EXISTENCE OF EXACT SOLUTIONS OF SINGULAR ORDINARY DIFFERENTIAL EQUATIONS NEAR THEIR APPROXIMATE SOLUTIONS

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With suitable restrictions on the functions u and F, the improper initial-value problem associated with

$$y' = F\left(\frac{y}{u}\right)u' \tag{1}$$

and

$$y(0+) = 0 \tag{1'}$$

has the solution tu for every number t satisfying

$$t = F(t). \tag{2}$$

If for the more general differential equation

$$y'(x) = f(x, y(x)) \tag{3}$$

there exist u and F such that (3) is in some sense asymptotic at 0 to (1), then one may ask for which solutions t of (2) there exists a solution of (3) that is asymptotic at 0 to tu in some useful manner. A partial answer to this question is more or less implicit in [2], the conclusion there being roughly that if (3) is dominated by an equation of the form (1) with a nonnegative F, *via* a certain set of inequalities and intervening functions, then to each solution t of (2) there corresponds a solution y of (3) on a deleted right-hand neighborhood of 0 that satisfies $|y| \leq t\,|u|$. This inequality does not imply, however, that y is asymptotic to tu in any very strong sense, and indeed the foregoing result does not even imply that different solutions t give rise to different solutions y. The purpose of the present paper is to show how one can apply the existence theory in [2] to obtain a more satisfactory, though still incomplete, answer to this question.

It will be more straightforward to eliminate (1) from the problem by thinking of the improper integral equation equivalent to (3) and (1'), namely

$$y = \int_{0+} f(x, y(x))\,dx, \tag{4}$$

as having approximate solutions, in a sense that must be made precise, and then asking for exact solutions asymptotic to these approximate solutions. This eliminates u and t as used above; however, they will appear again with different meanings when the theorems of [2] are applied.

Let P be the complex plane, and P_∞ its one-point compactification. Suppose that the function $f\colon (0, \infty) \times P \to P_\infty$ satisfies the Carathéodory condition that every

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restriction of f to a cross-section $(0, \infty) \times \{z\}$ is measurable, while almost every restriction of f to a cross-section $\{x\} \times P$ is continuous; this condition ensures that if $y: (0, \infty) \to P$ and y is measurable (in particular, if y is continuous), then $f(x, y(x))$ is measurable as a function of x (see [1, p. 665] or [2]). Let us now give precision to the notion of an approximate solution of (4). The definition will be shortened if we understand the phrase "h is integrable from $0+$ to x," when applied to a function $h: (0, a] \to P$ and a number x in the bounded interval $(0, a]$, to mean that if $0 < \delta \leq x$, then h is Lebesgue-integrable from δ to x, and $\lim_{\delta \to 0+} \int_\delta^x h$ exists and is finite. This limit will be denoted by $\int_{0+}^x h$.

Definition 1. The statement that y_0 is an *approximate solution with tolerance function* ε, on the bounded interval $(0, a]$, of the improper integral equation

$$y = \int_{0+} f(x, y(x))\,dx$$

means that

(i) $y_0: (0, a] \to P$,

(ii) $y_0(0+) = 0$,

(iii) y_0 is absolutely continuous,

(iv) $f(x, y_0(x))$ is integrable from $0+$ to a, and

(v) $\varepsilon: (0, a] \to [0, \infty)$, and if $x \in (0, a]$, then

$$\left| \int_{0+}^x f(x', y_0(x'))\,dx' - y_0(x) \right| \leq \varepsilon(x). \tag{5}$$

Given an approximate solution y_0 of (4), one can place various restrictions on the behavior of f in the vicinity of the graph of y_0, measuring these restrictions in terms of the tolerance function ε. The restriction described in the following definition is of this nature, and it has the advantage of being satisfied in many cases of interest, yet being strong enough to imply the existence of an exact solution of (4) near the given approximate solution.

Definition 2. If y_0 is an approximate solution with tolerance ε, on the bounded interval $(0, a]$, of the improper integral equation

$$y = \int_{0+} f(x, y(x))\,dx,$$

then f is said to be *restricted in growth near* y_0 if and only if there exist a nonnegative function H on $(0, a] \times [0, \infty)$, a finite-valued, nonnegative function K, and a number $t \geq 1$, such that

(i) if $x \in (0, a]$, then $H(x, 0) = 0$ and $H(x, r)$ is nondecreasing in r,

(ii) if $(x, w) \in (0, a] \times P$, then

$$|f(x, y_0(x) + w) - f(x, y_0(x))| \leq H(x, |w|), \tag{6}$$

(iii) if $(x, s) \in \operatorname{dom} K$, then

$$\int_{0+}^{x} H(x', s\,\varepsilon(x'))\,dx' \leq K(x, s)\,\varepsilon(x)\,, \tag{7}$$

where

$$\operatorname{dom} K = (0, a] \times \{s \geq 0\colon \text{ the integrand in (7) is integrable from } 0+ \text{ to } a\},$$

and

(iv) if $x \in (0, a]$, then $(x, t) \in \operatorname{dom} K$ and

$$1 + K(x, t) \leq t\,. \tag{8}$$

We are now in position to formulate a theorem that embodies the promised result.

THEOREM. *Suppose* y_0 *is an approximate solution with tolerance function* ε, *on the bounded interval* $(0, a]$, *of the improper integral equation*

$$y = \int_{0+} f(x, y(x))\,dx\,.$$

Suppose further that H, K, *and* t *satisfy the conditions in Definition* 2, *so that* f *is restricted in growth near* y_0. *Then on the interval* $(0, a]$ *there exists an exact solution* y *of this equation such that*

$$\begin{aligned} |(y - y_0)(q) - (y - y_0)(p)| \leq{}& \left|\int_p^q f(x, y_0(x))\,dx - (y_0(q) - y_0(p))\right| \\ &+ \int_p^q H(x, t\,\varepsilon(x))\,dx \end{aligned} \tag{9}$$

if $0 < p \leq q \leq a$, *and* $|y(x) - y_0(x)| \leq t\,\varepsilon(x)$ *if* $0 < x \leq a$.

Proof. Define a function $G\colon (0, \infty) \times P \to P_\infty$ by

$$G(x, w) = \begin{cases} f(x, y_0(x) + w) - y_0'(x) & \text{if } y_0 \text{ is differentiable at } x, \\ 0 & \text{otherwise}\,. \end{cases}$$

Let $G_0(x) = G(x, 0)$ and $K_0(x) = 1$, for $0 < x \leq a$, and let $u = \varepsilon$. Then, because of conditions (ii), (iii), and (iv) of Definition 1, G_0 is integrable from $0+$ to a, and in view of (5) we see that if $x \in (0, a]$, then

$$\left|\int_{0+}^{x} G_0\right| = \left|\int_{0+}^{x} f(x', y_0(x'))\,dx' - y_0(x)\right| \leq \varepsilon(x) = K_0(x)\,u(x)\,. \tag{10}$$

From (6) it follows that if $(x, w) \in (0, a] \times P$, then

$$|G(x, w) - G_0(x)| \leq H(x, |w|)\,. \tag{11}$$

The existence of the functions G_0, u, K_0, H, and K with the postulated properties, including in particular the three inequalities (10), (11), and (7) with u in place of ε, guarantees that the function G satisfies Hypothesis A of [2] (and in fact it is almost tantamount to this assertion). Further, the existence of the number t satisfying (8), with the 1 replaced by $K_0(x)$, says that G is "restricted in growth somewhere near 0," as defined in [2]. Since y_0 is absolutely continuous, so that y_0' is measurable, and f satisfies the Carathéodory condition, it follows that G satisfies the Carathéodory condition. This, together with the restrictedness in growth of G somewhere near 0, constitutes satisfaction of the hypothesis of Theorem 2 of [2]. The conclusion of that theorem is that there exists a continuous function

$\phi\colon (0, a] \to P$ such that $\phi = \int_{0+} G(x, \phi(x))\,dx$,

$$|\phi(q) - \phi(p)| \leq \left|\int_p^q G_0\right| + \int_p^q H(x, tu(x))\,dx \qquad \text{if } 0 < p \leq q \leq a,$$

and $|\phi(x)| \leq t\,u(x)$ if $0 < x \leq a$. If $y = y_0 + \phi$, then we may use y to verify the conclusion of the present theorem, thus completing the proof.

If y_0 and y_0^* are two approximate solutions of (4) for which the hypothesis of the theorem is true, and their tolerance functions ε and ε^* satisfy the conditions $\varepsilon = o(|y_0 - y_0^*|)$ and $\varepsilon^* = o(|y_0 - y_0^*|)$, then the corresponding exact solutions y and y^* are distinct; for if $y = y^*$, then

$$|y_0 - y_0^*| \leq |y - y_0| + |y^* - y_0^*| \leq t\varepsilon + t^*\varepsilon^* = o(|y_0 - y_0^*|),$$

which is impossible.

An example particularly apt for illustrating this theorem is the case in which (3) reads

$$y'(x) = 1 + [c + p(x)]\left[\frac{y(x)}{x}\right]^2, \tag{12}$$

where $c \in P$, $p\colon (0, \infty) \to P$, $p(0+) = 0$, p is integrable on every bounded subset of $(0, \infty)$, and $\|p\|(x) \equiv \sup_{(0,x]} |p| > 0$ if $0 < x$. Here the analogues of (1) and (2) are

$$y'(x) = 1 + c\left[\frac{y(x)}{x}\right]^2 \tag{13}$$

and

$$t = 1 + ct^2. \tag{14}$$

If m is a solution of (14), then mx is a solution of (13); therefore, by taking $y_0(x) = mx$ on any bounded interval (0, a], we get an approximate solution of (4) with a convenient tolerance given by $\varepsilon(x) = |m^2| \int_{0+}^{x} \|p\|$. A suitable H is given by

$$H(x, r) = |c + p(x)|\left[2|m|\frac{r}{x} + \left(\frac{r}{x}\right)^2\right],$$

and a suitable K by

$$K(x, s) = \alpha(x)s + \beta(x)s^2,$$

where

$$\alpha(x) = \frac{\int_{0+}^{x} |2cm + 2mp(x')| \left(\frac{1}{x'} \int_{0+}^{x'} \|p\| \right) dx'}{\int_{0+}^{x} \|p\|}$$

and

$$\beta(x) = \frac{\int_{0+}^{x} |m^2 c + m^2 p(x')| \left(\frac{1}{x'} \int_{0+}^{x'} \|p\| \right)^2 dx'}{\int_{0+}^{x} \|p\|}.$$

H and K are easily seen to satisfy (i), (ii), and (iii) of Definition 2.

Now consider the number ρ defined by

$$\rho = \limsup_{x \to 0+} \frac{\int_{0+}^{x} \frac{1}{x'} \left(\int_{0+}^{x'} \|p\| \right) dx'}{\int_{0+}^{x} \|p\|}.$$

Note that $0 \leq \rho \leq 1$ in all cases, and that $\rho = 0$ except for rather pathological functions p. In any case, if $|c|\rho < 1/2|m|$, then

$$\limsup \alpha(x) < 1 \quad \text{and} \quad \lim \beta(x) = 0,$$

so that for a sufficiently small a there exists a number $t \geq 1$ for which (iv) of Definition 2 holds; hence the improper integral equation

$$y(x) = x + \int_{0+}^{x} [c + p(x')] \left[\frac{y(x')}{x'} \right]^2 dx'$$

has a solution y on (0, a] such that $|y(x) - mx| \leq t\,|m^2| \int_{0+}^{x} \|p\| = o(x)$, and y also satisfies (12) almost everywhere, as well as (1'). If $|c|\rho \geq 1/2|m|$, then the theorem gives no information, under the particular choice of a tolerance function ε that was made above.

Finally, the theorem presented here could without difficulty be extended to include the case where the function f maps into a finite-dimensional Banach space, or even an infinite-dimensional Banach space in which a reasonable integral is defined.

REFERENCES

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