DEGREE OF APPROXIMATION BY RATIONAL FUNCTIONS AND POLYNOMIALS

J. L. Walsh

In a recent paper [1], Newman proved the striking result that the function $|x|$ can be uniformly approximated on the interval $[-1, 1]$ by rational functions of degree n, with an error $O(e^{-\sqrt{n}})$. This represents much more rapid convergence than the error $O(1/n)$ for best approximation by polynomials that is given (for the same function and the same interval) by the theory of Jackson, Bernstein, Montel, and de La Vallée Poussin.

The special rational functions used by Newman have the origin as a limit point of their poles. This can be shown without reference to the precise formulas involved; indeed, we now formulate a general theorem relevant to this topic.

THEOREM. *If the function* $f(z)$ *is approximable on a closed Jordan arc* C *to the order* $n^{-\alpha}$ $(\alpha > 0)$ *by rational functions* $Q_n(z)$ *(of degree* n*) whose poles have no limit point on* C*, then* $f(z)$ *is also approximable on* C *to the order* $n^{-\alpha}$ *by polynomials* $p_n(z)$ *of degree* n.

In Newman's case, this theorem applies to each closed subinterval of $[-1, 1]$ containing the origin in its interior, and it shows that rational functions of respective degrees n having no limit point of poles on the subinterval can not converge to $|x|$ with approximation of order $n^{-\alpha}$ $(\alpha > 1)$, on the subinterval. Thus Newman's rational functions must have at least one limit point of poles on each such subinterval; hence the origin must be a limit point of poles.

The theorem is an immediate consequence of the following two propositions (see [2, Section 9.7, Lemma 1] and [3, Theorem 1]).

LEMMA 1. *Let* C *be a nondegenerate, bounded continuum whose complement* K *is simply connected. Let* $w = \phi(z)$ *map* K *conformally onto the domain* $|w| > 1$, *with* $\phi(\infty) = \infty$, *and for* $R > 1$ *let* $C(R)$ *denote the preimage of the circle* $|w| = R$. *If* $Q(z)$ *is a rational function of degree* n *whose poles lie on the curve* $C(R_0)$ *or in its exterior, for some* $R_0 > 1$, *and if* $|Q(z)| \leq M$ *on* C, *then*

$$|Q(z)| \leq M\left(\frac{R_0 R - 1}{R_0 - R}\right)^n$$

on the curve $C(R)$, *for* $1 < R < R_0$.

LEMMA 2. *Let* $\{\varepsilon_n\}$ *be a sequence of positive numbers such that*

$$\varepsilon_{[\mu n]} = O(\varepsilon_n) \quad \textit{and} \quad r^n = O(\varepsilon_n)$$

for each positive μ *and each* r $(0 < r < 1)$. *Let* D *be a domain,* C *a Jordan arc in* D, $f(z)$ *a function defined on* C, *and* $\{f_n(z)\}$ *a sequence of functions that are analytic in* D. *If*

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$$|f(z) - f_n(z)| \leq A_1 \varepsilon_n \qquad (z \in C)$$

and

$$|f_n(z)| \leq A_2 M_0^n \qquad (z \in D)$$

for some constants A_1, A_2, M_0 *and for* $n = 1, 2, \cdots$, *then there exist polynomials* $p_n(z)$ *of degree* n $(n = 1, 2, \cdots)$ *such that for some constant* A_3

$$|f(z) - p_n(z)| \leq A_3 \varepsilon_n \qquad (z \in C).$$

We note that every sequence $\{\varepsilon_n\} = \{n^{-\alpha}\}$ $(\alpha > 0)$ satisfies the restriction in Lemma 2.

REFERENCES

1. D. J. Newman, *Rational approximation to* $|x|$, Michigan Math. J. 11 (1964), 11-14.

2. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain,* Amer. Math. Soc. Colloquium Publications 20 (1935).

3. ———, *Note on polynomial approximation on a Jordan arc,* Proc. Nat. Acad. Sci. U.S.A. 46 (1960), 981-983.

University of Maryland
College Park, Maryland 20740