UNKNOTTING POLYHEDRAL HOMOLOGY MANIFOLDS

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1. INTRODUCTION

Gugenheim [1] showed in 1953 that an n-dimensional polyhedron unknots piecewise linearly in Euclidean k-space E^k if $k \geq 2n + 2$. For piecewise linear manifolds, this result can be improved upon by one dimension—Zeeman's unknotting theorem [11, Theorem 24] includes the fact that every connected, closed, piecewise linear n-manifold $(n \geq 2)$ unknots piecewise linearly in E^{2n+1}, as well as the fact that every 1-connected, closed, piecewise linear n-manifold $(n \geq 3)$ unknots in E^{2n}. The principal object of the present paper is to establish results of this sort for a larger class of polyhedra that includes all polyhedral homology manifolds, and hence all triangulated, closed, topological manifolds.

We say that the polyhedron X *strongly unknots* in E^k if, given two imbeddings f and g of X into E^k which agree on a subpolyhedron Y of X, there exists an ambient isotopy of E^k which transforms f into g, while leaving pointwise-fixed the image of Y. We prove that an n-dimensional polyhedron X $(n \geq 2)$ strongly unknots in E^{2n+1} if $H^n(X - p) = 0$ for each point $p \in X$ (Theorem 3). It follows that every compact, connected polyhedral homology n-manifold $(n \geq 2)$ strongly unknots in E^{2n+1}. This result is then used to prove that if M is either a connected, orientable polyhedral homology n-manifold $(n \neq 2)$ with $H_1(M) = 0$, or a compact triangulated topological n-manifold $(n \neq 2)$ with nonempty boundary, then M unknots in E^{2n} (Theorem 5 and Corollary 5, respectively).

The proofs of these results make use of Zeeman's unknotting theorem, and they hinge upon the following question. If the n-dimensional polyhedron X collapses to the subpolyhedron Y, and Y unknots in E^k, is it true that X also unknots in E^k? It is always true if $k > 2n$ (Lemma 2), but it is generally false if $k \leq 2n$. However, in the critical case $k = 2n$, it is true provided that X satisfies a certain local unknotting condition (Theorem 4). The proof of Theorem 4 is based on the work of Lickorish [3] on the piecewise linear unknotting of cones.

2. DEFINITIONS AND BASIC FACTS

The subset X of a Euclidean space E^n is called a *polyhedron* if there exists a finite simplicial complex K in E^n such that $|K| = X$. The complex K is then called a *triangulation* of X. The map f of the polyhedron X into a Euclidean space is *piecewise linear* if the triangulation K can be chosen so that f is linear on each simplex of K.

A *piecewise linear set* is a subset Y of a Euclidean space such that each point of Y has a neighborhood (in Y) whose closure is a polyhedron. A map of the piecewise linear set Y into a Euclidean space is *piecewise linear* if its restriction to each subpolyhedron of Y is piecewise linear. *Throughout this paper, we shall work within the category of piecewise linear sets and maps.*

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An *isotopy* of Y is a level-preserving homeomorphism H of $Y \times I$ (I is the unit interval $[0, 1]$) onto itself such that, if H_t: $Y \to Y$ is defined for $t \in I$ by $H(y, t) = (H_t(y), t)$, then H_0 is the identity map. We shall also refer to the family H_t of homeomorphisms of Y as an isotopy. Two imbeddings f and g of X into Y are *ambient isotopic* if there exists an isotopy H of Y such that $H_1 f = g$. The polyhedron X is said to *unknot* in E^n if any two imbeddings of X into E^n are ambient isotopic.

For convenience, all manifolds will be *compact* and *without boundary*, unless otherwise specified. A *piecewise linear* n-*manifold* is then a polyhedron having a triangulation in which the link of every vertex is homeomorphic to the boundary of an n-simplex. A *triangulated* n-*manifold* is a polyhedron that is a topological n-manifold. The polyhedron X is a *polyhedral homology* n-*manifold* provided for each point $p \in X$ the integral homology group $H_i(X, X - p)$ is trivial for $i < n$ and infinite cyclic for $i = n$. Using exactness and excision, we can easily see that this is equivalent to the condition that the link of any vertex of any triangulation of X have the homology groups of the $(n - 1)$-sphere S^{n-1}. The polyhedron Y is an n-dimensional *pseudomanifold* [6] if for each triangulation K of Y

(a) every simplex of K is a face of some n-simplex of K,

(b) each $(n - 1)$-simplex of K is a face of exactly two n-simplexes of K, and

(c) for any two n-simplexes A and B of K, there exists a (finite) sequence $A = A_0, A_1, \cdots, A_r = B$ of n-simplexes of K such that A_i and A_{i-1} intersect in a common $(n - 1)$-face $(i = 1, \cdots, r)$.

Finally, we call the polyhedron X a *manifold with singular boundary* if there exist a connected, piecewise linear manifold M with nonempty boundary, and a piecewise linear map h of M onto X such that the *singular set*

$$S(h) = \mathrm{Cl}\,\{x \in M: h^{-1}h(x) \neq x\}$$

of h is contained in the *boundary* ∂M of M. Among these various types of manifolds, we have the following relations.

(1) If X is a triangulated n-manifold, then X is a polyhedral homology n-manifold [6, p. 237].

(2) Every connected polyhedral homology manifold is a pseudomanifold [6, p. 238].

(3) Every pseudomanifold is a manifold with singular boundary (Lemma 1 below).

However, the converses of these statements are all false.

We denote the *cone* with vertex a and base X by aX, or (to avoid confusion) by $a * X$.

We recall Whitehead's definition [8] of collapsing. If L is a subcomplex of the finite simplicial complex K, there is an *elementary simplicial collapse* from K to L if $K - L$ consists of two simplexes A and B with $A = aB$, where a is a vertex of A. Then $|K| = |L| \cup A$ and $|L| \cap A = a * \partial B$. The complex K *collapses simplicially* to L if there exists a (finite) sequence of elementary simplicial collapses going from K to L.

LEMMA 1. *If the polyhedron* X *is either an* n-*dimensional pseudomanifold or an* n-*manifold with singular boundary, then there exists a map* h *of an* n-*ball onto* X *such that* $S(h) \subset \partial B$.

Proof. We prove this simultaneously for the two cases, by a well-known kind of argument which we outline for completeness.

If X is an n-manifold with singular boundary, the following argument shows that X satisfies condition (c) of the definition of an n-dimensional pseudomanifold. Let $g: M \to X$ be a map of the piecewise linear n-manifold M onto X such that $S(g) \subset \partial M$, and let K be a triangulation of X such that h is simplicial with respect to K and some triangulation of M. The fact that $S(g) \subset \partial M$ implies that the inverse image of each n-simplex of K is an n-simplex of M. For any two n-simplexes A and B of X, the fact that M is an n-pseudomanifold implies the existence of a sequence $g^{-1}(A) = C_0, C_1, \cdots, C_r = g^{-1}(B)$ of n-simplexes of M such that each $C_i \cap C_{i-1}$ is a common (n - 1)-face of C_i and C_{i-1}. Clearly,

$$A = g(C_0), g(C_1), \cdots, g(C_r) = B$$

is the desired sequence of n-simplexes of K.

From this it follows easily that if L is an n-dimensional *proper* subcomplex of K, then L contains an (n - 1)-simplex that does not lie in $g(\partial M)$, in the case where X is an n-manifold with singular boundary, and is a face of exactly one n-simplex of L (here we use condition (b) in the definition of a pseudomanifold).

If $\triangle$ is an n-simplex of K, we can now use the above fact to collapse away the n-simplexes of $K - \{\triangle\}$ one by one, never removing an (n - 1)-simplex of $g(\partial M)$, until no n-simplexes remain. Thus we can collapse $X - \text{Int}\,\triangle$ to an (n - 1)-dimensional subpolyhedron which, in case X is an n-manifold with singular boundary, contains $g(\partial M)$. Since the union of two piecewise linear n-balls that intersect in an (n - 1)-ball common to their boundaries is a piecewise linear n-ball [11, Chapter 3], this clearly implies the desired result.

The Penrose-Whitehead-Zeeman imbedding theorem [4] includes the fact that every piecewise linear n-manifold can be imbedded in E^{2n}. Before proceeding to consider unknotting problems, we note that this is also true for an n-manifold with singular boundary.

THEOREM 1. *Every* n-*manifold with singular boundary can be piecewise linearly imbedded in* E^{2n}.

Proof. This is obvious if n = 1. We defer the case n = 2, and suppose first that $n \geq 3$. Let X be the n-manifold with singular boundary, and $f: M \to X$ a map of a piecewise linear manifold with nonempty boundary onto X such that $S(f) \subset \partial M$.

If $Y = f(\partial M)$, then by general position there is a map $g: X \to E^{2n}$ such that $g \mid Y$ is an imbedding and S(g) is a finite subset of X - Y. We can now eliminate S(g) by the technique used in the proof of the Penrose-Whitehead-Zeeman theorem.

Let C be a polygonal arc in X - Y that contains S(g). Let D be a simplicial cone over g(C) from a vertex in general position with respect to the vertices of g(X). Then $D \cap g(X) = g(C)$, because dim D = 2 and $n \geq 3$.

Choose triangulations of X and of E^{2n} with respect to which g is simplicial. If A and B are the simplicial neighborhoods of C and D, respectively, in the second barycentric subdivisions of these triangulations, then, using Whitehead's regular neighborhood theorem [11, Theorem 8], we see that

(1) A is an n-ball and B is a 2n-ball,

(2) $g(\partial A) \subset \partial B$ and $g(\text{Int } A) \subset \text{Int } B$,

(3) $g^{-1}(B) = A$, and

(4) $g \mid Cl(X - A)$ is an imbedding.

We can therefore easily alter g inside A so as to obtain an imbedding of X in E^{2n} (see [9, p. 67]).

In the case $n = 2$, we use the fact that there is a map $f: \Delta \to X$ of a 2-simplex onto S such that $S(f) \subset \partial\Delta$ (by Lemma 1). As above, let $g: X \to E^4 \subset S^4$ be a piecewise linear map such that $g \mid Y$ is an imbedding $(Y = f(\partial\Delta))$ and $S(g)$ is a finite subset of $X - Y$.

Let K and L be triangulations of X and S^4, respectively, with respect to which g is simplicial. If R is the simplicial neighborhood of $g(Y)$ in the second barycentric subdivision of L, then R is a 4-ball with 1-dimensional handles, since $g(Y)$ is a 1-dimensional polyhedron. Since 1-dimensional polyhedra unknot in S^4, it follows, by the regular neighborhood theorem, that $N = S^4 - \text{Int } R$ is a 4-ball with 2-dimensional handles. That is, N is the union of a 4-ball B_0 and a collection $B_1, \cdots, B_k$ of mutually disjoint 4-balls such that $B_0 \cap B_i$ is a solid torus T_i that is unknotted in both ∂B_0 and ∂B_i $(i = 1, \cdots, k)$.

Now it is easily verified that $g(X) \cap \partial N$ is a simple closed polygon J. Since $g(S(g)) \subset \text{Int } N$, $J' = g^{-1}(J)$ is a simple closed polygon in $X - Y$ and bounds a disk D' in $X - Y$ such that $S(g) \subset \text{Int } D'$.

There obviously exists an isotopy H_t of N such that $H_1(J)$ intersects none of the mutually disjoint solid tori

$$\partial B_1 - \text{Int } T_1, \cdots, \partial B_k - \text{Int } T_k$$

in ∂N. Since $H_1(J)$ then bounds a disk in the 4-ball $B_0 \subset N$, it follows that J bounds a disk D in N. We now redefine g so as to map D' homeomorphically onto D, leaving g unchanged on $X - \text{Int } D'$; thus we obtain an imbedding of X into E^4.

3. UNKNOTTING IN E^{2n+1}

Price [5] has shown that the n-dimensional polyhedron X unknots in E^{2n+1} if the integral cohomology group $H^n(X)$ is zero. In Theorem 2 we give a generalization of this, making use of Zeeman's unknotting theorem and Lemma 2 below. We then apply Theorem 2 to the problem of unknotting n-manifolds in E^{2n+1}.

We mentioned simplicial collapsing in Section 2. In the piecewise linear category, there is an *elementary collapse* from the polyhedron X to the subpolyhedron Y if $X = Y \cup A$, where A is a k-ball $(k > 0)$ and $Y \cap A$ is a $(k - 1)$-ball $B \subset \partial A$. X *collapses* to Y is there is a finite sequence of elementary collapses going from X to Y.

LEMMA 2. *Let* X *be an* n-*dimensional polyhedron and* Y *a subpolyhedron of* X. *If* X *collapses to* Y *and* Y *unknots in* E^{2n+1}, *then* X *unknots in* E^{2n+1}.

Proof. By induction on the number of elementary collapses from X to Y, we may assume that $X = Y \cup A$, where A is a k-ball $(k \leq n)$ and $Y \cap A = B$ is a $(k - 1)$-ball contained in ∂A.

If f and g are two imbeddings of X into E^{2n+1}, we may assume that $f \mid Y = g \mid Y$, since Y unknots in E^{2n+1}. By general position we may also assume that $f(A) \cap g(A) = g(B)$, so that $f(A) \cup g(A)$ is a k-ball.

Let Δ be a $(k-1)$-simplex in a k-plane $E^k \subset E^{2n+1}$, and denote by E^{k-1} the subspace of E^k determined by Δ. Let p and q be points in the two components of $E^k - E^{k-1}$. We want to define a homeomorphism

$$h: E^{2n+1} \to E^{2n+1}$$

such that $h(f(A) \cup g(A))$ is the suspension $S(\Delta) = p\Delta \cup q\Delta$, with

$$hf(A) = p\Delta, \quad hg(A) = q\Delta, \quad hg(B) = \Delta.$$

If $k = 1$, this is an elementary matter. If $n \geq 2$, the codimension of the ball $f(A) \cup g(A)$ is at least 3, so that Zeeman's theorem on the unknotting of combinatorial balls [10] (or Theorem (4) of Lickorish [3]) gives a homeomorphism $h': E^{2n+1} \to E^{2n+1}$ such that $h'(f(A) \cup g(A)) = S(\Delta)$. Using the facts that $Cl(E^k - S(\Delta))$ is homeomorphic to $S^{k-1} \times [0, 1)$ and that any homeomorphism on the boundary of a ball extends to the interior of the ball, we can define a homeomorphism $h'': E^k \to E^k$ such that

$$h''h'f(A) = p\Delta, \quad h''h'g(A) = q\Delta, \quad h''h'g(B) = \Delta.$$

From h' and h'' we obtain the desired homeomorphism $h: E^{2n+1} \to E^{2n+1}$.

If v is a point in general position in $E^{2n+1} - E^k$, then the $(k+1)$-ball $C = v * S(\Delta)$ intersects $hg(Y - B)$ in a finite set of points. It follows that if the point v' on the line segment from v to the barycenter of Δ is sufficiently close to Δ, then the $(k+1)$-ball $C' = v * S(\Delta)$ does not intersect $hg(Y - B)$. It is then an elementary (but tedious) matter to define an isotopy H'_t of E^{2n+1} such that $H'_t \mid hg(Y) = 1$ for each $t \in I$, and such that

$$H'_1 hf = hg: X \to E^{2n+1}.$$

H'_t sweeps $p\Delta$ across C to $q\Delta$, leaving $hg(Y)$ fixed.

If $H_t = h^{-1} H'_t h$ for each $t \in I$, then H_t is the desired isotopy of E^{2n+1} such that $H_t \mid g(Y) = 1$ and $H_1 f = g: X \to E^{2n+1}$.

THEOREM 2. *Let* X *be an* n*-dimensional polyhedron* $(n \geq 2)$. *If a triangulation of* X *contains an* n*-simplex* A *such that* $H^n(X - \text{Int } A) = 0$, *then* X *unknots in* E^{2n+1}.

Proof. Let $A_1, \cdots, A_k$ be the n-simplexes of the given triangulation of X, assuming that $H^n(X - \text{Int } A_k) = 0$. Denote by X_0 the $(n-1)$-skeleton of X, and let

$$X_i = X_0 \cup A_1 \cup \cdots \cup A_i \quad (i = 1, \cdots, k).$$

Since $X_{k-1} = X - \text{Int } A_k$, we see by exactness of the cohomology sequence of the pair (X_{k-1}, X_{i-1}) that $H^n(X_{i-1}) = 0$ $(i = 1, \cdots, k)$.

Since X_0 unknots in E^{2n+1} (actually in E^{2n}) by Gugenheim's theorem [1], we assume inductively that X_{i-1} unknots in E^{2n+1}. If B_i is an n-simplex in Int A_i, then $Y_i = X_i - \text{Int } B_i$ collapses to X_{i-1}; therefore Lemma 2 implies that Y_i unknots in E^{2n+1}. In order to show that X_i unknots in E^{2n+1}, it therefore suffices to consider two imbeddings f and g of X_i into E^{2n+1} such that $f \mid Y_i = g \mid Y_i$.

We show first that $E^{2n+1} - g(Y_i)$ is n-connected. That it is $(n-1)$-connected follows from general-position considerations (since the codimension of Y_i is $n+1$, any map of an n-complex into E^{2n+1} is homotopic to one that misses $g(Y_i)$). But

$H_n(E^{2n+1} - g(Y_i)) \approx H^n(Y_i)$, by Alexander duality, and $H^n(Y_i) = 0$ because Y_i deformation retracts to X_{i-1}. It therefore follows from the Hurewicz isomorphism theorem that $\pi_n(E^{2n+1} - g(Y_i)) = 0$.

Since $g(Y_i)$ intersects $f(B_i) \cup g(B_i)$ in $g(\partial B_i)$, it is clear that $g(Y_i)$ is link-collapsible on $f(B_i) \cup g(B_i)$. So let N be a regular neighborhood of $g(Y_i)$ mod $f(B_i) \cup g(B_i)$ in E^{2n+1} (see [2]). Then N is a piecewise linear $(2n+1)$-manifold with boundary that collapses to $g(Y_i)$, such that $g(Y_i - \partial B_i) \subset \text{Int } N$, and such that

$$N \cap f(B_i) = N \cap g(B_i) = g(\partial B_i) \subset \partial N.$$

If $Q = E^{2n+1} - \text{Int } N$, it follows that Q is an n-connected, piecewise linear $(2n+1)$-manifold with boundary, and $f \mid B_i$ and $g \mid B_i$ are proper imbeddings of B_i into Q such that $f \mid \partial B_i = g \mid \partial B_i$. Therefore $f \mid B_i$ and $g \mid B_i$ are homotopic in Q leaving ∂B_i fixed.

Since B_i is 0-connected and Q is 1-connected, Zeeman's unknotting theorem [11, Theorem 24] now provides an isotopy H_t of Q such that $H_t \mid \partial Q = 1$ for each $t \in I$, and $H_1 f \mid B_i = g \mid B_i$. Extending H_t by

$$H_t \mid E^{2n+1} - Q = 1 \qquad \text{for each } t \in I,$$

we then have the desired isotopy H_t of E^{2n+1} such that $H_1 f = g\colon X_i \to E^{2n+1}$. Hence the proof of Theorem 2 is complete by induction on k.

COROLLARY 1. *If* $n \geq 2$, *every* n-*manifold with singular boundary unknots in* E^{2n+1}.

Proof. If X is an n-manifold with singular boundary, then by Lemma 1 there exists a map f of an n-ball B onto X such that $S(f) \subset \partial B$. Let $\triangle$ be an n-simplex in Int B, in a triangulation with respect to which f is simplicial. If $A = f(\triangle)$, then $X - \text{Int } A$ collapses to the $(n-1)$-dimensional polyhedron $f(\partial B)$. Therefore $H^n(X - \text{Int } A) = 0$, and Theorem 2 applies.

COROLLARY 2. *If* $n \geq 2$, *every connected polyhedral homology* n-*manifold, and hence every connected triangulated topological* n-*manifold, unknots in* E^{2n+1}.

This follows immediately from Corollary 1 and the results in Section 2.

COROLLARY 3. *If* X *is an* n-*dimensional polyhedron such that* $H^n(X)$ *is a finite cyclic group of prime order, then* X *unknots in* E^{2n+1}.

Proof. We may assume that $n \geq 2$, since every collapsible 1-dimensional polyhedron unknots in E^3. By Theorem 2, it suffices to show that each triangulation K of X contains a simplex A such that $H^n(X - \text{Int } A) = 0$. We prove this by induction on the number k of n-simplexes in K, assuming its truth for a complex containing $k - 1$ n-simplexes.

Let A be any n-simplex of K. Then, by exactness of the cohomology sequence of the pair $(X, X - \text{Int } A)$, $H^n(X - \text{Int } A)$ is either trivial or finite cyclic of prime order. If $H^n(X - \text{Int } A) \neq 0$, then by induction the complex $K' = K - \{A\}$ triangulating $X - \text{Int } A$ contains an n-simplex B such that

$$H^n(X - \text{Int } A - \text{Int } B) = H^n(K' - B) = 0.$$

Here $H^n(K' - B)$ denotes simplicial cohomology of the simplicial complex $K' - B$.

We want to show that $H^n(X - \text{Int } B) = H^n(K - B) = 0$. Consider the following portion of the exact Mayer-Vietoris sequence of the simplicial triad $(K; K - A, K - B)$:

$$\to H^n(K) \to H^n(K - A) \oplus H^n(K - B) \to H^n(K' - B) = 0.$$

Since $H^n(K - A) \neq 0$, this implies that $H^n(K - B) = 0$, since the finite prime cyclic group $H^n(K)$ cannot be mapped homomorphically onto the direct sum of two nontrivial groups.

Question. Does Corollary 3 hold for an n-dimensional polyhedron X if $H^n(X)$ is an arbitrary cyclic group? It is easy to construct examples for which our method of proof fails. For instance, let X_1 consist of two n-balls B_1 and B_2 attached to S^{n-1} by maps $\partial B_1 \to S^{n-1}$ and $\partial B_2 \to S^{n-1}$ of degrees p and q, respectively, and let X_2 consist of $S_1^{n-1} \vee S_2^{n-1}$ (two (n - 1)-spheres with a single point in common) together with two n-balls, one attached to S_1^{n-1} by a map of degree p, and the other attached to S_2^{n-1} by a map of degree q. If the two integers p and q are relatively prime, then $H^n(X_1) = Z$ and $H^n(X_2) = Z_{pq}$, while if A is any n-simplex of X_i, then $H^n(X_i - \text{Int } A)$ is either Z_p or Z_q $(i = 1, 2)$.

In the next section, we need the following concept. We recall that the polyhedron X *strongly unknots* in E^k if for each subpolyhedron Y of X and each pair of imbeddings f and g of X into E^k such that $f \mid Y = g \mid Y$, there exists an isotopy H_t of E^k such that $H_t \mid f(Y)$ is the identity for each $t \in I$ and $H_1 f = g: X \to E^k$ (that is, the isotopy H_t transforms the imbedding f into g, while leaving pointwise-fixed the image of the subpolyhedron Y).

THEOREM 3. *If* X *is an* n-*dimensional polyhedron* $(n \geq 2)$ *such that* $H^n(X - p) = 0$ *for every point* $p \in X$, *then* X *strongly unknots in* E^{2n+1}.

Proof. We observe first that if Z is a proper subpolyhedron of X and $p \in X - Z$, then the fact that $H^n(X - p) = 0$ implies that $H^n(Z) = 0$. This follows immediately from the exact cohomology sequence of the pair $(X - p, Z)$.

Let Y be a subpolyhedron of X, and let

$$f, g: X \to E^{2n+1}$$

be two imbeddings of X such that $f \mid Y = g \mid Y$. Choose a triangulation K of X with a subcomplex L that is a triangulation of Y.

Denote by $A_1, \cdots, A_k$ the simplexes of $K - L$, listed in increasing order of dimension (that is, $i < j$ if $\dim A_i < \dim A_j$). If

$$X_i = Y \cup A_1 \cup \cdots \cup A_i \qquad (i = 1, \cdots, k),$$

then each X_i is a subpolyhedron of X.

Suppose that we have inductively constructed an isotopy H_t^i of E^{2n+1} such that

$$H_t^i \mid g(Y) = 1 \qquad \text{and} \qquad H_1^i f \mid X_i = g \mid X_i$$

for each $t \in I$. Let B_{i+1} be a simplex of the same dimension as A_{i+1}, lying in $\text{Int } A_{i+1}$, and let

$$Y_{i+1} = X_{i+1} - \text{Int } A_{i+1}.$$

Since Y_{i+1} collapses to X_i, the proof of Lemma 2 provides an isotopy G_t^{i+1} of E^{2n+1} such that $G_t^{i+1} \mid g(X_i) = 1$ for each $t \in I$, and

$$G_1^{i+1} H_1^i f \mid Y_{i+1} = g \mid Y_{i+1}.$$

Since $H^n(Y_{i+1}) = 0$, the construction in the proof of Theorem 2 yields an isotopy F_t^{i+1} of E^{2n+1} such that $F_t^{i+1} \mid g(Y_{i+1}) = 1$ for each $t \in I$ and

$$F_1^{i+1} G_1^{i+1} H_1^i f \mid X_{i+1} = g \mid X_{i+1}.$$

If $H_t^{i+1} = F_t^{i+1} G_t^{i+1} H_t^i$ for each $t \in I$, it follows that H_t^{i+1} is an isotopy of E^{2n+1} such that

$$H_t^{i+1} \mid g(Y) = 1 \quad \text{and} \quad H_1^{i+1} f \mid X_{i+1} = g \mid X_{i+1}$$

for each $t \in I$. Hence, by induction on k, we obtain the desired isotopy H_t of E^{2n+1} such that $H_t \mid g(Y) = 1$ for each $t \in I$ and $H_1 f = g \colon X \to E^{2n+1}$; this completes the proof of Theorem 3.

Now let X be an n-manifold with singular boundary, and let $h \colon B \to X$ be a map of an n-ball B onto X with $S(h) \subset \partial B$. If p is a point of X, and Y is the complement of the open star of p in a triangulation of X, then $H^n(X - p) \approx H^n(Y)$. Let $A \subset \text{Int } B$ be an n-ball such that $Y \subset h(B - \text{Int } A)$. Since $h(B - \text{Int } A)$ collapses to $h(\partial B)$ and $H^n(h(\partial B)) = 0$, it follows from the exact cohomology sequence of the pair $(h(B - \text{Int } A), Y)$ that $H^n(X - p) \approx H^n(Y) = 0$. Moreover, if Z is a subpolyhedron of X, then it follows from the exact cohomology sequence of the pair $(X - p, Z - p)$ that $H^n(Z - p) = 0$. Theorem 3 therefore implies the following generalization of Corollary 1.

COROLLARY 4. *Every subpolyhedron of an* n*-manifold with singular boundary* $(n \geq 2)$, *and hence every subpolyhedron of a connected polyhedral homology* n*-manifold* $(n \geq 2)$, *strongly unknots in* E^{2n+1}.

Considering the classical knots in E^3, we see that the restriction $n \geq 2$ is necessary in Theorems 2 and 3 and their corollaries.

4. UNKNOTTING IN E^{2n}

We shall now consider the question as to when a polyhedral homology n-manifold unknots in E^{2n}. For this we need a result for E^{2n} along the lines of Lemma 2 for E^{2n+1}. However, the straightforward analogue of Lemma 2 for imbeddings of an n-dimensional polyhedron in E^{2n} is false. For example, the cone on a disjoint pair of (n - 1)-spheres is a collapsible n-dimensional polyhedron that knots in E^{2n} for $n \geq 1$ (see [5]). To eliminate this intrinsically local type of knotting, we must impose an additional condition.

THEOREM 4. *Let* X *be an* n*-dimensional polyhedron* $(n \neq 2)$ *that collapses to a subpolyhedron* Y *such that* Y *unknots in* E^{2n}. *If every subpolyhedron of the link of each vertex of each triangulation of* X *strongly unknots in* S^{2n-1}, *then* X *unknots in* E^{2n}.

This result is dimensionally the best possible, since the n-dimensional annulus $S^{n-1} \times I$ knots in E^{2n-1} — it can be imbedded in E^{2n-1} in such a way that its two boundary (n - 1)-spheres are linked — whereas it collapses to the sphere S^{n-1}, which unknots in E^{2n-1} if $n \geq 3$ [10].

Proof. Suppose first that $n = 1$. Since the link of each vertex of X unknots in S^1, and since no finite set containing three or more points unknots in S^1, we see that each component of X is either a point, an arc, or a simple closed polygon. Since Y has the same types of components, the fact that Y unknots in E^2 implies that Y is a (finite) disjoint union of points and arcs (recall that S^1 knots in E^2 by the orientation phenomenon). The same is then true of X, so it follows by standard topology of the plane that X unknots in E^2. We may therefore assume that $n \geq 3$.

From the fact that X collapses to the subpolyhedron Y, it follows that X has a triangulation K that collapses simplicially to a subcomplex L which triangulates Y [11, Theorem 4]. Thus there are subcomplexes

$$K = K_0 \supset K_1 \supset \cdots \supset K_s = L$$

such that

$$|K_i| = |K_{i+1}| \cup A_i \quad \text{and} \quad |K_{i+1}| \cap A_i = a_i * \partial B_i \qquad (i = 0, 1, \cdots, s-1),$$

where A_i is a k-simplex $(1 \leq k \leq n)$ of K_i, a_i is a vertex of A_i, and B_i is the (k - 1)-face of A_i opposite a_i.

The link $\mathrm{lk}(a_i, K_i)$ of a_i in the subcomplex K_i is a subpolyhedron of $\mathrm{lk}(a_i, K)$, and therefore it strongly unknots in S^{2n-1}, by hypothesis. In order to prove Theorem 4 by induction on the number s of elementary simplicial collapses, it therefore suffices to establish the following assertion.

ASSERTION. *Let* X *be an* n-*dimensional polyhedron* $(n \geq 3)$, Y *a subpolyhedron, and* (K, L) *a triangulation of the pair* (X, Y). *Suppose that*

$$X = Y \cup A \quad \textit{and} \quad Y \cap A = a * \partial B,$$

where A *is a* k-*simplex of* K, a *is a vertex of* A, *and* B *is the* (k - 1)-*face of* A *opposite* a. *If* Y *unknots in* E^{2n} *and* lk (a, K) *strongly unknots in* S^{2n-1}, *then* X *unknots in* E^{2n}.

Since Y unknots in E^{2n}, it suffices to consider two imbeddings f and g of X into E^{2n} such that $f \mid Y = g \mid Y$. By general position we may assume that $f(A) \cap g(A) - f(a * \partial B)$ is a finite set of points. Hence there exists a regular neighborhood A' of $f(a * \partial B)$ in f(A) such that

$$A' \cap g(A) = g(a * \partial B) = f(a * \partial B).$$

Since $n \geq 3$ and $k \leq n$, the codimension $2n - k$ of the k-ball f(A) is at least three, so that some homeomorphism of E^{2n} onto itself carries f(A) onto a k-simplex [10]. It is therefore an elementary matter, similar to the construction of the isotopy in the proof of Lemma 2, to define an isotopy F_t of E^{2n} such that $F_t \mid g(Y) = 1$ for each $t \in I$, and $F_1(f(A)) = A'$. We are therefore justified in assuming that $f(A) \cap g(A) = f(a * \partial B) = g(a * \partial B)$.

Now let

$$C = g(\mathrm{st}(a, K)) \cup f(A),$$

$$D = g(\mathrm{lk}(a, K)) \cup f(B).$$

Note that C is piecewise linearly a cone with base D. Since every cone is link-collapsible on its base, and since

$$\mathrm{Cl}(g(X) - C) \cap C \subset D,$$

there exists a regular neighborhood N of C mod $D \cup \mathrm{Cl}(g(X) - C)$ in E^{2n} [2, p. 722]. Being a regular neighborhood of a cone, N is of course a 2n-ball, and $D \subset \partial N$, $C - D \subset \mathrm{Int}\, N$.

Let $X' = \mathrm{lk}(a, K)$ and $Y' = \mathrm{lk}(a, L) = X' - \mathrm{Int}\, B$. Then the restrictions of f and g to the cone $aX' = \mathrm{st}(a, K)$ are *proper* imbeddings of aX' in the ball N; that is, $f^{-1}(\partial N) = g^{-1}(\partial N) = X'$. Moreover, f and g agree on the subcone aY' of aX', because $Y \cap aX' = aY'$.

Since X' strongly unknots in S^{2n-1} by hypothesis, we may now apply Lemma 3 below, which is based on Lickorish's results [3] on the piecewise linear unknotting of cones. We obtain an isotopy H_t of the 2n-ball N such that $H_t \mid g(aY') = 1$ for all $t \in I$, and $H_1 f = g: aX' \to N$.

Since $Y \cap N = aY'$, $X - Y \subset aX'$, and $H_t \mid g(aY') = 1$, Lemma 4 below allows us to extend H_t to an isotopy of E^{2n} such that $H_t \mid g(Y) = 1$ and $H_1 f = g: X \to E^{2n}$; this completes the proof of the assertion, and hence that of Theorem 4.

Tindell [7] has shown that every compact piecewise linear n-manifold ($n \neq 2$) with nonempty boundary unknots in E^{2n}. Before proving Lemmas 3 and 4, we note that this holds for triangulated topological manifolds.

COROLLARY 5. *If* M *is a compact triangulated topological* n-*manifold with nonempty boundary, and* $n \neq 2$, *then* M *unknots in* E^{2n}.

Proof. Since each component of M is an arc if $n = 1$, we assume that $n \geq 3$. Consider first the double 2M of M. It is a triangulated n-manifold without boundary, and hence it is a polyhedral homology n-manifold. Hence the link of any vertex in any triangulation of 2M is a connected polyhedral homology $(n-1)$-manifold [6, p. 239]. It therefore follows from Corollary 4 that every subpolyhedron of the link of any vertex of M strongly unknots in S^{2n-1}. Since $\partial M \neq \emptyset$, M collapses to an $(n-1)$-dimensional subpolyhedron, which unknots in E^{2n}, by Gugenheim [1]. Consequently, Theorem 4 applies.

LEMMA 3. *Let* X *be an* $(n-1)$-*dimensional polyhedron that strongly unknots in* S^{k-1}, *where* $k - n \geq 3$. *Let* Y *be a subpolyhedron of* X, *and let* f *and* g *be two proper imbeddings of the cone* aX *into the* k-*ball* B *such that* $f \mid aY = g \mid aY$. *Then there exists an isotopy* H *of* B *such that* $H_t \mid g(aY) = 1$ *for each* $t \in I$ *and* $H_1 f = g: aX \to B$.

Proof. Let α be a homeomorphism of B onto the cone bS^{k-1} on a $(k-1)$-sphere with vertex b. If we could find an isotopy H'_t of bS^{k-1} such that $H'_t \mid \alpha g(aY) = 1$ for each $t \in I$, and such that

$$H'_1 \alpha f = \alpha g: aX \to bS^{k-1},$$

then $H_t = \alpha^{-1} H'_t \alpha$ would be the desired isotopy of B. We may therefore assume that B coincides with the cone bS^{k-1}.

We first apply Theorem (1) of Lickorish [3]: *any two proper imbeddings of a cone into a ball (of codimension at least 3) that agree on the base of the cone are equivalent via an isotopy of the ball which leaves fixed the boundary of the ball.* We thus obtain a homeomorphism $h: bS^{k-1} \to bS^{k-1}$ such that $hg(aY) = b * g(Y)$ and $h \mid S^{k-1} = 1$.

Since X strongly unknots in S^{k-1}, there exists an isotopy ϕ_t of S^{k-1} such that $\phi_t \mid g(Y) = 1$ for each $t \in I$, and such that

$$\phi_1 hf \mid X = hg \mid X\colon X \to S^{k-1}.$$

By conewise extension (joining $\phi_t\colon S^{k-1} \to S^{k-1}$ with $b \to b$), we obtain an isotopy Φ_t of bS^{k-1} such that $\Phi_t \mid b * g(Y) = 1$ for each $t \in I$, and

$$\Phi_1\, hf \mid X = hg \mid X\colon X \to S^{k-1}.$$

The two proper imbeddings Φ_1 hf and hg of the cone aX into the ball bS^{k-1} agree, both on the subcone aY (because $hg(aY) = b * g(Y)$) and on the base X of aX.

Consequently, Theorem (2) of [3] yields an isotopy Ψ_t of bS^{k-1} such that $\Psi_t \mid hg(aY) = 1$ for each $t \in I$, and

$$\Psi_1\, \Phi_1\, hf = hg\colon aX \to bS^{k-1}.$$

Finally, we define the isotopy H_t of $B = bS^{k-1}$ by

$$H_t = h^{-1}\Psi_t \Phi_t h \qquad (\text{each } t \in I).$$

Then $H_t \mid g(aY) = 1$, because $\Phi_t \mid hg(aY) = \Psi_t \mid hg(aY) = 1$ for each $t \in I$, and

$$h_1 f = h^{-1}(\Psi_1\, \Phi_1\, hf) = h^{-1}\, hg = g$$

on aX; this completes the proof of Lemma 3.

LEMMA 4. *Let* B *be an* n-*ball in* E^n, *and* Y *a polyhedron in* E^n. *If* H_t *is an isotopy of* B *such that* $H_t \mid Y \cap B = 1$ (*that is, if* H_t *leaves* $Y \cap B$ *fixed*), *then* H_t *can be extended to an isotopy of* E^n *that leaves* Y *fixed.*

Proof. The plan of the proof is as follows. In E^n - Int B we shall construct a "blister" such as would form on the surface ∂B of the ball B if it were burned at each point of $\partial B - Y$, but at no point of $Y \cap \partial B$, this blister being so thin that it intersects Y only in points where its surface touches $\partial B \cap Y$. The isotopy H_t will then be extended across the blister so that it is the identity on the outer surface of the blister. H_t can then be defined as the identity outside the ball B with its blister.

To construct the blister, we start with a collar

$$c\colon \partial B \times I \to E^n - \text{Int } B$$

for B in E^n such that $c(b, 0) = b$ for each $b \in \partial B$. The idea is to pinch each fiber $c(y \times I)$ to the point $y \in \partial B$ if $y \in Y$, without pinching to a point any of the other fibers of the collar.

We may assume that there exists a point $p \in \partial B$ such that $c(p \times I) \cap Y = \phi$ (by shortening the fibers of c, if necessary). Let f be a piecewise linear homeomorphism of $\partial B - p$ onto E^{n-1}, where E^{n-1} denotes the hyperplane $E^{n-1} \times 0 \subset E^{n-1} \times E^1 = E^n$, and let

$$F = f \times 1\colon (\partial B - p) \times I \to E^{n-1} \times I \subset E^n.$$

Define $Z = Fc^{-1}(Y)$. We shall first construct a blister on E^{n-1} in $E^{n-1} \times I$, and then transfer this blister to the surface of the ball B by the map cF^{-1}. We shall do this by defining a piecewise linear function $\phi\colon E^{n-1} \to I$ such that $\phi^{-1}(0) = Z \cap E^{n-1}$ and such that

$$Z \cap \{(x, s) \in E^{n-1} \times I: s \leq \phi(x)\} = Z \cap E^{n-1}.$$

To construct the function ϕ, we use the elementary fact that for each finite simplicial complex L in E^n, there exists a positive number $\varepsilon(L)$ such that, if g is a map of $|L|$ into E^n, linear on each simplex of L and moving no vertex of L a distance greater than $\varepsilon(L)$, then $g: |L| \to E^n$ is an imbedding.

Now let K be a locally finite triangulation of $E^{n-1} \times I$, which contains a subcomplex triangulating the polyhedron Z. Suppose that the simplicial neighborhood of $Z \cap E^{n-1}$ in E^{n-1}, consisting of all simplexes of K in E^{n-1} that intersect Z, is a regular neighborhood N of $Z \cap E^{n-1}$ in E^{n-1} (take a second barycentric subdivision of K, if necessary). Then each simplex of N is either contained in $Z \cap E^{n-1}$ or ∂N, or it is the join of a simplex in $Z \cap E^{n-1}$ and a simplex in ∂N.

Finally, let L be the subcomplex of K such that $|L| = Z \cup N$, let $\varepsilon > 0$ be the number $\varepsilon(L)$ referred to above, and assume that $\varepsilon \leq 1$. Since Z is compact and does not intersect E^{n-1} - Int N, we may suppose that ε is sufficiently small so that

$$Z \cap ((E^{n-1} - \text{Int } N) \times [0, \varepsilon]) = \emptyset.$$

For each vertex v of the subcomplex K' of K that spans E^{n-1}, define

$$\phi(v) = \begin{cases} 0 & \text{if } v \in Z \cap E^{n-1}, \\ \varepsilon & \text{otherwise.} \end{cases}$$

Extending ϕ linearly to the simplexes of K', we obtain a piecewise linear map $\phi: E^{n-1} \to I$ with the desired properties; that is, it is clear that $\phi^{-1}(0) = Z \cap E^{n-1}$, and it is easily verified that the "blister"

$$\{(x, s) \in E^{n-1} \times I: s \leq \phi(x)\}$$

intersects Z in $Z \cap E^{n-1}$.

We now define $\psi: \partial B \to I$ by

$$\psi(b) = \begin{cases} \phi f(b) & \text{if } b \neq p, \\ \varepsilon & \text{if } b = p. \end{cases}$$

Then ψ is piecewise linear because $\phi(E^{n-1} - N) = \varepsilon$. If

$$A' = \{(b, s) \in \partial B \times I: s \leq \psi(b)\},$$

then $A = c(A')$ is the desired blister on the ball B such that $A \cap Y = \partial B \cap Y$.

We define the isotopy G_t on $\partial B \times I$ for each $t \in [0, 1]$ by

$$G_t(b, s) = \begin{cases} (b, s) & \text{if } s \in [t, 1], \\ (H_{t-s}(b), s) & \text{if } s \in [0, t]. \end{cases}$$

Let $\alpha: \partial B \times I \to A'$ be a piecewise linear map such that $\alpha(b \times I) = (b, 0)$ if $b \in Y \cap \partial B$, and such that $\alpha \mid b \times I$ is a homeomorphism onto $b \times [0, \psi(b)]$ otherwise.

We can now extend the isotopy H_t to the blister A by defining

$$H_t \mid A = c\alpha G_t \alpha^{-1} c^{-1}$$

for each $t \in I$. Since $G_t \mid (\partial B \cap Y) \times I = 1$ for each t and $A \cap Y = \partial B \cap Y$, the mapping $H_t \mid A$ is thereby well-defined. This extension is continuous and well-defined on ∂B, because the composition

$$\partial B \to \partial B \times 0 \xrightarrow{G_t} \partial B \times 0 \to \partial B$$

is equal to $H_t \mid \partial B$ for each t. Furthermore, each H_t is the identity on the "outer boundary"

$$c(\{(b, s) \in \partial B \times I: s = \psi(b)\})$$

of the blister A, because the composition

$$\partial B \to \partial B \times 1 \xrightarrow{G_t} \partial B \times 1 \to \partial B$$

is the identity for each $t \in I$.

We can therefore extend H_t to the remainder of E^n by defining

$$H_t \mid E^n - A - B = 1$$

for each $t \in I$. Since $A \cap Y = \partial B \cap Y$ and $H_t \mid B \cap Y = 1$, we see that $H_t \mid Y = 1$; this completes the proof of Lemma 4.

We next apply Theorem 4 to obtain a result on the unknotting of polyhedral homology n-manifolds in E^{2n}. Zeeman's unknotting theorem includes the fact that every connected, simply connected, piecewise linear n-manifold $(n \geq 3)$ unknots in E^{2n}. The following theorem is a generalization of this.

THEOREM 5. *If* M *is a connected, orientable polyhedral homology* n-*manifold* $(n \neq 2)$ *such that* $H_1(M) = 0$, *then* M *unknots in* E^{2n}.

Since S^2 knots in E^4, the restriction $n \neq 2$ is necessary. The only connected, polyhedral homology 1-manifold is the 1-sphere S^1, which knots in E^2 (the orientation phenomenon), but of course $H_1(S^1) = Z$. We therefore assume, in the following proof, that $n \geq 3$.

Proof. By Lemma 1, there exists a map h of an n-ball A onto M such that $S(h) \subset \partial A$. Let B be an n-ball interior to A, and define

$$X = h(A - \text{Int } B), \qquad Y = h(\partial A).$$

Then X collapses to Y, and Y unknots in E^{2n} (by Gugenheim [1], since $\dim Y = n - 1$).

The link of each vertex of any triangulation of M is a connected polyhedral homology (n - 1)-manifold [6, p. 239]. It therefore follows from Corollary 4 that every subpolyhedron of the link of any vertex of any triangulation of M strongly unknots in S^{2n-1}. We note here that the unknotting results in this paper, which are stated for imbeddings in Euclidean spaces, hold also for imbeddings in spheres, because each of the isotopies that we construct is the identity outside a compact set.

It now follows from Theorem 4 that X unknots in E^{2n}. In order to show that M unknots in E^{2n}, it therefore suffices to consider two imbeddings f and g of M into E^{2n} such that $f \mid X = g \mid X$.

First we deduce from the fact that M is orientable (meaning that $H^n(M) \approx Z$) with $H_1(M) = 0$ that $H^{n-1}(Y) = 0$. Note that $H^{n-1}(M) \approx H_1(M) = 0$, by Poincaré duality. Let us regard M as a CW-complex with a single n-cell, and with Y as its (n - 1)-skeleton. Then the cellular cochain complex of M has the form

$$0 \to C^0 \to \cdots \to C^{n-1} \xrightarrow{\delta} C^n \approx Z \to 0.$$

Since $H^n(M) = Z$, we conclude that $\delta = 0$. It follows that $H^{n-1}(Y) \approx H^{n-1}(M) = 0$.

Since Y is (n - 1)-dimensional, $E^{2n} - f(Y)$ is (n - 1)-connected, by general-position arguments. Now $H_n(E^{2n} - f(Y)) \approx H^{n-1}(Y) = 0$, by Alexander duality, and it follows from the Hurewicz theorem that $E^{2n} - f(Y)$ is n-connected.

If R is a regular neighborhood of $f(X) = g(X)$ modulo $fh(B) \cup gh(B)$, and $Q = E^{2n} - \operatorname{Int} R$, it follows that Q is n-connected, because R collapses to f(Y). Therefore the two proper imbeddings $f \mid h(B)$ and $g \mid h(B)$ of the n-ball h(B) into Q are homotopic leaving the boundary fixed. Since $n \geq 3$, Zeeman's unknotting theorem supplies an isotopy H_t of Q such that $H_t \mid \partial Q = 1$ for each $t \in I$, and $H_1 f \mid h(B) = g \mid h(B)$. Defining $H_t \mid R = 1$ for each $t \in I$, we then have an isotopy of E^{2n} such that $H_1 f = g \colon M \to E^{2n}$.

For a sample application of Theorem 5, let M^3 be the Poincaré 3-manifold such that $H_1(M^3) = 0$ but $\pi_1(M^3)$ is the binary icosahedral group [6, p. 218]. Then M^3 is orientable, because $\pi_1(M^3)$ is a finite group; therefore Theorem 5 shows that M^3 unknots in E^6. The suspension $S(M^3)$ is not a topological manifold, but it is an orientable, simply connected polyhedral homology 4-manifold, and by Theorem 5 it unknots in E^8.

5. FINAL REMARKS

In the preceding sections, we have for convenience discussed unknotting of polyhedra only in Euclidean spaces. In this final section, we indicate the extent to which the ambient manifold can be generalized in each of our results.

A polyhedron X is said to *unknot* in the piecewise linear manifold M if any two homotopic imbeddings of X into M are ambient isotopic.

Lemma 2 *holds, with* E^{2n+1} *replaced by an arbitrary piecewise linear* (2n + 1)-*manifold without boundary*. The reason for this is that in the inductive step of our proof we work inside a (2n + 1)-ball, whether or not the ambient manifold is E^{2n+1}. Similarly, *Theorem* 4 *holds with* E^{2n} *replaced by an arbitrary piecewise linear* 2n-*manifold, and the same is true of Corollary* 5.

In our proofs of Theorems 2 and 3, we used Lemma 2, together with the fact that if Y is an n-dimensional polyhedron in E^{2n+1} with $H^n(Y) = 0$, then $E^{2n+1} - Y$ is n-connected. Consider now an n-dimensional polyhedron Y in a (2n + 1)-manifold M. If M is n-connected, we easily deduce, using the Poincaré duality isomorphism

$$H_i(M, M - Y) \approx H^{2n+1-i}(Y),$$

exactness, and the Hurewicz isomorphism theorem, that M - Y is n-connected. Therefore *Theorems* 2 *and* 3, *together with Corollaries* 1 *to* 4, *all hold with* E^{2n+1} *replaced by an* n-*connected piecewise linear* (2n + 1)-*manifold without boundary.*

By a similar argument, we see that *Theorem* 5 *holds with* E^{2n} *replaced by an* n-*connected* 2n-*manifold.* Of course, in view of the Poincaré conjecture for piecewise linear manifolds, we see that these extensions of Theorems 2, 3, and 5 are of additional interest only if the ambient manifold is not compact.

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