and*

(b) $\int_B \sqrt{M_\mu'^* M_\mu'}\, d\mu$ *is independent of* μ, *for all* $B \in \mathscr{B}$.

Proof. (a) Of course, by $\sqrt{M_\mu'^* M_\mu'}$ we mean the unique nonnegative hermitian matrix H such that $H^2 = M_\mu'^* \cdot M_\mu'$. Thus, looking at the diagonal elements of the product, we see that

$$|H_{ij}| \leq \sqrt{\sum_{k=1}^{n} |M'_{\mu_{ik}}|^2} \leq \sum_{k=1}^{n} |M'_{\mu_{ik}}|,$$

and that the last quantity is in $L_1(\Omega, \mathscr{B}, \mu)$. Since H is clearly measurable, it is in $L_1(\Omega, \mathscr{B}, \mu)$.

(b) Let μ and ν be two positive measures such that $M \ll \mu$ and $M \ll \nu$; then $\mu + \nu$ is also a positive measure such that $M \ll \mu + \nu$, and $\mu \ll \mu + \nu$ and $\nu \ll \mu + \nu$. We shall show that

$$\int_B \sqrt{M_\mu'^* \cdot M_\mu'}\, d\mu = \int_B \sqrt{M_{\mu+\nu}'^* \cdot M_{\mu+\nu}'}\, d(\mu + \nu),$$

and the theorem will follow by symmetry. But the last equality follows by the usual chain rule for Radon-Nikodým derivatives. ■

We are thus led to define the *total matricial variation* $|M|$ of an $n \times m$-matrix-valued measure M as the $m \times m$ nonnegative hermitian matrix-valued measure such that $|M|(B) = \int_B \sqrt{M_\mu'^* \cdot M_\mu'}\, d\mu$, for each $B \in \mathscr{B}$ and for some nonnegative measure μ on $\mathscr{B}$ such that $M \ll \mu$.

As is easily seen, $\mathscr{R}(|M|'_\mu) = \mathscr{R}(M'_\mu)$ a.e. μ, and a convenient choice of μ is trace $|M|$. Now let σ = trace $|M|$. By the polar decomposition 2.17, there exists a partial isometry $V(\omega)$ such that $M'_\sigma(\omega) = V(\omega) \cdot |M|'_\sigma(\omega)$. It is clear from the construction that V and $|M|'_\sigma$ are measurable. By integration we therefore get the following proposition.

4.3 HAHN-JORDAN DECOMPOSITION. *Let* M *be any* $m \times n$ - *matrix-valued measure. Then* $M(B) = \int_B V\, d|M|$ $(B \in \mathscr{B})$, *where* V *is partial-isometry valued and* $|M|$ *is the total matricial variation of* M.

When M is a complex-valued measure, V reduces to a complex-valued function with absolute value 1 for all ω. The Hahn-Jordan Decomposition becomes $M(B) = \int_B e^{i\phi(\omega)} d|M|$, where ϕ is a real-valued function. When M is real-valued, $e^{i\phi(\omega)}$ takes only the values ± 1, and $M(B) = |M|(B \cap A) - |M|(B \cap A')$, where $A = \{\omega : e^{i\phi(\omega)} = 1\}$ is the carrier of the positive part of M. We thus recover the classical form.

It is easy to see that even if $\int_B \Psi\, dM = 0$ for all $B \in \mathscr{B}$, the function Ψ need not vanish a.e. M. Example:

$$M(B) = \int_B \begin{bmatrix} 1 & \omega \\ \omega & \omega^2 \end{bmatrix} d\omega, \quad \Psi(\omega) = \begin{bmatrix} \omega^2 & -\omega \\ -\omega & \omega \end{bmatrix} \quad (\omega \in [0, 1], B \subseteq [0, 1]).$$

This suggests the introduction of a new notion of equivalence of matrix-valued functions with respect to a matrix-valued measure, based on the following lemma.

4.4 LEMMA. *Let* M *be an* $m \times n$ - *matrix-valued measure, and let* μ *be a nonnegative measure such that* $M \ll \mu$. *Also, let* J_μ *be a* $\mathscr{B}$-*measurable* $m \times n$ - *matrix-valued function such that* $J_\mu(\omega)$ *is a projection along* $\mathscr{N}(M'_\mu(\omega))$ *for almost all* ω (μ-*measure*). *Then, for each* $\mathscr{B}$-*measurable* $\ell \times m$ - *matrix-valued function* Φ, $\int_B \Phi\, dM = 0$ *for all* $B \in \mathscr{B}$ *if and only if* $\Phi \cdot J_\mu = 0$ a.e. μ.

This lemma follows immediately from 2.13 and the fact that $\mathscr{N}(J_\mu(\omega)) = \mathscr{N}(M'_\mu(\omega))$ for almost all ω (μ-measure).

Let $\sigma(M)$ = trace $|M|$; then $\sigma(M)$ is a nonnegative measure such that $M \ll \sigma(M)$. Let $J_M = M'_{\sigma(M)} \cdot M'^{\#}_{\sigma(M)}$. By (2.4), $J_M(\omega)$ is the orthogonal projection onto $\mathscr{N}(M'_{\sigma(M)}(\omega))^{\perp}$, and hence $\mathscr{N}(J_M(\omega)) = \mathscr{N}(M'_{\sigma(M)}(\omega))$. Hence, by 4.4, $\int_B \Phi\, dM = 0$ for all B in $\mathscr{B}$ if and only if $\Phi \cdot J_M = 0$ a.e. $\sigma(M)$.

We now define M-equivalence. If Φ and Ψ are $\ell \times m$ - and $k \times m$ - matrix-valued functions, respectively, then we say that Φ and Ψ are *equivalent with respect to* M ($\Phi \equiv \Psi$ (mod M)) if and only if $\Phi \cdot J_M = \Psi \cdot J_M$ a.e. $\sigma(M)$.

4.5 THEOREM. (a) *If* $\int_B \Phi\, dM$ *exists and* $\Phi \equiv \Psi$ (mod M), *then* $\int_B \Psi\, dM$ *exists and* $\int_B \Phi\, dM = \int_B \Psi\, dM$.

(b) $\int_B (\Phi - \Psi)\, dM = 0$ *for all* B *in* $\mathscr{B}$ *if and only if* $\Phi \equiv \Psi$ (mod M).

5. ABSOLUTE CONTINUITY AND THE RADON-NIKODÝM THEOREM

A direct generalization of absolute continuity would amount to the assertion that N is absolutely continuous with respect to M if $N(B) = 0$ whenever $B \in \mathscr{B}$ and $|M|(B) = 0$. But this definition is inadequate for our purposes; for instance, it does not distinguish between matrices of different ranks. A more appropriate definition: N is *absolutely continuous* with respect to M ($N \ll M$) if

$$\mathscr{R}(|N|(B)) \subseteq \mathscr{R}(|M|(B)) \quad \text{for all } B \text{ in } \mathscr{B}.$$

It is clear that if $N \ll M$, then N is absolutely continuous with respect to M in the first sense mentioned above. Also, the following elementary properties are easily verified for either concept.

5.1 LEMMA. (a) *The relation* $\ll$ *is reflexive and transitive.*

(b) $N \ll M$ *if and only if* $N \ll |M|$.

(c) $N \ll M$ *if and only if* $|N| \ll M$.

This concept of absolute continuity is still not strong enough to yield a Radon-Nikodým theorem, as the following example shows.

5.2 *Example.* Let μ be Lebesgue measure on the family $\mathscr{B}$ of Borel subsets of $\Omega = [0, 1]$, and let

$$M(B) = \int_B \begin{bmatrix} 1 & \omega \\ \omega & \omega^2 \end{bmatrix} \mu(d\omega), \quad N(B) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \mu(B) \qquad (B \in \mathscr{B}).$$

Using 3.2(a), we easily see that $\mathscr{R}(M(B)) = \mathscr{C}^2 = \mathscr{R}(N(B))$ for $B \in \mathscr{B}$ such that $\mu(B) > 0$, and that $\mathscr{R}(M(B)) = \{0\} = \mathscr{R}(N(B))$ if $\mu(B) = 0$. Thus $N \ll M$. But there is no 2×2-matrix-valued function Φ such that $N'_\mu = \Phi \cdot M'_\mu$, since N'_μ has rank 2 and M'_μ has rank 1. Thus there is no Φ such that $N_\mu(B) = \int_B \Phi \cdot dM$ for all $B \in \mathscr{B}$.

We now introduce a stronger concept. We say that N is *strongly absolutely continuous with respect* to M ($N \lll M$) if and only if there exists a nonnegative measure μ such that $N \ll \mu$, $M \ll \mu$, and $\mathscr{R}(N'_\mu(\omega)) \subseteq \mathscr{R}(M'_\mu(\omega))$ for almost all ω (μ-measure). By methods similar to those used in the proof of 4.2, it can be shown that if $N \lll M$, then for *any* nonnegative measure μ such that $N \ll \mu$, $M \ll \mu$, we have the inclusion $\mathscr{R}(N'_\mu(\omega)) \subset \mathscr{R}(M'_\mu(\omega))$ for almost all ω (μ-measure). It is also easy to verify that the assertions of Lemma 5.1 are valid for the relation $\lll$. The following lemma gives some of the other properties.

5.3 LEMMA. (a) *If* N, P $\lll$ M *and if* A *and* B *are constant matrices such that* $AN + BP$ *is defined, then* $AN + BP \lll M$.

(b) *If* $N \lll M$, *then* $N \ll M$.

The proof of (a) is straightforward, and (b) follows from 3.2 and the fact that $\mathscr{R}(M'_\mu(\omega)) = \mathscr{R}(|M|'_\mu(\omega))$ for almost all ω (μ-measure).

For the concept of strong absolute continuity, the Radon-Nikodým theorem holds, and in fact becomes almost a triviality:

5.4 RADON-NIKODÝM THEOREM. *Let* M *and* N *be* $\ell \times n$- *and* $m \times n$-*matrix-valued measures on* $\mathscr{B}$: $N \lll M$ *if and only if there exists an* $\ell \times m$-*matrix-valued function* Φ *such that* Φ *is in* $L_1(\Omega, \mathscr{B}, M)$ *and such that*

$N(B) = \int_B \Phi \cdot dM$ *for all* B *in* $\mathscr{B}$. *This* Φ *is unique* (mod M); *in fact*, $\Phi = N'_\mu \cdot (M'_\mu)^\#$ (mod M), *where* μ *is any nonnegative measure such that* $M \ll \mu$.

Proof. The uniqueness of Φ (mod M) follows from 4.5(b). The assertion that $N(B) = \int_B \Phi \cdot dM$ for all B in $\mathscr{B}$ is clearly equivalent to the relation $N'_\mu = \Phi M'_\mu$ a.e. μ, which implies $\mathscr{R}(N'_\mu(\omega)) \subseteq \mathscr{R}(M'_\mu(\omega))$ for almost all ω (μ-measure). Suppose, therefore, that $\mathscr{R}(N'_\mu(\omega)) \subseteq \mathscr{R}(M'_\mu(\omega))$ for almost all ω (μ-measure), and let $\Phi = N'_\mu \cdot (M'_\mu)^\#$. Then

$$\int_B \Phi\, dM = \int_B N'_\mu \cdot M'^{\#}_\mu M'_\mu\, d\mu = \int_B N'_\mu \cdot P_{\mathscr{R}(M'_\mu)}\, d\mu = \int_B N'_\mu\, d\mu = N(B). \blacksquare$$

If $N \lll M$, we therefore define the *Radon-Nikodým derivative* $dN \cdot dM^\#$ of N with respect M by the condition $(dN \cdot dM^\#)(\omega) = N'_\mu(\omega) \cdot M'_\mu(\omega)^\#$ up to sets of μ-measure zero.

Next we wish to extend the classical result that if $\nu \ll \mu$, then $\mu \ll \nu$ if and only if $\frac{d\nu}{d\mu} > 0$ a.e. μ, and that in this event $\frac{d\mu}{d\nu} = 1/\left(\frac{d\nu}{d\mu}\right)$.

5.5 THEOREM. *Let* M *and* N *be matrix-valued measures on* $\mathscr{B}$ *such that* $N \lll M$. *Then* $M \lll N$ *if and only if*

$$\operatorname{rank}(dN \cdot dM^\#(\omega)) = \operatorname{rank}(M'_\mu(\omega)) \text{ a.e. } \mu,$$

and in this event $dM \cdot dN^\# = (dN \cdot dM^\#)^\#$.

Proof. If $M \lll N$, then $\mathscr{R}(N'_\mu(\omega)) = \mathscr{R}(M'_\mu(\omega)) = \mathscr{N}(M'_\mu(\omega)^\#)^\perp$ for almost all ω (μ-measure). Thus $\operatorname{rank}(dN \cdot dM^\#(\omega)) = \operatorname{rank}(M'_\mu(\omega)^\#) = \operatorname{rank}(M'_\mu(\omega))$ for almost all ω (μ-measure) as desired. Suppose next that $\operatorname{rank}(dN \cdot dM^\#) = \operatorname{rank}(M'_\mu)$ a.e. μ. Since $\mathscr{R}(N'_\mu) \subseteq \mathscr{N}(M'^{\#}_\mu)^\perp$ a.e. μ, it follows that $\mathscr{R}(N'_\mu) = \mathscr{N}(M'^{\#}_\mu)^\perp$ a.e. μ. Thus $\mathscr{R}(M'_\mu) = \mathscr{R}(N'_\mu)$ a.e. μ, and hence $M \lll N$. The formula for $dM \cdot dN^\#$ follows from (2.8) and (2.6). $\blacksquare$

Unfortunately, it is not the case that if $N \lll M \lll N$ and $\Phi \equiv dN \cdot dM^\#$ (mod M), then $\Phi^\# \equiv dM \cdot dN^\#$ (mod N). This is shown in the next example:

5.6 *Example.* Let $\Omega = \{1\}$ consist of one point, and set

$$N\{1\} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M\{1\} = \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix}, \quad \mu\{1\} = 1, \quad \Phi = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \Phi^\# = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \end{bmatrix}.$$

Then $N = \Phi M$ but $M \neq \Phi^\# N$, and hence $\Phi \equiv dN \cdot dM^\#$ (mod M) but $\Phi^\# \not\equiv dM \cdot dN^\#$ (mod N).

Related to this behavior is the fact that the conditions $\Phi \equiv \Psi$ (mod M) and $N \lll M$ do not imply that $\Phi \equiv \Psi$ (mod N), as is shown by the following example.

5.7 *Example.* In example 5.6, $\Phi \equiv N$ (mod N), but $\Phi \not\equiv N$ (mod M). To see this, note that $K = N$ is a projection such that $\mathscr{N}(K) = \mathscr{N}(N)$ and $\Phi \cdot K = N \cdot K$, while

$J = \frac{1}{13}\begin{bmatrix} 9 & 6 \\ 6 & 4 \end{bmatrix}$ is a projection such that $\mathcal{N}(J) = \mathcal{N}(M)$ but $\Phi \cdot J \neq N \cdot J$. Thus, taking $\Psi = N$, we see that $\Phi \not\equiv \Psi \pmod{M}$ even though $\Phi \equiv \Psi \pmod{N}$ and $M \lll N$.

However, we do have the following positive result.

5.8 THEOREM. *If* $\Phi \equiv \Psi \pmod{M}$ *and* $N \lll M$, *and if* $N'_\mu(\omega)$ *and* $M'_\mu(\omega)$ *are normal matrices for almost all* ω *(*μ*-measure), then* $\Phi \equiv \Psi \pmod{N}$.

Proof. Clearly, $\mathcal{R}(N'^*_\mu(\omega)) \subseteq \mathcal{R}(M'^*_\mu(\omega))$ for almost all ω (μ-measure), since for a normal matrix N, $\mathcal{R}(N) = \mathcal{R}(N^*)$. Hence $\mathcal{N}(M'_\mu(\omega)) \subseteq \mathcal{N}(N'_\mu(\omega))$ for almost all ω (μ-measure). Therefore, if $\Phi \cdot P_{\mathcal{N}(M'_\mu)^\perp} = \Psi \cdot P_{\mathcal{N}(M'_\mu)^\perp}$ a.e. μ, then

$\Phi \cdot P_{\mathcal{N}(N'_\mu)^\perp} = \Psi \cdot P_{\mathcal{N}(N'_\mu)^\perp}$ a.e. μ. ∎

We close this section with a brief discussion of the Besicovitch derivatives (see [1]). The following example shows that in general the Besicovitch pointwise procedure (or the more abstract derivation procedure using nets; see [13, p. 152]) is not satisfactory.

5.9 *Example.* Suppose that (i) μ, M, and N are defined as in Example 5.2, (ii) for each $\omega \in (0, 1]$, the B_n^ω $(n \geq 1)$ are symmetric neighborhoods of ω such that $\mu(B_n^\omega) \to 0$, (iii) $D_n^M(\omega) = M(B_n^\omega)/\mu(B_n^\omega)$, $D_n^N(\omega) = N(B_n^\omega)/\mu(B_n^\omega)$. Then it follows from [13, p. 151] that

$$\lim_{n\to\infty} D_n^M = M'_\mu \text{ a.e. } \mu \quad \text{and} \quad \lim_{n\to\infty} D_n^N = N'_\mu \text{ a.e. } \mu .$$

One would expect that

$$N(B_n^\omega)\cdot M(B_n^\omega)^\# = D_n^N(\omega)\cdot D_n^M(\omega)^\#$$

converges to $N'_\mu(\omega)\cdot M'_\mu(\omega)^\#$ for almost all ω (μ-measure). Unfortunately, this is not so, for,

$$\operatorname{rank}\{N(B_n^\omega)\cdot M(B_n^\omega)^\#\} = 2, \qquad \operatorname{rank}\{N'_\mu(\omega)\cdot M'_\mu(\omega)^\#\} = 1,$$

and hence by 2.12 $D_n^{M\#}$ does not converge a.e. μ.

The following theorem provides a sufficient condition for Besicovitch differentiability.

5.10 THEOREM. *Let* M *and* N *be matrix-valued measures on the* σ*-algebra* $\mathcal{B}$ *of all Borel subsets of Euclidean* k*-space* Ω. *Let* B_n^ω *be open spheres with common center* ω *and with radii* $r_n \to 0$, *and suppose that with the notation of* 5.9(iii),

$$\lim_{n\to\infty} \operatorname{rank} D_n^M = \operatorname{rank} M'_\mu \text{ a.e. } \mu,$$

where μ *is any nonnegative measure on* $\mathcal{B}$ *such that* $M, N \ll \mu$. *Then* $N \ll M$ *implies*

$$\lim_{n\to\infty} N(B_n^\omega)M(B_n^\omega)^\# = (dN\cdot dM^\#)(\omega) \text{ a.e. } \mu .$$

The proof follows immediately from 2.12. We note that in general

$$\liminf_{n \to \infty} \operatorname{rank} D_n^M \geq \operatorname{rank} M'_\mu,$$

and that if M'_μ has full rank a.e. μ, then the condition is automatically satisfied.

6. SINGULARITY AND THE LEBESGUE DECOMPOSITION

If M and N are complex-valued measures, then M and N are mutually singular ($M \perp N$) if and only if there exists a nonnegative measure μ such that $M, N \ll \mu$ and $\frac{dM}{d\mu} \cdot \frac{dN}{d\mu} = 0$ a.e. μ. The following matricial definition is therefore reasonable.

6.1 *Definition.* If M and N are matrix-valued measures, then M and N are *mutually singular* ($M \perp N$) if and only if there exists a nonnegative measures μ such that

$$M, N \ll \mu \quad \text{and} \quad \mathscr{R}(M'_\mu) \cap \mathscr{R}(N'_\mu) = \{0\} \text{ a.e. } \mu .$$

As we have shown above (see 4.2), $\mathscr{R}(M'_\mu)$ is essentially independent of μ, since

$$M'_{\mu+\nu} = M'_\mu \frac{d\mu}{d(\mu + \nu)} \text{ a.e. } \mu + \nu .$$

Hence, if $M \perp N$ and μ is any nonnegative measure such that $M, N \ll \mu$, then $\mathscr{R}(M'_\mu) \cap \mathscr{R}(N'_\mu) = \{0\}$ a.e. μ.

Unfortunately, Definition 6.1 is inadequate for our purposes: as will be seen from the next example, it does not insure the uniqueness of a Lebesgue decomposition.

6.2 *Example.* Let $\Omega = \{1\}$ have one point, and set

$$N\{1\} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}, \quad M\{1\} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}.$$

Then we can write two distinct decompositions $N = N_1 + N_2$ with $N_1 \lll M$ and $N_2 \perp M$, namely

$$N\{1\} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \quad \text{and} \quad N\{1\} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}.$$

The following definition does yield a *unique* Lebesgue decomposition, as we shall see in Theorem 6.7.

6.3 *Definition.* Let H be a nonnegative hermitian matrix-valued measure, and let M and N be matrix-valued measures on $\mathscr{B}$. Then we shall say that M and N are *mutually* H-*singular* ($M \perp_H N$) if and only if $M, N \lll H$ and

$$(6.4) \qquad \int_B (dM \cdot dH^{\#})\, dH\, (dN \cdot dH^{\#})^* = 0 \quad \text{for all } B \text{ in } \mathscr{B}.$$

It is easy to verify that if μ is a nonnegative measure and $H \ll \mu$, then (6.4) is equivalent to each of the two conditions

$$(6.5) \qquad (dM \cdot dH^{\#}) H'_{\mu} (dN \cdot dH^{\#})^{*} = 0 \text{ a.e. } \mu,$$

$$(6.6) \qquad M'_{\mu} \cdot H'^{\#}_{\mu} \cdot N'^{*}_{\mu} = 0 \text{ a.e. } \mu.$$

The following result clarifies the connection between the two concepts of singularity.

6.7 LEMMA. *Let* M *and* N *be matrix-valued measures on* $\mathscr{B}$. *Then the following conditions are equivalent:*

(a) $M \perp N$.

(b) $M \perp_H N$, *where* $H = |M| + |N|$.

(c) *There exists a nonnegative hermitian matrix-valued measure* H *on* $\mathscr{B}$ *such that* $M \perp_H N$.

Proof. Let (a) hold. Then

$$\mathscr{R}(M'_{\mu}) = \mathscr{R}(|M|'_{\mu}) \text{ a.e. } \mu, \quad \mathscr{R}(N'_{\mu}) = \mathscr{R}(|N|'_{\mu}) \text{ a.e. } \mu, \quad H'_{\mu} = |M|'_{\mu} + |N|'_{\mu} \text{ a.e. } \mu.$$

Thus $\mathscr{R}(|M|'_{\mu}) \cap \mathscr{R}(|N|'_{\mu}) = \{0\}$ a.e. μ, and using 2.16 and (6.6), we see that $M \perp_H N$. Thus (b) holds.

That (b) implies (c) is trivial.

To show that (c) implies (a), we note that

$$\mathscr{R}(M'_{\mu}) \subseteq \mathscr{R}(H'_{\mu}) \text{ a.e. } \mu, \quad \mathscr{R}(N'_{\mu}) \subseteq \mathscr{R}(H'_{\mu}) \text{ a.e. } \mu,$$

$$M'_{\mu} \cdot H'^{\#}_{\mu} \cdot N'^{*}_{\mu} = 0 \text{ a.e. } \mu \quad \text{(by (6.6))}.$$

Hence $\mathscr{R}(M'_{\mu}) \cap \mathscr{R}(N'_{\mu}) = \{0\}$ a.e. μ, by 2.16. ■

The following four elementary properties of singularity are easy to establish, and we shall omit their proofs.

(6.8) $M \perp_H N$ *if and only if* $N \perp_H M$.

(6.9) *If* $M \perp_H R$, $N \perp_H R$, *and* A *and* B *are constant matrices such that* $AM + BN$ *is defined, then* $AM + BN \perp_H R$.

(6.10) *If* $M \perp_H N$ *and* $R \lll M$, *then* $R \perp_H N$.

(6.11) *If* $M \perp_H N$ *and* $M \lll N$, *then* $M = 0$.

Next we discuss the concept of the "carrier" of a matrix-valued measure. If ν is a complex-valued measure and μ is a nonnegative measure such that $\nu \ll \mu$, then the carrier of ν with respect to μ is the set $C = \left\{ \omega : \frac{d\nu}{d\mu}(\omega) \neq 0 \right\}$, which is only defined up to null sets (μ-measure). Now we can deal just as readily with the indicator function 1_C of C, instead of C. The appropriate generalization of an indicator function is a projection-valued function. Now let M be a matrix-valued measure, and let H be a nonnegative hermitian matrix-valued measure such that $M \lll H$. We shall now define a projection-valued function that will correspond to the indicator function of the carrier of M with respect to H. First we need the following result, which follows from 2.15.

6.12 LEMMA. *Let* M *and* H *be defined as in* 6.3, *and let* τ = trace H. *Then*

(a) $\mathscr{R}(dM \cdot dH^{\#}) \cap (\mathscr{R}(M_\tau'))^{\perp}) = \{0\}$ a.e. τ,

(b) $\mathscr{R}(dM \cdot dH^{\#}) + (\mathscr{R}(M_\tau'))^{\perp}) = C^n$ a.e. τ.

Lemma 6.12 yields the existence of a projection $J_{H,M}(\omega)$ onto $\mathscr{R}(dM \cdot dH^{\#}(\omega))$ along $\mathscr{R}(M_\tau'(\omega))^{\perp}$ for almost all ω (τ-measure). We shall call $J_{H,M}$ the H-*carrier function of* M. It can be expressed in the explicit form

(6.13) $$J_{H,M}(\omega) = M_\tau'(\omega)^* (M_\tau'(\omega) H_\tau'(\omega)^{\#} M_\tau'(\omega)^*)^{\#} M_\tau'(\omega) H_\tau'(\omega)^{\#} \text{ for almost all } \omega \ (\tau\text{-measure}).$$

6.14 LEBESGUE DECOMPOSITION. *Let* M *and* N *be matrix-valued measures on* $\mathscr{B}$, *and let* H *be a nonnegative hermitian matrix-valued measure on* $\mathscr{B}$ *such that* M, N $\lll$ H *and* $\int_\Omega (dN \cdot dH^{\#}) J_{H,M} dH$ *exists. Then*

(i) *there exist unique matrix valued measures* N_a *and* N_s *on* $\mathscr{B}$ *such that*

$$\text{(a) } N = N_a + N_s, \quad \text{(b) } N_a \lll M, \quad \text{(c) } N_s \perp_H M;$$

(ii) *for all* $B \in \mathscr{B}$, N_a *and* N_s *are given by the formulas*

$$N_a(B) = \int_B (dN \cdot dH^{\#}) J_{H,M} dH, \qquad N_s(B) = \int_B (dN \cdot dH^{\#})(I - J_{H,M}) dH.$$

Proof. The uniqueness follows easily from the properties (6.8) to (6.11). It therefore remains to show that N_a and N_s as defined in (ii) satisfy (a), (b), and (c) of (i). (a) is obvious. To prove that $N_a \lll M$ and $N_s \perp_H M$, we must show that

$$\mathscr{R}(N_{a,\tau}') \subseteq \mathscr{R}(M_\tau') \quad \text{and} \quad N_{s,\tau}' \cdot H_\tau'^{\#} \cdot M_\tau'^{*} = 0 \quad \text{a.e. } \tau,$$

where τ = trace H. We see that $N_{a,\tau}' = N_\tau' H_\tau'^{\#} J_{H,M} H_\tau'$ and thus

$$\mathscr{R}(N_{a,\tau}') \subseteq \mathscr{R}(J_{H,M} H_\tau') = \mathscr{R}(M_\tau' H_\tau'^{\#} \cdot H_\tau') = \mathscr{R}(M_\tau') \text{ a.e. } \tau,$$

the last equality being since $\mathscr{R}(M_\tau') \subseteq \mathscr{R}(H_\tau')$ a.e. τ. Likewise,

$$N_{s,\tau}' H_\tau'^{\#} M_\tau'^{*} = N_\tau' H_\tau'^{\#} (I - J_{H,M}) H_\tau' H_\tau'^{\#} M_\tau'^{*} \text{ a.e. } \tau.$$

But $H_\tau' H_\tau'^{\#} M_\tau'^{*} = (M_\tau' \cdot P_{\mathscr{R}(H_\tau')})^* = M_\tau'^{*}$ a.e. τ, since $\mathscr{R}(M_\tau') \subseteq \mathscr{R}(H_\tau')$. Also, $(I - J_{H,M}) M_\tau'^{*} = 0$ a.e. τ, since $I - J_{H,M}$ is a projection onto $\mathscr{R}(M_\tau')^{\perp} = \mathscr{N}(M_\tau'^{*})$. ∎

In [2, IV, Theorem 2] Cramér obtained the Lebesgue decomposition of a nonnegative hermitian measure with respect to Lebesgue measure over the real line. This is an easy consequence of our last theorem.

6.15 COROLLARY (CRAMÉR'S THEOREM). *Let* N *be a nonnegative hermitian matrix-valued measure, and let* μ *be a nonnegative* σ*-finite measure on the family of Borel subsets of* $(-\infty, \infty)$. *Then there exist unique matricial measures* N_a *and* N_s *such that* $N = N_a + N_s$, $N_a \lll \mu I$, $N_s \perp \mu I$, *and* N_a *and* N_s *are nonnegative hermitian measures.*

Proof. It is sufficient to show that with $M = \mu I$, the N_a and N_s given in Theorem 6.14 are independent of H. Let H be defined as in 6.14, and let τ = trace H. Then, a.e. τ,

$$J_{H,M} = \begin{cases} I & \text{if } \frac{d\mu}{d\tau} \neq 0, \\ 0 & \text{if } \frac{d\mu}{d\tau} = 0, \end{cases}$$

and thus

$$N_a(B) = \int_{B \cap \left(\frac{d\mu}{d\tau} \neq 0\right)} N'_\tau H'^{\#}_\tau H'_\tau \, d\tau \qquad (B \in \mathscr{B}).$$

But if $\frac{d\mu}{d\tau} \neq 0$, then $H'^{\#}_\tau H'_\tau = I$; since $M'_\tau = \frac{d\mu}{d\tau} I$ and $M \lll H$. Therefore $N_a(B) = N\left(B \cap \left(\frac{d\mu}{d\tau} \neq 0\right)\right)$. That the last quantity is independent of τ follows from the fact that $N \ll \tau$. ■

7. APPLICATIONS TO STOCHASTIC PROCESSES

In this section, we shall indicate briefly how the ideas developed in this paper help to illuminate some results in the theory of multivariate, weakly stationary stochastic processes. We assume that the reader is familiar with the subject (see [8], for example).

The following two results are stated by Masani [8; 8.7, 8.8]:

7.1 THEOREM. *Suppose that* (i) $f, g \in H^q$, (ii) $\hat{g} = (g \mid M_\infty^{(f)})$, (iii) $\Phi_{\hat{g}} \in L_{2,M_{ff}}$ *is the isomorph of* $\hat{g}$, *in other words,*

$$\hat{g} = \int_0^{2\pi} \Phi_{\hat{g}} \, dE f_0 .$$

Then

(a) $M_{gf}(B) = M_{\hat{g}f}(B) = \int_B \Phi_{\hat{g}} \, dM_{ff} \qquad (B \in \mathscr{B}),$

(b) $M_{\hat{g}\hat{g}}(B) = \int_B \Phi_{\hat{g}} \, dM_{ff} \Phi_{\hat{g}}^* = \int_B \Phi_{\hat{g}} \, dM_{f\hat{g}} \qquad (B \in \mathscr{B}).$

7.2 COROLLARY. g *is subordinate to* f *if and only if there exists* Φ *in* $L_{2,M_{ff}}$ *such that for each* $B \in \mathscr{B}$

$$M_{gf}(B) = \int_B \Phi \, dM_{ff}, \qquad M_{gg}(B) = \int_B \Phi \, dM_{ff} \Phi^* .$$

In these results, the functions $\Phi_{\hat{g}}$ and Φ are not identified. In the light of our earlier discussion, we now see that they are essentially Radon-Nikodým derivatives, in fact, that

$$\Phi_{\hat{g}} \equiv dM_{\hat{g}\hat{g}} \cdot dM^{\#}_{f\hat{g}} \pmod{M_{f\hat{g}}}, \qquad \Phi \equiv dM_{gf} \cdot dM^{\#}_{ff} \pmod{M_{ff}} .$$

Our concept of singularity likewise enables us to state a multivariate generalization of results due to Kolmogorov [7, Theorems 12, 14].

7.3 THEOREM. *Let* x, y, *and* z *be* q*-variate weakly stationary stochastic processes such that* $x = y + z$. *Let* M_x, M_y, *and* M_z *denote their spectral measures. If* y *and* z *are orthogonal processes, then* y *and* z *are subordinate to* x *if and only if* $M_y \perp_{M_x} M_z$. *If* y *and* z *are subordinate to* x, *then* y *and* z *are orthogonal if and only if* $M_y \perp_{M_x} M_z$.

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University of California, Santa Barbara, California 93106
and
University of Kansas, Lawrence, Kansas 66044