THE BOUNDARY BEHAVIOUR OF TSUJI FUNCTIONS

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1. INTRODUCTION

Suppose that f(z) is meromorphic in $|z| < 1$. We denote by

$$f^*(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

the spherical derivative of f(z). If Γ is any rectifiable Jordan curve or arc in $|z| < 1$, then

$$L(\Gamma) = \int_\Gamma f^*(z)\,|dz| \tag{1.1}$$

is the length of the image of Γ, under the associated map of $|z| < 1$ onto the Riemann sphere.

We denote the disk $|z| < 1$ by D, its circumference $|z| = 1$ by C, and the circles $|z| = r$ by C_r, for $0 < r < 1$. Also, for each nonnegative number ℓ we denote by $T_1(\ell)$ the class of functions f(z) such that

$$\limsup_{r \to 1} L(C_r) = \limsup_{r \to 1} \int_0^{2\pi} f^*(re^{i\theta})\, r\, d\theta \leq \ell, \tag{1.2}$$

and we write $T_1 = \bigcup_{\ell \geq 0} T_1(\ell)$.

The class T_1 was introduced by Tsuji [11], and it was further considered by Collingwood and Piranian [3], who called the functions $f \in T_1$ *Tsuji functions.* Let $S(\theta, \alpha)$ denote the straight-line segment joining the two points $e^{i\theta}$ and $(1 - e^{i\alpha}\cos\alpha)e^{i\theta}$, and let $\Lambda(\theta, \alpha)$ denote the length of the spherical image of $S(\theta, \alpha)$ by f(z), so that

$$\Lambda(\theta, \alpha) = \int_{S(\theta, \alpha)} f^*(z)\,|dz|.$$

If f(z) approaches a limit as $z \to e^{i\theta}$ along $S(\theta, \alpha)$, we denote the limit by $w(\theta, \alpha)$. If in each open triangle in D having one vertex at $e^{i\theta}$ and containing $S(\theta, \alpha)$ the function f(z) assumes infinitely often all values on the sphere, with at most two exceptions, then $S(\theta, \alpha)$ is called a *segment of Julia.* If for a fixed θ and all α $(-\pi/2 < \alpha < \pi/2)$ $S(\theta, \alpha)$ is a segment of Julia, then $e^{i\theta}$ is called a *Julia point.* With this notation and terminology, we can state a theorem of Tsuji [11] as follows.

THEOREM A. *If* $f \in T_1$, *then, for each* α *in* $|\alpha| < \pi/2$, $\Lambda(\theta, \alpha)$ *is an integrable function of* θ, *and for almost all* θ, $\Lambda(\theta, \alpha)$ *is an integrable function of* α.

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Moreover, for almost all θ *the relation* $w(\theta, \alpha) = w(\theta, \beta)$ *holds whenever both limits exist, and* $S(\theta, \gamma)$ *is a segment of Julia if* $\Lambda(\theta, \gamma) = \infty$.

(We note that Bagemihl's theorem on disjoint arc cluster sets for arbitrary functions [2] strengthens immediately the first part of the second sentence in Theorem A: *the set of points* $e^{i\theta}$ *at which* f *has more than one asymptotic value is at most countable.*)

Collingwood and Piranian [3] gave several illuminating examples showing that Tsuji functions can in fact have segments of Julia. In particular, they constructed a meromorphic function $f(z) \in T_1$ for which each point of C is a Julia point, and a meromorphic function $g(z) \in T_1$, of bounded characteristic, for which each point of E is a Julia point, where E is an arbitrarily prescribed set of measure 0 on C. They also proved that the regular function

$$f(z) = \exp\left(\frac{1+z}{1-z}\right)^2$$

is a Tsuji function and has each of the two segments $S(0, \mp\pi/4)$ as a segment of Julia. It should be said that the examples of Collingwood and Piranian also lead to meromorphic Tsuji functions f(z) with arbitrarily large growth as measured by the characteristic T(r, f).

Finally, Collingwood and Piranian stated three conjectures concerning regular Tsuji functions. The first two of these are disproved elsewhere [6], through the construction of a regular function $f(z) \in T_1$ with infinitely many Julia points. The third asserts that *a regular normal Tsuji function has no segments of Julia.*

In this paper we shall prove this conjecture, and rather more. Suppose that f(z) is meromorphic in a simply connected domain Δ. Following Lehto and Virtanen (see [8], [9]), we say, for $0 < K < \infty$, that $f(z) \in N(K)$ in Δ if

$$f^*(z) \leq K \frac{d\sigma_z}{|dz|},$$

where $d\sigma_z$ refers to the hyperbolic metric with respect to Δ. If $f \in N(K)$ for some K, we say that f(z) is *normal* in Δ. Since the hyperbolic metric decreases with expanding domain Δ, it follows that if $f \in N(K)$ in Δ, then $f \in N(K)$ in every subdomain of Δ. This remark is frequently useful. We shall be concerned mainly with the case where Δ is some subdomain of D. We say that a function f meromorphic in D *belongs to* N(K) *at* ζ on C if

$$f^*(z) \leq K/(1 - |z|^2)$$

at all points z of D in some disk $|z - \zeta| < \varepsilon$. If $f \in N(K)$ for some K, at ζ, we say that f is *normal* at ζ. If $f \in N(K)$ for every K, at ζ, we say f is *subnormal* at ζ, and if f does not belong to N(K) for any K, at ζ, we say that f(z) is *abnormal* at ζ. We note that since $(1 - |z|^2)^{-1}$ is the hyperbolic metric of D, a function f is normal in D if and only if it is normal at every point ζ of C. We also note that the set of points of C at which f is normal is an open subset of C. Thus the set of points of C where f is abnormal is closed. The points or arcs of C at which f is normal or abnormal will be called *normal* or *abnormal arcs* or *points* of C, if there is no ambiguity about the function f.

The theory of normal functions was developed by Lehto and Virtanen [8], [9], who proved the following proposition.

THEOREM B. *Suppose that* $f(z)$ *is meromorphic in* D. *Then* ζ *is an abnormal boundary point of* D *if and only if there exists a sequence of points* $z_n \in D$ *such that* $z_n \to \zeta$ *and*

$$(1 - |z_n|^2) f^*(z_n) \to \infty . \tag{1.3}$$

Further, if $\{z_n\}$ *satisfies* (1.3), $\varepsilon > 0$, *and*

$$S(z_n, \varepsilon) = \left\{ z: \left| \frac{z - z_n}{1 - \bar{z}_n z} \right| < \varepsilon \right\},$$

then for each fixed ε *the function* f *assumes in* $S(z_n, \varepsilon)$ *at least one of any three preassigned values in the closed plane, for all sufficiently large* n.

Conversely, if $f(z)$ satisfies this latter condition in $S(z_n, \varepsilon)$ for every ε, then (1.3) holds as $z \to \zeta$ through a sequence $\{z'_n\}$ such that the hyperbolic distance of z'_n from z_n tends to zero. We remark that from the theory of normal families (and from Ahlfors' theory of covering surfaces) various other properties of $f(z)$ in the regions $S(z_n, \varepsilon)$ can be deduced. For instance, if $\{\Delta_\nu\}$ $(\nu = 1, \cdots, 5)$ is a set of simply connected domains on the Riemann sphere whose closures are disjoint, then for $n > n_0(\varepsilon)$, $f(z)$ gives a schlicht map of a subdomain of $S(z_n, \varepsilon)$ onto at least one of the domains Δ_ν (see for example [5, p. 156]). Hence the area of the image of $S(z_n, \varepsilon)$ on the Riemann sphere remains above a positive absolute constant for large n, and so the area of the image of the intersection of D with any neighbourhood of ζ is infinite if ζ is abnormal. We also note that by Theorem B any abnormal point ζ of C is necessarily a Picard-point (that is, $f(z)$ assumes in each neighbourhood of ζ all values in the closed plane, with at most two exceptions; the exceptional values are called *Picard values*).

For most of our purposes, a hypothesis weaker than (1.2) will suffice. We shall say that $f(z) \in T_2(\ell)$ if f is meromorphic in D and if there exists a sequence of closed Jordan curves $\Gamma_m \subset D$ whose interiors D_m expand to D as $m \to \infty$ and for which

$$\limsup_{m \to \infty} L(\Gamma_m) \leq \ell , \tag{1.4}$$

where $L(\Gamma)$ is given by (1.1). We also write $T_2 = \bigcup_{\ell > 0} T_2(\ell)$. The classes T_2 and $T_2(\ell)$ are evidently invariant under a conformal map of $|z| < 1$ onto itself, while T_1 and $T_1(\ell)$ are not [3, Theorem 4].

2. STATEMENT OF THE MAIN RESULTS

THEOREM 1. *If* $f \in T_2(\ell)$, *then* f *is continuous and subnormal at all normal boundary points* ζ *of* C. *In particular, if* f *is normal in* D, *then* f *is continuous on the whole of* C, *in the metric of the Riemann sphere. Further, the image of the normal points of* C *by* $f(z)$ *lies on a path of length at most* ℓ *on the Riemann sphere.*

COROLLARY 1. *Each normal meromorphic function* $f(z)$ *in* T_2 *(and a fortiori each normal function in* T_1*) is continuous on* $|z| = 1$, *and therefore it can have no segments of Julia.*

This proves Conjecture 3 of Collingwood and Piranian [3] even for meromorphic functions. We also prove the following two results.

THEOREM 2. *If* f *is normal in* D, *then the limit*

$$L = \lim_{r \to 1} L(C_r)$$

exists as a finite or infinite limit. If L *is finite, then* L *is the length of the image of* C *by the function* f(z), *when* f *is extended to* C *by continuity.*

THEOREM 3. *If* $f \in T_2$ *and* ζ *denotes a point of* C, *then the following four statements are equivalent:*

(i) f *is normal at* ζ;

(ii) f *omits at least three values, in some neighbourhood of* ζ;

(iii) *the image by* f(z) *of some neighbourhood of* ζ *in* D *has finite area in the metric of the Riemann sphere;*

(iv) f *is continuous on an arc of* C *containing* ζ.

The *range* $R(\zeta)$ of f(z) at ζ is defined as the set of values that f(z) assumes at least once (and therefore infinitely often) in every neighbourhood of ζ. It follows from Theorem 3(iv) that if f is normal at ζ, then $R(\zeta)$ reduces either to $f(\zeta)$ or to the null set; and that otherwise, by (ii), the complement of $R(\zeta)$ is empty or consists of one or two points.

All these possibilities can occur. Thus Collingwood and Piranian showed [3, Theorem 5] that

$$f(z) = \exp\left(\frac{1+z}{1-z}\right)^2 \in T_1$$

and that f(z) has a segment of Julia ending at z = 1, so that the point z = 1 is abnormal. Here the range clearly excludes the points $w = 0, \infty$. If

$$F(z) = f + \frac{1}{f} = \frac{f^2+1}{f},$$

then $F'(z) = f'(1 - f^{-2})$, so that

$$\frac{|F'|}{1+|F|^2} = \frac{|f'| \cdot |f^2 - 1|}{|f|^2 + |f^2+1|^2} < A \frac{|f'|}{1+|f|^2},$$

where A is an absolute constant. Thus $F(z) \in T_1$; moreover, F(z) is regular and assumes every finite value infinitely often near z = 1, since F(z) = a whenever $f^2 + 1 - af = 0$, and since the roots of this equation in f are finite and different from zero. A similar argument shows that if

$$G = F + 1/F,$$

then $G \in T_1$ and G assumes every value, including infinity, infinitely often near z = 1.

The function f(z) = z belongs to $T_1(2\pi)$ and assumes no value more than once, so that its range is empty at every boundary point. Elsewhere [6, Theorem 2], we give an example of a normal function whose range consists of the point 0 at a sequence of boundary points, and it is not difficult to modify this example so that the corresponding set of points is uncountable.

The first of the following two theorems gives a simple condition under which f(z) is necessarily normal. The second theorem shows that the first cannot be extended to T_2.

THEOREM 4. *If* $f \in T_1(\ell)$ *with* $\ell < \pi$, *then* f *is normal in* $|z| < 1$. *Thus, if* $f \in T_1(0)$, *then* f *is constant.*

THEOREM 5. *There exist nonconstant functions* f(z) *in* $T_2(0)$.

A nonconstant function f in $T_2(0)$ cannot have any normal boundary point ζ, since $f(e^{i\theta})$ would have to be constant in a neighbourhood of ζ, by Theorem 1. Also, if $\{\Gamma_m\}$ is an "expanding sequence of curves" in $|z| < 1$ for which (1.4) holds (with $\ell = 0$), and if γ_m is the image of Γ_m on the Riemann sphere and w_0 is a limit point of a sequence of points w_m on γ_m, then for some increasing sequence $\{m_p\}$, each circular neighbourhood N of w_0 contains all except finitely many of the γ_{m_p}. It follows from the argument principle that f(z) assumes inside Γ_m equally often all values outside N. In particular, f(z) assumes in D infinitely often all values except possibly w_0. Thus f(z) *in* $T_2(0)$ *can have no Picard value other than* w_0. Also, any path Γ tending to C meets Γ_m for all sufficiently large m and so contains a sequence of points z_{m_p} ($z_{m_p} \in \Gamma_{m_p}$) such that $f(z_{m_p}) \to w_0$. *Thus* f(z) *can have no asymptotic value other than* w_0. *If the continua* γ_m *have a second limit point* w_1, *then* f(z) *can have no Picard value and no asymptotic value.* We shall provide two examples showing that each of these situations can actually occur; first, a regular function in $T_2(0)$ for which ∞ is an asymptotic value at every point of C, and then a meromorphic function in $T_2(0)$ having no Picard value or asymptotic value. We shall also show that Iversen's Theorem holds for $T_2(0)$, in the generalised sense that the Picard value, if it exists, is necessarily asymptotic along a spiral path Γ tending to C from D.

2.1. In order to state our next result, we need to introduce an extra hypothesis. Suppose that f is meromorphic in $|z| < 1$, and let A be an open arc of C or the whole of C. We say that f is *tame* on A if there exists a set E, dense on A, such that each point ζ of E is the endpoint of an asymptotic path of f, that is, a Jordan arc γ lying in D (except for the endpoint ζ) such that f approaches a finite or infinite limit (called *asymptotic value*) as $z \to \zeta$ along γ. The condition that f is tame has been discussed for regular functions by G. R. MacLane [10], who proved the following theorem.

THEOREM C. *If* f *is regular in* D *and*

$$\int_0^1 (1 - r)\, T(r, f)\, dr < \infty$$

(where T(r, f) is the Nevanlinna characteristic of f(z)), *then* f *is tame on* C. *More generally, if* A *is an arc of* C *and*

$$\int_0^1 (1 - r) \log^+ |f(re^{i\theta})|\, dr < \infty$$

for a set of points $\zeta = e^{i\theta}$ *dense on* A, *then* f *is tame on* A.

It also follows from Theorem A that if $f \in T_1$, then f is tame on C. On the other hand, our examples for Theorem 5 (see Section 7.1) show that f in T_2 need not have any asymptotic values and so need not be tame on any arc of C. A simple example by MacLane shows that a normal *meromorphic* function also need have no asymptotic values [10], though by Theorem C a normal *regular* function is necessarily tame on C, since such a function satisfies the condition

$$T(r, f) = O\left(\log \frac{1}{1-r}\right) \quad \text{as } r \to 1 .$$

If f is meromorphic and of bounded characteristic in $|z| < 1$, then f has radial limits p.p. on C, so that again f is tame on C. We can now state a result that is not only fundamental to the proof of Theorem 1, but has some independent interest.

THEOREM 6. *Suppose that* $f(z) \in T_2(\ell)$ *and that* f *is tame on an arc* $A = \{\zeta = e^{i\theta} \mid \theta_1 < \theta < \theta_2\}$ *of* C. *Then there exist asymptotic paths at each point* $\zeta = e^{i\theta}$ *of* A. *In particular, given any continuous positive function* $\varepsilon(x)$ $(0 < x < 1)$, *we may choose such paths* $\gamma_1(\theta)$ *and* $\gamma_2(\theta)$ *to lie in the regions*

$$D_1(\theta) = \{z = re^{i(\theta+\phi)} \mid 0 < \phi < 1,\ 1 - \varepsilon(\phi) < r < 1\}$$

and

$$D_2(\theta) = \{z = re^{i(\theta-\phi)} \mid 0 < \phi < 1,\ 1 - \varepsilon(\phi) < r < 1\},$$

respectively (except for the endpoint ζ*).*

The corresponding asymptotic values $\phi_1(\theta)$ *and* $\phi_2(\theta)$ *are continuous on the left and on the right, respectively, and they are uniquely determined by* θ. *If* $\phi(\theta)$ *denotes the asymptotic value corresponding to the asymptotic paths at* $e^{i\theta}$, *for the values* θ *in* $0 \leq \theta \leq 2\pi$ *for which such a path exists, then all the values* $\phi(\theta)$ *lie on a path of length at most* ℓ *in the metric of the Riemann sphere. Finally, the set of values* $\zeta_n \in C$ *at which* f *has distinct asymptotic values* w_n *and* w_n' *is countable, and* $\sum d(w_n, w_n') \leq \ell$, *where* $d(w_1, w_2)$ *denotes the great-circle distance from* w_1 *to* w_2 *on the Riemann sphere.*

Theorem 6 sharpens in particular the second part of Theorem A, even in its amended form. For if $f \in T_1$, then f is tame on C, and by Theorem 6 it has asymptotic values along suitable paths at every point ζ of C. On the other hand, Theorem 6 asserts nothing concerning the existence of rectifiable (let alone rectilinear) asymptotic paths or about paths with rectifiable images. We shall consider these problems for normal points of C in Theorems 8 to 10 in Section 6.

Theorem 3 gives five possibilities for the range of a function in T_2; while four of these may occur for a large number of ζ, there is a restriction in the fifth case.

THEOREM 7. *Suppose that* $f \in T_2(\ell)$ *and that* f *is tame on* C. *Then the set* E *of points of* C *where the range of* f *has a complement consisting of exactly two points is countable; moreover, if* $\{\zeta_n\}$ *is an enumeration of* E *and* w_n, w_n' *are the exceptional values at* ζ_n, *then*

$$\sum d(w_n, w_n') \leq \ell .$$

COROLLARY. *If* $f \in T_1$ *and* $f \neq a, b$ *in* D, *where* $a \neq b$, *then* f *is normal except at a finite number of points of* C. *Thus at most a finite number of points of* C *can be endpoints of segments of Julia.*

This result proves a special case of Collingwood and Piranian's first conjecture, which was that for a regular function $f \in T_1$ at most a finite number of points of C are endpoints of segments of Julia. While the full conjecture is false (as we pointed out earlier) the conjecture is correct with the additional assumption that f omits some value besides ∞ in D. This condition is in fact satisfied by the example

$$f(z) = \exp\left(\frac{1+z}{1-z}\right)^2$$

of Collingwood and Piranian. It would be interesting to know whether the condition that f is tame on C can be omitted in Theorem 7.

Theorem 7 also suggests the question whether with the hypotheses of that theorem there can be infinitely many segments of Julia at one of the points ζ_n. If not, we should have a proof of the second conjecture of Collingwood and Piranian (namely, that a holomorphic function in T_1 cannot have more than a finite number of segments of Julia), under the stronger hypothesis that f omits two distinct values instead of only one.

The remainder of our paper now proceeds as follows. In the next two sections we prove Theorems 6 and 1, respectively. In Sections 5 and 6, we investigate local properties of functions in T_2 at the normal arcs of C, and we prove Theorems 2 and 3. Finally, in Section 7 we prove in turn our remaining global results, namely Theorems 4, 5, and 7.

3. PROOF OF THEOREM 6

We now begin the proof of Theorem 6, which is fundamental to our further results. Suppose that f is meromorphic in D, and let E be a set of points of C that are endpoints of asymptotic paths. A finite system of paths $\gamma_1, \gamma_2, \cdots, \gamma_n = \gamma_1$ is called an *ordered system of asymptotic paths on* E if it satisfies the following four conditions.

(i) γ_ν lies in $|z| < 1$ except for one endpoint ζ_ν on E.

(ii) The paths $\gamma_\nu, \gamma_{\nu+1}$ have no common points in $r_0 \leq |z| < 1$, for some $r_0 < 1$.

(iii) If r is near enough to 1 so that all the paths γ_ν meet the circle C_r, and if $z_\nu = re^{i\alpha_\nu}$ denotes the last point of γ_ν on C_r (so that the arc of γ_ν from z_ν to ζ_ν lies in $r < |z| < 1$, except for its endpoints), then $\alpha_1 < \alpha_2 < \cdots < \alpha_n = \alpha_1 + 2\pi$.

(iv) f(z) approaches a limit w_ν as $z \to \zeta_\nu$ along γ_ν.

The value w_ν is called the *asymptotic value associated with* γ_ν. For any two complex numbers w_1 and w_2 (one of them may be infinite), we define $d(w_1, w_2)$ to be the minimum length of curves joining the points associated with w_1, w_2 on the Riemann sphere. We define the spherical boundary variation of f on E as

$$v_E(f) = \sup\left\{\sum_{\nu=1}^{n-1} d(w_\nu, w_{\nu+1})\right\},$$

where the w_ν are the asymptotic values associated with an ordered system of asymptotic paths on E, and where the supremum is taken over all such systems.

LEMMA 1. *If* $f \in T_2(\ell)$ *and* E *is a set of points of* C *that are endpoints of asymptotic paths of* f, *then* $v_E(f) \le \ell$.

Lemma 1 becomes intuitively clear if we consider an expanding sequence $\{\Gamma_m\}$ of rectifiable Jordan curves that satisfy condition (1.4), together with ordered sets on E that are associated with asymptotic values whose variation is nearly $v_E(f)$ (see Figure 1). The details of the proof are as follows.

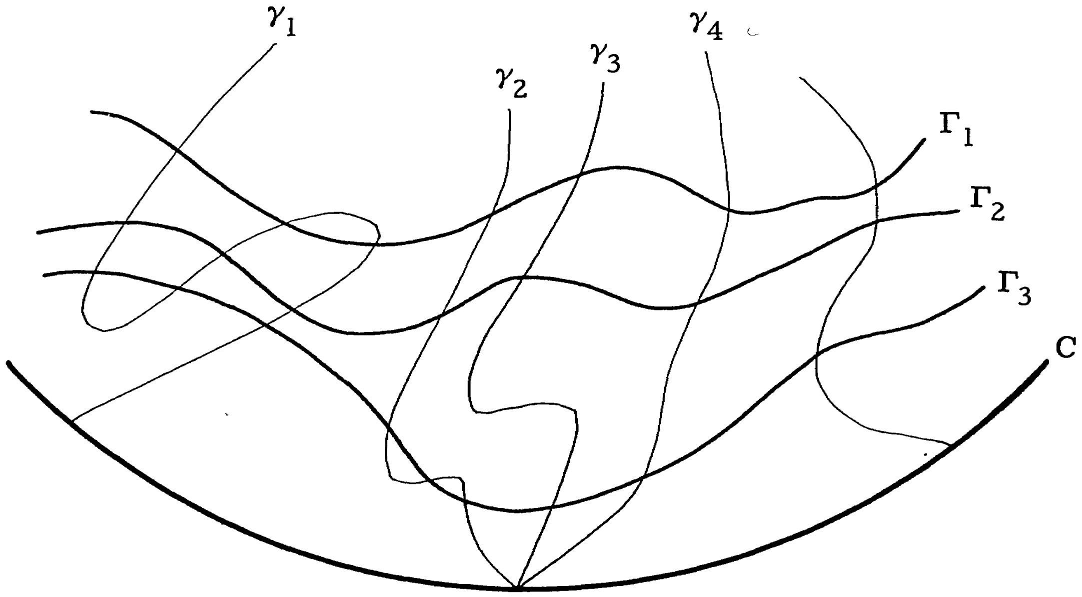

Figure 1.

Let $\{\gamma_\nu\}$ $(\nu = 1, \cdots, n)$ be an ordered system of asymptotic paths on E, and let $\{w_\nu\}$ denote the set of associated asymptotic values. Since $f \in T_2(\ell)$, there exists a Jordan curve Γ lying in the annulus $r_0 < |z| < 1$ such that

$$\int_\Gamma f^*(z)\,|dz| \le \ell + \varepsilon .$$

We may assume that r_0 is so near 1 that the paths γ_ν satisfy the conditions (i) to (iv), and so that if z_ν is a point of γ_ν in this annulus and $f(z_\nu) = w'_\nu$, then

$$d(w'_\nu, w_\nu) < \varepsilon/n .$$

We choose for z_ν a point of intersection of Γ with γ_ν. Such a point must exist, since by hypothesis Γ separates $|z| = r_0$ from $|z| = 1$; and in view of (iv), the arcs $z_\nu z_{\nu+1}$ of Γ are disjoint except for endpoints, for $\nu = 1, \cdots, n-1$. We also suppose $z_n = z_1$. Thus

$$\int_\Gamma f^*(z)\,|dz| = \sum_{\nu=1}^{n-1} \int_{z_\nu}^{z_{\nu+1}} f^*(z)\,|dz| \ge \sum_{\nu=1}^{n-1} d(w'_\nu, w'_{\nu+1})$$

$$\ge \sum_{\nu=1}^{n-1} (d(w_\nu, w_{\nu+1}) - 2\varepsilon/n) .$$

Hence

$$\sum_{\nu=1}^{n-1} d(w_\nu, w_{\nu+1}) \leq \int_\Gamma f^*(z)\,|dz| + 2\varepsilon \leq \ell + 3\varepsilon .$$

Here ε can be chosen as small as we please, and so Lemma 1 follows.

LEMMA 2. *Under the hypotheses of Lemma* 1, *the set of points of* E *that are endpoints of two paths with distinct asymptotic values is finite or countable. If this set is a sequence* $\{\zeta_n\} = \{e^{i\theta_n}\}$, *and if* w_n, w_n' *are distinct asymptotic values at* ζ_n, *then*

$$\sum d(w_n, w_n') \leq \ell .$$

The first part of Lemma 2 follows immediately from Bagemihl's theorem on disjoint arc cluster sets [2]. However, our hypotheses permit an extremely simple proof: Let $\zeta_1, \cdots, \zeta_N$ be N points of E such that distinct asymptotic values w_n and w_n' exist at ζ_n $(1 \leq n \leq N)$. Using the sequence $\{\Gamma_m\}$ as in the proof of Lemma 1, together with appropriate pairs of asymptotic paths terminating at $\zeta_1, \cdots, \zeta_N$, we see that the inequality $d(w_n, w_n') > \ell/\sqrt{N}$ can not hold for more than $\sqrt{N}$ indices n. Therefore pairs of distinct asymptotic values can occur at only countably many points on C. The proof of the second part of the lemma is similar.

LEMMA 3. *Suppose that, under the hypotheses of Lemma* 2, *the sequence* $\{\theta_n\}$ *converges monotonically to some value* θ. *Then* $\{w_n\}$ *and* $\{w_n'\}$ *converge to a common limit* w, *which depends only on* θ *and on whether* $\{\theta_n\}$ *is increasing or decreasing.*

The proof follows the same pattern as in the preceeding lemmas.

The following lemma establishes the existence of the paths $\gamma_i(\theta)$ (i = 1, 2) in Theorem 6.

LEMMA 4. *Suppose that* $f \in T_2(\ell)$ *and that* $\{\gamma_n\}$ *is a sequence of asymptotic paths (with endpoints* ζ_n*) such that either every finite system* $\{\gamma_n\}_1^N$ *or every finite system* $\{\gamma_n\}_N^1$ *is an ordered system. Let* $\{w_n\}$ *denote the corresponding sequence of asymptotic values, and let*

$$\lim w_n = w, \qquad \lim \zeta_n = \zeta = e^{i\theta} .$$

Then there exists an asymptotic path γ *with endpoint* ζ *and with the asymptotic value* w. *In the notation of Theorem* 6, *the path* γ *may be assumed to lie in* $D_1(\theta)$ *or in* $D_2(\theta)$ *(depending on the orientation of the system* $\{\gamma_n\}$*), if* $\zeta_n \neq \zeta$ *for each* n. *If the image of each path* γ_n *has finite spherical length, then* γ *can be chosen so that its image has finite spherical length.*

Proof. To be definite, suppose that the sets $\{\gamma_n\}_1^N$ form ordered systems. For each n, let γ_n denote an asymptotic path that lies in $D_2(\theta)$, except for its endpoint on C. Choosing a subsequence of $\{\gamma_n\}$, if necessary, we may assume that the image of γ_n under f lies within a spherical distance less than 2^{-n} from the point w. (In case the path γ_n has a rectifiable image, we may also suppose that the image has spherical length less than 2^{-n}.)

Since $f \in T_2(\ell)$, there exists an expanding sequence $\{\Gamma_m\}$ of Jordan curves tending to C and satisfying (1.4). For each m for which some arc of Γ_m joins the paths γ_n and γ_{n+1}, let L_{mn} denote the minimum of the spherical lengths of the images under f of such connecting arcs of Γ_m. Clearly, if we write

$$L_n = \liminf_{m \to \infty} L_{mn},$$

then $\sum_{n=1}^{\infty} L_n \leq \ell$. Therefore we can piece together appropriate arcs of the curves Γ_m and the paths γ_n so that the path γ thus obtained has the desired properties (see Figure 2).

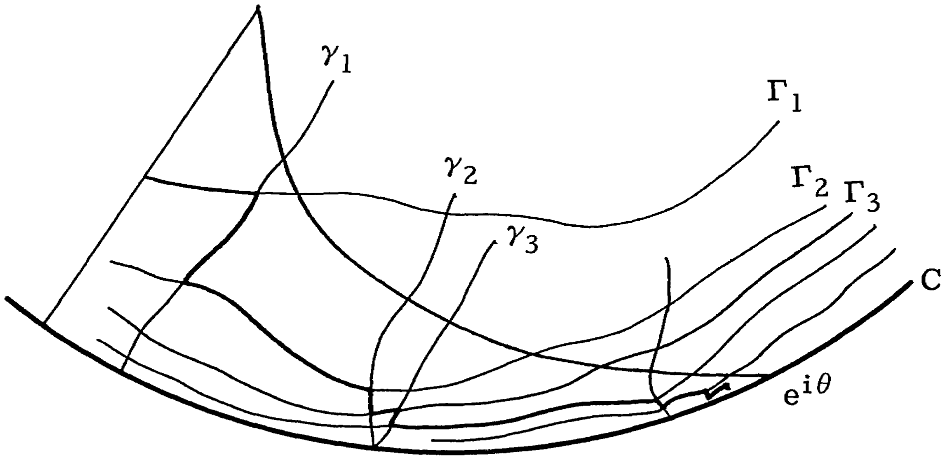

Figure 2.

To complete the proof of Theorem 6, let $f \in T_2(\ell)$, and let G denote any family of asymptotic paths that constitutes the union of an increasing sequence of finite systems of asymptotic paths. The ordered systems induce an order in G, and the asymptotic values associated with the paths in G define a function w on the space G. By Lemmas 1, 2, and 3, the total variation (in the spherical metric) of the function w on G does not exceed ℓ. If a sequence $\{\gamma_n\}$ of asymptotic paths in G is monotonic in terms of the order in G, then Lemma 4 permits us to adjoin to G an asymptotic path γ that intersects the paths γ_n in their natural order. This concludes the proof of Theorem 6.

4. PROOF OF THEOREM 1

In order to prove Theorem 1, we need a key result of Lehto and Virtanen [9]. It is best expressed in a conformally invariant form, as follows.

LEMMA 5. *Suppose that* $f(z) \in N(K)$ *in a Jordan domain* $\tilde{D}$ *of the closed plane, and that* $f(z)$ *is continuous on an arc* γ *of the boundary of* $\tilde{D}$. *For* $0 < \alpha < 1$, *let* $T_\alpha(\gamma)$ *be the domain consisting of all points of* $\tilde{D}$ *where the harmonic measure of* γ *with respect to* $\tilde{D}$ *is at least* α. *Then, for each* $\eta > 0$, *there exists* $\varepsilon > 0$, *depending only on* α, η, *and* K, *such that if for some point* a *in the closed plane*

$$d(f(z), a) < \varepsilon \qquad \text{on } \gamma,$$

then

$$d(f(z), a) < \eta \qquad \text{in } T_\alpha(\gamma).$$

From this, we shall deduce the following proposition.

LEMMA 6. *If* $f(z) \in T_2(\ell)$ *and the boundary points* $\zeta = e^{i\theta}$ *are normal for* $\theta_1 \leq \theta \leq \theta_2$, *where* $\theta_1 < \theta_2$, *then* $f(z)$ *has radial limits at* $e^{i\theta}$ *for a set of values* θ *that is dense in the interval* $[\theta_1, \theta_2]$.

It is clearly sufficient to show that f has a radial limit at $e^{i\theta}$ for some θ in $[\theta_1, \theta_2]$. Let $\{\Gamma_m\}$ be an expanding sequence of Jordan curves that tend to C and satisfy (1.4). We choose $\eta = \pi/4$ and $\alpha = 1/2$, in Lemma 5. Suppose that $f \in N(K)$ in the sector $0 < |z| < 1$, $\theta_1 < \arg z < \theta_2$, and let $\varepsilon = \varepsilon(K)$ be the corresponding value of ε in Lemma 5. Let q be an integer such that $(\ell + 1)/q < \varepsilon/2$, and define

$$\phi_\nu = \theta_1 + \nu(\theta_2 - \theta_1)/q \qquad (\nu = 0, 1, \cdots, q).$$

For sufficiently large m, f maps Γ_m onto a curve of spherical length at most $\ell + 1$. Hence, for at least one ν, there is an arc Γ'_m of Γ_m that joins the radii $\arg z = \phi_\nu$ and $\arg z = \phi_{\nu+1}$ in the sector $\phi_\nu \leq \arg z \leq \phi_{\nu+1}$, and whose image has spherical length less than $\varepsilon/2$. We choose ν so that this is true for infinitely many m. We suppose that Γ'_m joins $z_m = r_m e^{i\theta_\nu}$ and $z'_m = r'_m e^{i\theta_{\nu+1}}$ and lies (except for endpoints) in the sector $\phi_\nu < \arg z < \phi_{\nu+1}$. Let Δ_m be the Jordan domain bounded by Γ'_m and the straight-line segments $z_m 0$ and $0 z'_m$. By choosing a subsequence, if necessary, we can suppose that $\{f(z_m)\}$ converges to a, say, so that

$$d(f(z_m), a) < \varepsilon/2 \qquad \text{on } \Gamma'_m$$

for some large m. Since the spherical length of the image of Γ'_m is at most $\varepsilon/2$, we deduce that

$$d(f(z), a) \leq d(f(z), f(z_m)) + d(f(z_m), a) < \varepsilon$$

on Γ'_m. Thus, by Lemma 1,

$$d(f(z), a) \leq \pi/2, \tag{4.1}$$

in the domain $T_{1/2}(\Gamma'_m)$ defined as in Lemma 5 with respect to the domain Δ_m.

On letting $m \to \infty$, we see that (4.1) still holds in the domain $T_{1/2}(\gamma)$ defined similarly with respect to the sector $\theta_\nu < \arg z < \theta_{\nu+1}$, $|z| < 1$, where γ is the arc $z = e^{i\theta}$ $(\theta_\nu \leq \theta \leq \theta_{\nu+1})$. In fact, every point of $T_{1/2}(\gamma)$ lies in $T_{1/2}(\Gamma'_m)$ for all sufficiently large m, since the harmonic measure of a fixed arc increases with expanding domain. This implies that

$$\left|\frac{f(z) - a}{1 + \bar{a} f(z)}\right| < 1$$

in $T_{1/2}(\gamma)$, so that by Fatou's Theorem f(z) possesses radial limits (and in fact angular limits) p.p. on γ. This proves Lemma 6.

To complete the proof of Theorem 1, we need another result, which Lehto and Virtanen [9] deduce from Lemma 5.

LEMMA 7. *Suppose that* $f(z)$ *is normal in a domain* Δ *and that* $f(z) \to a_j$ *as* z *approaches a boundary point* ζ *of* Δ *along paths* γ_j *in* Δ $(j = 1, 2)$. *Then* $a_1 = a_2$, *and* $f(z) \to a_1$ *uniformly as* $z \to \zeta$ *between* γ_1 *and* γ_2.

We can now complete the proof of Theorem 1. Suppose that $\zeta = e^{i\theta_0}$ is a normal boundary point for $f(z) \in T_2(\ell)$. Then it follows from Lemma 6 that f(z) has radial limits at a dense set of points $e^{i\theta}$ in an interval (θ_1, θ_2) containing θ_0. Hence, by Theorem 6, f(z) has an asymptotic value $\phi_1(\theta_0)$ as $z \to e^{i\theta_0}$ along some path $\gamma_1(\theta_0)$ of that theorem and an asymptotic value $\phi_2(\theta_0)$ as $z \to e^{i\theta_0}$ along $\gamma_2(\theta_0)$. In view of Lemma 7, it follows that $\phi_1(\theta_0) = \phi_2(\theta_0) = \phi(\theta_0)$, say, and that

$$f(z) \to \phi(\theta_0) \tag{4.2}$$

uniformly as $z \to e^{i\theta_0}$ outside the domains $D_1(\theta_0)$ and $D_2(\theta_0)$.

This implies that (4.2) holds as $z \to e^{i\theta_0}$ in any manner from $|z| < 1$. For suppose contrary to this that $\{z_n\}$ is a sequence such that $|z_n| < 1$, $z_n \to e^{i\theta_0}$, and $f(z_n) \not\to \phi(\theta_0)$. Then we choose the function $\varepsilon(\phi)$ of Theorem 6 so that z_n lies outside $D_1(\theta)$ and $D_2(\theta)$ for each n, and so we obtain a contradiction. In fact, if

$$\phi_n = |\arg z_n - \theta_0|,$$

it is merely necessary to choose the function $\varepsilon(\phi)$ so that $\varepsilon(\phi_n) < 1 - |z_n|$ for each n. This proves that f(z) is continuous at the normal boundary points on $|z| = 1$.

Also, it follows from Theorem 6 that the boundary values $\phi(\theta)$ at these points lie on a path of length at most ℓ on the Riemann sphere.

It remains to show that if ζ is normal, then

$$(1 - |z|^2)f^*(z) \to 0$$

as $z \to \zeta$ in any manner from D, so that f is subnormal at ζ. We may suppose without loss in generality that $f(\zeta) = 0$, since this may be achieved by a rotation of the Riemann sphere in the w-plane, where $w = f(z)$. Then

$$|f(z)| < \varepsilon \tag{4.3}$$

if $|z| < 1$ and $|z - \zeta| < \delta$, say. Suppose now that $|z_0 - \zeta| < \delta/2$. Then (4.3) certainly holds in the circle

$$|z_0 - z| < (1 - |z_0|)/2.$$

Hence, by Cauchy's inequality,

$$f^*(z_0) \leq |f'(z_0)| < \frac{\varepsilon}{(1 - |z_0|)/2} < 4\varepsilon/(1 - |z_0|^2).$$

Since this is true for all z_0 sufficiently near ζ in $|z_0| < 1$, we deduce that ζ is subnormal. This completes the proof of Theorem 1.

5. BEHAVIOUR AT THE NORMAL BOUNDARY POINTS: PRELIMINARY RESULTS

Suppose that $f \in T_2$. We proceed to investigate more closely the behaviour of f at the points of C where f is normal.

LEMMA 8. *Suppose that* f(z) *is regular in* D *and remains (in the plane metric) continuous on* C. *Suppose further that the function* $f(e^{i\theta})$ $(0 \leq \theta \leq 2\pi)$ *has finite variation* ℓ. *Then*

$$\ell(r) = \int_0^{2\pi} |f'(re^{i\theta})|\, r\, d\theta \leq r\ell \qquad (0 < r < 1). \tag{5.1}$$

The result is (probably) known, but for the sake of completeness I include the following simple proof, which was communicated to me by C. Pommerenke. Suppose that $\theta_0 = 0 < \theta_1 < \cdots < \theta_k = 2\pi$, and consider the function

$$u(z) = \sum_{\nu=0}^{k-1} |f(ze^{i\theta_{\nu+1}}) - f(ze^{i\theta_\nu})|.$$

Then u(z) is subharmonic in $|z| < 1$ and continuous in $|z| \leq 1$, and so u(z) attains its maximum in $|z| \leq 1$ at a point $\zeta = e^{i\theta}$ on C. Thus

$$u(z) \leq \sum_{\nu=0}^{k-1} |f(e^{i(\theta+\theta_{\nu+1})}) - f(e^{i(\theta+\theta_\nu)})| \leq \ell,$$

by hypothesis. If r is fixed, we can choose the θ_ν so that

$$\ell(r) \leq u(r) + \varepsilon \leq \ell + \varepsilon.$$

Since ε is arbitrary, we deduce that $\ell(r) \leq \ell$. Finally, we note that $|f'(z)|$ is subharmonic, so that the mean value $\ell(r)/r$ increases with r. This gives the required result.

The next two lemmas require some further notation. For each α $(0 < \alpha < \pi/2)$ and each ρ $(0 < \rho \leq \cos\alpha)$, we write

$$D(\theta, \alpha, \rho) = \{z: 0 < |1 - ze^{-i\theta}| \leq \rho,\ |\arg(1 - ze^{-i\theta})| \leq \alpha\}$$

and

$$M(\theta, \alpha, \rho, f) = M(\theta, \alpha, \rho) = \sup_{z \in D(\theta,\alpha,\rho)} |f'(z)|. \tag{5.2}$$

Also, we shall say that a sequence of arcs

$$\gamma_n = \{z: z = r_n(\theta)e^{i\theta},\ \theta_n \leq \theta \leq \theta_n'\} \qquad (n = 1, 2, \cdots)$$

smoothly approximates the arc γ: $z = e^{i\theta}$ $(\theta_0 \leq \theta \leq \theta_0')$ of C provided that

(i) $0 < r_n(\theta) < 1$ $(\theta_n \leq \theta \leq \theta_n')$,

(ii) $|r_n'(\theta)| \leq K$ $(\theta_n \leq \theta \leq \theta_n'$; K independent of n),

(iii) $\theta_n \to \theta_0$ and $\theta_n' \to \theta_0'$ as $n \to \infty$,

(iv) $r_n(\theta) \to 1$ and $r_n'(\theta) \to 0$ as $n \to \infty$, p.p. in θ.

LEMMA 9. *Under the hypotheses of Lemma* 8, $f'(z) \to f'(e^{i\theta})$ p.p. *in* θ *as* $z \to e^{i\theta}$ *in* $D(\theta, \alpha, \zeta)$, *for fixed* α *and* ρ. *Further,* (5.3) $$\int_0^{2\pi} M(\theta, \alpha, \rho)\, d\theta \leq K(\alpha)\ell,$$

where $K(\alpha)$ *is independent of* ρ, ℓ, *and* f.

LEMMA 10. *Under the hypotheses of Lemmas* 8 *and* 9, $f(e^{i\theta})$ *is absolutely continuous for* $0 \leq \theta \leq 2\pi$, *and if the sequence of arcs* γ_n *smoothly approximates the arc* $z = e^{i\theta}$, $\theta_0 \leq \theta \leq \theta_0'$ *of* C, *then*

(5.4) $$\int_{\gamma_n} |f'(z)|\,|dz| \to \int_{\theta_0}^{\theta_0'} |f'(e^{i\theta})|\, d\theta \quad \text{as } n \to \infty.$$

The results of Lemmas 9 and 10 that are not already known are simple consequences of known results. In the first instance, (5.3) is a form of the Hardy-Littlewood Maximum Theorem [7] applied to the subharmonic functions $|f'(z)|^{1/2}$ and (5.1). It follows from (5.1) that $f'(z)$ is of bounded characteristic and so possesses p.p. in θ a limit $\phi(\theta)$ as $z \to e^{i\theta}$ in any $D(\theta, \alpha, \rho)$. Also,

$$\int_{\gamma_n} |f'(z)|\,|dz| = \int_{\theta_n}^{\theta_n'} |f'[r_n(\theta)e^{i\theta}]|\,|r_n'(\theta) + i r_n(\theta)|\, d\theta .$$

Here the integrand is (for varying n and θ) uniformly bounded by $|K + i|\, M(\theta, 0, 1)$, and this function is integrable over $[0, 2\pi]$, by (5.3). Also, the integrand tends to $|\phi(\theta)|$ p.p. in θ for $\theta_1 < \theta < \theta_2$, and to zero for θ outside this range, so that by Lebesgue's dominated-convergence theorem,

(5.5) $$\int_{\gamma_n} |f'(z)|\,|dz| \to \int_{\theta_0}^{\theta_0'} |\phi(\theta)|\, d\theta$$

and similarly

$$\int_{\gamma_n} f'(z)\,dz = f[r_n(\theta_n')e^{i\theta_n'}] - f[r_n(\theta_n)e^{i\theta_n}] \to \int_{\theta_0}^{\theta_0'} \phi(\theta)\, i e^{i\theta}\, d\theta .$$

But by the continuity of f(z), we see (at least if the γ_n are chosen so that $r_n(\theta) = 1 - 1/n$, for instance) that the left-hand side tends to $f(e^{i\theta_0'}) - f(e^{i\theta_0})$. Thus $f(e^{i\theta})$ is absolutely continuous, and the integral of $i\phi(\theta)e^{i\theta}$ (and hence the derivative $ie^{i\theta} f'(e^{i\theta})$ of $f(e^{i\theta})$) exists p.p. and is equal to $i\phi(\theta)e^{i\theta}$. Thus $\phi(\theta) = f'(e^{i\theta})$ p.p. in θ, and this proves the first statement of Lemma 10. Also, (5.5) now yields (5.4), since $|f'(e^{i\theta})| = \phi(\theta)$ p.p.

LEMMA 11 (Fejér and Riesz [4]). *With the hypothesis of Lemma* 8,

$$\int_{-1}^{1} |f'(re^{i\theta})|\, dr \leq \ell/2 \qquad (0 \leq \theta \leq 2\pi).$$

5.1. From the global results represented by Lemmas 8 to 11 we proceed to deduce the local theorems we require.

LEMMA 12. *Let* D_0 *be a subdomain of* D *whose frontier consists of the arc* γ: $z = e^{i\theta}$ $(\theta_1 \leq \theta \leq \theta_1')$ *of* C, *together with a crosscut joining* $e^{i\theta_1}$ *and* $e^{i\theta_1'}$ *in* D. *Suppose that* $w = f(z)$ *is regular in* D_0 *and continuous in* $\overline{D}_0$, *and that* f(z) *maps* γ *onto a path of finite length in the* w-*plane. Then*

$$\int_{r_0}^{1} |f'(re^{i\theta})|\, dr < \infty$$

if $\theta_1 < \theta < \theta_1'$ *and* r_0 *is sufficiently near* 1.

The following proof was suggested to me by J. G. Clunie. We may suppose without loss in generality that $\theta = 0$ and that $\theta_1 < -2\delta < 0 < 2\delta < \theta_1'$. Then, if r_0 is sufficiently near 1 and z_0 lies in the set

$$D_1 = \{z_0 \colon r_0 < |z_0| < 1,\ |\arg z_0| \leq \delta\},$$

$|f(z)|$ is bounded by some constant M, in the disk $|z - z_0| < 1 - |z_0|$, and so

$$|f'(z_0)| \leq M/(1 - |z_0|) \tag{5.6}$$

in the set D_1. We now set $z_1 = e^{-i\delta}$ and $z_2 = e^{i\delta}$, and we consider the function

$$\phi(z) = (z - z_1)(z - z_2) f(z) \tag{5.7}$$

on the frontier C_1 of D_1. On γ, $\phi(z)$ is the product of two functions of bounded variation of θ, so that $\phi(z)$ has bounded variation as a function of θ and maps γ onto a path of finite length. On the remaining part of C_1,

$$|\phi'(z)| \leq |z - z_1|\,|z - z_2|\,|f'(z)| + |2z - z_1 - z_2|\,|f(z)|,$$

and the right-hand side is uniformly bounded, in view of (5.6). Hence $\phi(z)$ maps the boundary of D_1 onto a curve of finite length ℓ.

Let $z = z(s)$ give a symmetrical map of $|s| < 1$ onto D_1, so that the real axes correspond. Then $\phi[z(s)]$ is regular in $|s| < 1$ and continuous in $|s| \leq 1$, and maps $|s| = 1$ onto a curve of length ℓ. Hence, by Lemma 11,

$$\int_{-1}^{1} |\phi'[z(s)]|\,|z'(s)|\, ds \leq \ell/2,$$

that is, $\int_{r_0}^{1} |\phi'(r)|\, dr \leq \ell/2$. We now apply (5.7) and note that for $r_0 < z < 1$,

$$|f'(z)| = \left|\frac{\phi'(z)}{(z - z_1)(z - z_2)} - \frac{\phi(z)(2z - z_1 - z_2)}{(z - z_1)^2 (z - z_2)^2}\right| \leq K_1 |\phi'(z)| + K_2,$$

where K_1 and K_2 are constants. Thus

$$\int_{r_0}^{1} |f'(r)|\, dr \leq K_1 \ell/2 + K_2(1 - r_0),$$

and this proves Lemma 12.

LEMMA 13. *Suppose that* f(z) *satisfies the hypotheses of Lemma* 12 *and that* $\theta_1 < \theta_0 < \theta_0' < \theta_1'$ *and* $0 < \alpha < \pi/2$. *Then the function* $f(e^{i\theta})$ *is absolutely continuous for* θ *in the interval* $[\theta_0, \theta_0']$ *and so has a derivative* $ie^{i\theta} f'(e^{i\theta})$ p.p. *in this interval. Further,* p.p. *in* $[\theta_0, \theta_0']$,

$$f'(z) \to f'(e^{i\theta})$$

as $z \to e^{i\theta}$ *in any domain* $D(\theta, \alpha, \rho)$ *for* $0 < \alpha < \pi/2$. *Finally, if* $0 < \alpha < \pi/2$, $M(\theta, \alpha, \rho)$ *is defined by* (5.2), *and* ρ *is sufficiently small, then*

$$\int_{\theta_0}^{\theta_0'} M(\theta, \alpha, \rho)\, d\theta < \infty .$$

Let θ_2 and θ_2' be chosen so that $\theta_1 < \theta_2 < \theta_0 < \theta_0' < \theta_2' < \theta_1'$, and let

$$D_2 = \{z: z = re^{i\theta},\ r_0 < r < 1,\ \theta_2 < \theta < \theta_2'\}.$$

Then it follows from Lemma 12 that f(z) is continuous on the boundary C_2 of D_2 and maps this boundary onto a path of finite length ℓ, say. Suppose now that $z = z(s)$ maps $|s| < 1$ onto D_2. The map is analytic and conformal on an arc

$$\Gamma_0 = \{s: s = e^{i\phi},\ \phi_0 \leq \phi \leq \phi_0'\}$$

that corresponds to the arc $\gamma_0 = \{z: z = e^{i\theta},\ \theta_0 \leq \theta \leq \theta_0'\}$ of $|z| = 1$. Hence we can apply Lemmas 9 and 10 to

$$F(s) = f[z(s)]$$

instead of f(z). Also, $\frac{s}{z}\frac{dz}{ds}$ remains continuous, positive, and bounded above and below on Γ_0. If $D_1(\phi, \alpha, \rho)$ and $M_1(\phi, \alpha, \rho)$ are defined with respect to $|s| < 1$ and F(s), and $D(\theta, \alpha, \rho)$ and $M(\theta, \alpha, \rho)$ are defined with respect to $|z| < 1$ and f(z), we see that, given α such that $0 < \alpha < \pi/2$, then, for

$$\alpha' = \frac{1}{2}\left(\alpha + \frac{\pi}{2}\right) \quad \text{and} \quad \rho' = \cos \alpha' ,$$

we can choose ρ so small that $D(\theta, \alpha, \rho)$ corresponds to a subdomain of $D_1(\phi, \alpha', \rho')$, where $z = e^{i\phi}$ corresponds to $z = e^{i\theta}$. Thus

$$\int_{\theta_0}^{\theta_0'} M(\theta, \alpha, \rho)\, d\theta < \text{constant} \times \int_{\phi_0}^{\phi_0'} M_1(\phi, \alpha', \rho')\, d\phi < \infty .$$

Also, since F'(s) has an angular limit $f'(e^{i\phi})$ p.p. in ϕ in the interval $[\phi_0, \phi_0']$, the derivative

$$f'(z) = F'(s)\frac{ds}{dz}$$

has the angular limit $\frac{ds}{dz} F'(e^{i\phi}) = f'(e^{i\theta})$ p.p. in θ.

This completes the proof of Lemma 13.

6. STATEMENT AND PROOF OF THE LOCAL THEOREMS

THEOREM 8. *Suppose that* $f \in T_2$ *and that* f *is normal at all points of the arc* γ: $\zeta = e^{i\theta}$ $(\theta_1 \leq \theta \leq \theta_2)$. *Then* $f(e^{i\theta})$ *is absolutely continuous on* $[\theta_1, \theta_2]$ *in the metric of the Riemann sphere. Further, if the sequence of arcs* γ_n *smoothly approximates* γ, *then*

$$\int_{\gamma_n} f^*(z)\,|dz| \to \int_\gamma f^*(e^{i\theta})\,d\theta \qquad \text{as } n \to \infty.$$

COROLLARY. *If* f *is normal in* D, *then* $f(e^{i\theta})$ *is absolutely continuous in* $[0, 2\pi]$ *and*

$$L(C_r) = \int_0^{2\pi} f^*(re^{i\theta})\,r\,d\theta \to \int_0^{2\pi} f^*(e^{i\theta})\,d\theta \qquad \text{as } r \to 1.$$

In particular, Theorem 2 *holds.*

THEOREM 9. *With the hypotheses of Theorem* 8,

$$S(\theta_1, \theta_2, f) = \int_0^1 \int_{\theta_1}^{\theta_2} [f^*(re^{i\theta})]^2\,r\,dr\,d\theta < \infty. \tag{6.1}$$

Conversely, if $f \in T_2$ *and* (6.1) *holds, then* f *is normal at* $\zeta = e^{i\theta}$ *for* $\theta_1 < \theta < \theta_2$.

Finally, we prove a result which shows what curves in D map onto curves of finite length on the Riemann sphere.

THEOREM 10. *With the hypotheses of Theorem* 8, *suppose that* $\theta_1 < \theta < \theta_2$ *and that* γ_1 *is an arc of the form*

$$z = e^{i\theta} + t\,e^{i\phi(t)} \qquad (0 \leq t \leq t_0),$$

lying in D *except for the endpoint* $e^{i\theta}$, *and such that* $\phi(t)$ *is absolutely continuous and* $|t\phi'(t)|$ *is essentially bounded in* $(0, t_0)$. *Then*

$$\int_{\gamma_1} f^*(z)\,|dz| < \infty.$$

For instance, all crosscuts in D with continuously turning tangents and with endpoints at normal points of D have images of finite length on the Riemann sphere.

6.1. *Proof of Theorems* 8 *and* 2. In order to prove Theorem 8, we suppose first that $|f(e^{i\theta})| < 2$ on γ. Then it follows from Lemma 13 that $f(e^{i\theta})$ is absolutely continuous on γ in the ordinary metric, and that $f'(z)$ has the angular limit $f'(e^{i\theta})$ p.p. on γ. Also, since f is continuous on γ, it follows that

$$f^*(z) = \frac{|f'(z)|}{1 + |f(z)|^2} \to f^*(e^{i\theta}) = \frac{|f'(e^{i\theta})|}{1 + |f(e^{i\theta})|^2} \qquad \text{p.p. on } \gamma,$$

as $z \to e^{i\theta}$ in any Stolz angle $D(\theta, \alpha, \rho)$. Again,

$$\int_{\gamma_n} f^*(z)\,|dz| = \int_{\theta_n}^{\theta_n'} \frac{|f'(r_n e^{i\theta})|\,|r_n'(\theta) + i r_n(\theta)|\,d\theta}{1 + |f(r_n e^{i\theta})|^2}.$$

In view of the definition of smooth approximants, the integrand tends to $f^*(e^{i\theta})$ as $n \to \infty$, p.p. in θ, and for fixed θ and varying n it is dominated by

$$K \sup_{0<r<1} f^*(re^{i\theta}) = KM(\theta),$$

say. Also, if

$$M_\rho(\theta) = \sup_{1-\rho<r<1} f^*(re^{i\theta}),$$

and if ρ is fixed and sufficiently small, then

$$M(\theta) \leq M_\rho(\theta) + K' \qquad (\theta_1 \leq \theta \leq \theta_2)$$

and

$$\int_{\theta_1}^{\theta_2} M(\theta)\,d\theta < \int_{\theta_1}^{\theta_2} M_\rho(\theta)\,d\theta + 2\pi K' < \infty,$$

by Lemma 13. Thus, by Lebesgue's dominated-convergence theorem, we deduce Theorem 8.

If $|f| > 1/2$ on γ, we deduce the result similarly by considering $1/f$ instead of f. In the general case, we divide γ into a finite number of arcs in each of which either $|f| < 2$ or $|f| > 1/2$. The general result follows by addition when we make a corresponding dissection of the arcs γ_n. Finally, we deduce the corollary by choosing for $\{\gamma_n\}$ a sequence of circles $z = r_n e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$, where $r_n \to 1$. Theorem 2 follows at once from the corollary. For if f is normal and does not belong to T_2, then $L = \infty$ in Theorem 2.

6.2. *Proof of Theorem* 9. Let D_r be the sector $0 < |z| < r$ $(\theta_1 \leq \arg z \leq \theta_2)$, and let γ_r be the arc $z = re^{i\theta}$ $(\theta_1 \leq \theta \leq \theta_2)$. Then, by Theorem 8,

$$\int_{\gamma_r} f^*(re^{i\theta})\,d\theta < K \qquad (0 < r < 1).$$

Also, if $f(e^{i\theta_1}) \neq \infty$, it follows from Lemma 12 that if r_0 is close to 1, then

$$\int_{r_0}^{1} f^*(te^{i\theta_1})\,dt \leq \int_{r_0}^{1} |f'(te^{i\theta_1})|\,dt < \infty,$$

and so

$$\int_0^1 f^*(te^{i\theta_1})\,dt < \infty,$$

since $f^*(te^{i\theta_1})$ is continuous in $0 \leq t \leq r_0$. If $f(e^{i\theta_1}) = \infty$, we obtain the same result by considering $1/f$ instead of f. Thus, if g_r denotes the boundary of D_r, it follows that the length L_r of the image of g_r by f(z) on the Riemann sphere is at most K' for $0 < r < 1$, where K' is a constant. Let w_1, w_2, w_3 be three distinct complex numbers not assumed by f(z) on the segments $\arg z = \theta_1$ and $\arg z = \theta_2$ or on γ. Since the images of these sets on the Riemann sphere have finite length, such points w_j exist. Also, the number of roots n_j of the equation $f(z) = w_j$ in D_1 must be finite. Now it follows from Ahlfors' theory of covering surfaces ([1], see also [6, p. 148 *et seq.*]) that if S_r is the area in the spherical metric of the image of D_r by f(z), then

$$S_r < \pi(n_1 + n_2 + n_3 + hL_r) = O(1) \quad \text{as } r \to 1,$$

where h is a constant depending only on w_1, w_2, w_3. This proves (6.1). From Theorem B it follows that if f is not normal at $\zeta = e^{i\theta}$, where $\theta_1 < \theta < \theta_2$, then (6.1) cannot hold.

6.3. *Proof of Theorem* 10.

LEMMA 14. *With the hypothesis of Theorem* 10, *let*

$$M(t) = \sup_{|z|<1,\ |z-e^{i\theta}|=t} f^*(z).$$

Then, if ε *is sufficiently small,*

$$\int_0^{\varepsilon} M(t)\,dt < \infty.$$

We may suppose without loss in generality that $f(e^{i\theta}) \neq \infty$, since otherwise we can consider 1/f instead of f. We also suppose ε so small that if $|e^{i\theta'} - e^{i\theta}| \leq 2\varepsilon$, then $\theta_1 \leq \theta' \leq \theta_2$ and $f(e^{i\theta'})$ is finite. Let $\phi = \phi(t)$ be the number such that $0 < \phi < \pi/2$ and

$$|e^{i(\theta+\phi)} - e^{i\theta}| = 2\sin\phi/2 = t \quad (0 < t < \varepsilon). \tag{6.2}$$

We note that the arc $|z - e^{i\theta}| = t$ $(|z| < 1)$ lies in

$$D(\theta + \phi, \alpha, \rho) \cup D(\theta - \phi, \alpha, \rho)$$

provided that

$$\alpha \geq \frac{\phi}{2}, \quad \alpha \geq \frac{\pi}{4} - \frac{3\phi}{4}, \quad \text{and } \rho \geq 2t\cos\left(\frac{\pi+\phi}{4}\right),$$

which is certainly the case if $\alpha = \pi/4$, and $\rho = 2\varepsilon$. We now choose ε so small that

$$\int_{-\varepsilon}^{\varepsilon} M(\theta + \phi, \pi/4, 2\varepsilon)\,d\phi < \infty,$$

in the notation of Lemma 13. Then t and ϕ are related by (6.2), $0 < t < \varepsilon$,

$$M(t) \leq M(\theta + \phi, \pi/4, 2\varepsilon) + M(\theta - \phi, \pi/4, 2\varepsilon),$$

and hence

$$\int_0^{\varepsilon} M(t)\,dt \le \int_0^{\varepsilon} \left\{ M\left(\theta + \phi, \frac{\pi}{4}, 2\varepsilon\right) + M\left(\theta - \phi, \frac{\pi}{4}, 2\varepsilon\right) \right\} \left|\frac{dt}{d\phi}\right| d\phi$$

$$\le \int_{-\varepsilon}^{\varepsilon} M\left(\theta + \phi, \frac{\pi}{4}, 2\varepsilon\right) d\phi < \infty .$$

This proves Lemma 14.

To complete the proof of Theorem 10, we suppose without loss in generality that $t_0 \le \varepsilon$, where ε is the quantity in Lemma 14 (otherwise, $f^*(z)$ is certainly bounded on the closed subarc $[\varepsilon, t_0]$ of γ_1, which is a compact subset of D). By hypothesis,

$$z = e^{i\theta} + t e^{i\theta(t)}, \qquad |dz| = |1 + t\phi'(t)|\, dt < K dt$$

on γ_1. Thus, by Lemma 14,

$$\int_{\gamma_1} f^*(z)\, |dz| \le K \int_0^{\varepsilon} M(t)\, dt < \infty .$$

This proves Theorem 10.

6.4. *Proof of Theorem* 3. Suppose now that $f \in T_2$ and that f is abnormal at ζ, so that condition (i) of Theorem 3 does not hold. Then, by Theorem B, (ii) and (iii) are also false, and so is (iv); since if f is continuous at ζ, f(z) cannot assume any value other than $f(\zeta)$ infinitely often near ζ. Next, if (i) holds, so that f is normal at ζ, then f remains continuous at ζ and at all points of C near ζ, since f is necessarily normal at points near ζ. Thus (iv) and so the weaker condition (ii) holds. Also, (iii) follows from Theorem 9. This proves Theorem 3.

7. PROOF OF THEOREMS 4, 5, AND 7

We proceed to prove the remainder of our global results, namely Theorems 4, 5, and 7. We prove Theorem 4 under the somewhat stronger assumption that

$$\text{(7.1)} \qquad L(C_r) \le \pi \qquad (r_0 < r < 1).$$

The argument is similar to one given by Lehto [8].

Let Γ_r be the image on the Riemann w-sphere of C_r by the mapping induced by f(z). Then the length of the curve Γ_r on the sphere is at most π for $r_0 < r < 1$, and hence (see for example [5, p. 152]) the complement of Γ_r contains a hemisphere H_r. Also, unless Γ_r is the union of two great semicircles, we may assume that Γ_r lies in the interior of the complementary hemisphere H_r'. On the other hand, if for some r the curve Γ_r contains an arc of a circle, then f(z) can be analytically continued into the whole closed plane, by simultaneous reflection in $|z| = r$ and Γ_r, so that f is rational and so certainly normal in $|z| < 1$. Thus we may ignore this case, and assume that in the range (7.1) Γ_r has a complementary domain Δ_r which contains a closed hemisphere in its interior and so contains at least one of each pair of diametrically opposite points w and $\tilde{w}$. In particular, the equations f(z) = w and $f(z) = \tilde{w}$ cannot both have roots on $|z| = r$, if (7.1) holds.

Next, if $f(z_1) = f(z_2) = w$, where $r_0 < |z_1| < |z_2| < 1$, then $f(z) \neq \tilde{w}$ for $|z_1| < |z| < |z_2|$. For we suppose without loss in generality that $f(z) \neq w$ in this range, which could otherwise be subdivided. Write $|z_j| = |r_j|$ $(j = 1, 2)$. Then Γ_{r_1} and Γ_{r_2} contain w, so that $\tilde{w}$ lies in Δ_r for $r = r_1, r_2$. Suppose now that we can find an r_3 such that $r_1 < r_3 < r_2$ and Γ_{r_3} contains $\tilde{w}$, so that Δ_{r_3} contains w. Since Δ_r always contains at least one of w and $\tilde{w}$, it follows by continuity that we can find r_1' and r_2' such that $r_1 < r_1' < r_3 < r_2' < r_2$ and Δ_r contains both w and $\tilde{w}$ for $r = r_1'$ and r_2'. Hence these two values are assumed equally often by $f(z)$ in $|z| \leq r_1'$ and $|z| \leq r_2'$, which contradicts the assumption that the equation $f(z) = \tilde{w}$ but not the equation $f(z) = w$ has a root in $r_1' < |z| < r_2'$.

We deduce at once that if the equation $f(z) = w$ has infinitely many roots in $|z| < 1$, then $f(z) = \tilde{w}$ has only a finite number of such roots. By taking for w and $\tilde{w}$ in turn the values $0, \infty$; $1, -1$; $i, -i$, we see that there are at least three values that f assumes only a finite number of times in $|z| < 1$, and so, by Theorem 3 (ii), f is normal at each point of C and so in $|z| < 1$. If now $\ell = 0$, then, by Theorem 1, $f(z)$ is constant on C and so in D.

7.1. *The class* $T_2(0)$: *Proof of Theorem* 5. Let $\{\lambda_n\}$ be a sequence of positive integers such that

$$\lambda_{n+1}/\lambda_n \to \infty, \tag{7.2}$$

and consider

$$f(z) = \sum_{n=1}^{\infty} a_n \lambda_n^{\alpha} z^{\lambda_n},$$

where $1 < \alpha < \infty$, where for each n

$$a_n = -1, 0, 1, \text{ or } 2,$$

and where infinitely many a_n are different from zero. Then for $|z| = r$ $(0 < r < 1)$ we have the inequality

$$|f(z)| \leq \sum_{m=1}^{\infty} 2m^{\alpha} r^m = O(1 - r)^{-\alpha-1}.$$

Thus

$$T(r, f) = O\left\{\log \frac{1}{1 - r}\right\},$$

and hence $f(z)$ is tame on C, in view of Theorem C.

We choose $r_n = \exp(-\lambda_n^{-1})$, and for $|z| = r_n$ we write

$$f(z) = \sum_{1}^{n-1} a_m \lambda_m^{\alpha} z^{\lambda_m} + a_n \lambda_n^{\alpha} z^{\lambda_n} + \sum_{n+1}^{\infty} a_m \lambda_m^{\alpha} z^{\lambda_m}$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3 .$$

Let p be an integer such that $p > \alpha$. Then

$$(7.3)\qquad \begin{aligned} \left|\Sigma_3\right| &\le 2 \sum_{n+1}^{\infty} \lambda_m^{\alpha} \exp\left(-\frac{\lambda_m}{\lambda_n}\right) < 2p! \sum_{n+1}^{\infty} \lambda_m^{\alpha}\left(\frac{\lambda_n}{\lambda_m}\right)^p \\ &= 2p!\,\lambda_n^p \sum_{n+1}^{\infty} \lambda_m^{\alpha-p} = O(\lambda_n^p \lambda_{n+1}^{\alpha-p}) = o(\lambda_n^{\alpha}), \end{aligned}$$

in view of (7.2). Again

$$(7.4)\qquad \left|\Sigma_1\right| \le 2 \sum_{m=1}^{n-1} \lambda_m^{\alpha} = O(\lambda_{n-1}^{\alpha}) = o(\lambda_n^{\alpha}),$$

in view of (7.2). Thus, on $|z| = r_n$,

$$(7.5)\qquad |f(z)| = \left|a_n \lambda_n^{\alpha} z^{\lambda_n} + o(\lambda_n^{\alpha})\right| = \frac{[|a_n| + o(1)]\lambda_n^{\alpha}}{e}.$$

By taking $\alpha + 1$ instead of α in the above argument, we see that

$$(7.6)\qquad |z f'(z)| = \frac{[|a_n| + o(1)]\lambda_n^{\alpha+1}}{e},$$

so that if $a_n \ne 0$, we have on $|z| = r_n$ the estimate

$$f^*(z) = \frac{|f'(z)|}{1 + |f(z)|^2} = \left(\frac{e}{|a_n|} + o(1)\right) \lambda_n^{1+\alpha},$$

and the last member tends to 0 as $n \to \infty$, since $\alpha > 1$. Thus,

$$L(C_{r_n}) \to 0 \qquad \text{as } n \to \infty,$$

so that $f \in T_2(0)$. Since f is also tame on C, it possesses asymptotic values at every point of C, by Theorem 6, and these asymptotic values must all be infinite in view of (7.5).

To construct a function in $T_2(0)$ without asymptotic values and without Picard values, we set

$$a_n = 1 \text{ if } n \text{ is odd}, \qquad a_n = 2 \text{ if } n \text{ is even},$$

and

$$g(z) = \sum_{1}^{\infty} \lambda_n^{\alpha} z^{\lambda_n}, \qquad \phi(z) = f(z)/g(z).$$

Using (7.5), together with its analogue for g, we see that

$$|\phi(z)| = a_n + o(1) \qquad (|z| = r_n),$$

and therefore ϕ can have no asymptotic value. We set

$$\psi(z) = g(z)[\phi(z) - a_n] = \sum_{m=1}^{\infty} (a_m - a_n)\lambda_m^{\alpha} z^{\lambda_m},$$

so that

$$z\psi'(z) = \sum_{m=1}^{\infty} (a_m - a_n)\lambda_m^{\alpha+1} z^{\lambda_m}.$$

We apply (7.3) and (7.4) to $\psi(z)$ and $z\psi'(z)$ on the circle $|z| = r_n$, and deduce that on this circle

$$|\psi(z)| = O(\lambda_n^p \lambda_{n+1}^{\alpha-p}) + O(\lambda_{n-1})^{\alpha},$$

$$|\psi'(z)| = O(\lambda_n^p \lambda_{n+1}^{\alpha+1-p}) + O(\lambda_{n-1})^{\alpha+1}.$$

We now choose $\lambda_n = 2^{K^n}$ where K is a positive integer such that $K > \frac{\alpha}{\alpha - 1}$. Then if p is sufficiently large,

$$\lambda_n^p(\lambda_{n+1})^{\alpha-p} = 2^{K^n[p+(\alpha-p)K]} \to 0 \qquad \text{as } n \to \infty.$$

Also,

$$\lambda_{n-1}^{\alpha} = 2^{\alpha K^{n-1}} = o(\lambda_n^{\alpha-1}).$$

Thus, on $|z| = r_n$, $\psi(z) = o(\lambda_n^{\alpha-1})$, and similarly $\psi'(z) = o(\lambda_n^{\alpha})$.

Hence, on $|z| = r_n$,

$$\phi'(z) = \frac{g(z)\psi'(z) - \psi(z)g'(z)}{[g(z)]^2} = \frac{o(\lambda_n^{2\alpha})}{\lambda_n^{2\alpha}} = o(1),$$

in view of (7.5) and (7.6) applied to $g(z)$. Hence

$$\phi^*(z) = o(1) \qquad \text{on } |z| = r_n,$$

so that $\phi(z) \in T_2(0)$.

Finally, we note that if w_0 is a Picard value for a function $f \in T_2(0)$, then w_0 is necessarily asymptotic in the more general sense that there exists a path Γ tending to C (but not necessarily with a definite endpoint on C) such that

$$f(z) \to w_0 \qquad \text{as } z \text{ tends to C on } \Gamma.$$

We may without loss in generality suppose that $w_0 = \infty$, so that $f(z)$ is regular for $r_0 \le |z| < 1$, say. Let $\{\Gamma_n\}$ be an expanding sequence of Jordan curves such that the spherical lengths of their images tend to 0, and such that

$$|f(z)| > n \qquad \text{on } \Gamma_n. \tag{7.7}$$

Let z_n be a point of Γ_n, and suppose that n_0 is an integer such that for $n > n_0$, Γ_n lies in the annulus $r_0 < |z| < 1$ and that $M(r_0, f) < n_0$. Consider the component K_n of the set $|f(z)| > n$ containing the point z_n. This component cannot meet $|z| = r_0$, and so it must have limit points on C. Thus there is a path δ_n in K_n joining z_n to a point of Γ_{n+1} and so to z_{n+1}, in view of (7.7). We now take the path consisting of the union of δ_{n_0}, δ_{n_0+1}, $\cdots$ and obtain a path Γ through all the points z_n for $n > n_0$ on which $f(z) \to \infty$. We can clearly allow Γ to spiral in turn around the various continua Γ_n and so to have the whole of C as its limiting set.

7.2. *Proof of Theorem* 7. The following form of Iversen's theorem for Tsuji functions yields Theorem 7 as an immediate consequence of Theorem 6.

THEOREM 11. *Suppose that* $f \in T_2$, *that* f *is tame on an arc* Γ: $z = e^{i\theta}$ $(\theta_1 < \theta < \theta_2)$ *of* C, *and that* $f(z) \neq w_0$ *in a neighbourhood of an interior point* $\zeta_0 = e^{i\theta}$ *of* Γ. *Then either* f *is normal at* ζ_0 *or* w_0 *is an asymptotic value at* ζ_0.

To prove Theorem 11, let $\phi_1(\theta)$ and $\phi_2(\theta)$ be asymptotic values at ζ_0 whose existence is asserted in Theorem 6. If one of these is equal to w_0, there is nothing to prove. Otherwise, we suppose that f is abnormal at ζ_0 and also (without loss in generality) that $w_0 = \infty$, so that $\phi_1(\theta)$ and $\phi_2(\theta)$ are finite. Since f is abnormal at ζ_0, f is unbounded there, and so there exists a sequence $\{z_n\}$ of points in D such that $z_n \to \zeta_0$ and $|f(z_n)| > n$.

Also, we may assume that z_n lies outside $D_1(\theta) \cup D_2(\theta)$ for each n, where $D_1(\theta)$ and $D_2(\theta)$ are the domains of Theorem 6. We then construct the paths $\gamma_1(\theta)$ and $\gamma_2(\theta)$ of Theorem 6, in $D_1(\theta)$ and $D_2(\theta)$, respectively, so that f is regular on γ_1 and γ_2 (except at ζ_0). Finally, we join the distinct endpoints of $\gamma_1(\theta)$ and $\gamma_2(\theta)$ by a Jordan arc on which f is regular. We thus obtain a closed Jordan curve γ that lies in D (except for ζ_0) and contains all but a finite number of the points z_n in its interior. Further, $|f(z)|$ is bounded by a constant M on γ and is regular and unbounded inside γ.

Consider now the component D_n of the set $|f(z)| > n$ containing z_n, where $n > M$. Then, for large n, z_n and so D_n lies inside γ, and $\overline{D}_n$ contains a frontier point of γ, which must be ζ_0, since $|f| < M$ on the other frontier points of γ. It now follows from the Phragmen-Lindelöf principle that f(z) cannot be bounded in D_n. Thus D_n contains a point z'_{n+1} such that

$$|f(z'_{n+1})| > n + 1,$$

and we can join z_n to z'_{n+1} on a path Γ_n in D_n on which $|f(z)| > n$. We now construct the component D_{n+1} of the set $|f(z)| > n + 1$ containing z'_{n+1}, and proceeding as before, we obtain a path that joins z'_{n+1} to a point z'_{n+2} in this component, where $|f(z'_{n+2})| > n + 2$. In this way we construct a path inside γ along which $f(z) \to \infty$. Since f is bounded inside γ, outside any fixed neighbourhood of ζ_0, this path must tend to ζ_0 and so has ∞ as an asymptotic value at ζ_0, as required. This proves Theorem 11.

Theorem 7 now follows at once from the last statement in Theorem 6. For if ζ_n is a point on C where f is abnormal and has two Picard values w_n and w'_n, then, by Theorem 11, f has two asymptotic values w_n and w'_n at ζ_n; by Theorem 6, the set of all such points ζ_n is countable and $\sum d(w_n, w'_n) \leq \ell$.

The corollary follows immediately; for if $f \in T_1(\ell)$ for some positive ℓ, then f is tame on C. Also, if $f \neq a, b$, then every abnormal point of C belongs to E. If there are k such points, we deduce from Theorem 7 that $k \cdot d(a, b) \leq \ell$. Thus E has at most finitely many points, and only these can be endpoints of segments of Julia.

It follows from Theorem 7 that the points ζ_n of E must be isolated abnormal points. For $f(z) \neq w_n, w_n'$ in some neighbourhood U of ζ_n, and therefore, at any other abnormal point ζ in U, f(z) has w_n and w_n' as Picard values, and so there are at most $\ell/d(w_n, w_n')$ such points ζ in U, by Theorem 7.

An interesting open question concerns the hypothesis that f is tame on an arc of C containing ζ_0. In fact, for the proof of Theorems 6 and 11 it is only necessary that there exist sequences $\{\zeta_n'\}$ and $\{\zeta_n''\}$ of points of C approaching ζ_0 from both sides at which f(z) has asymptotic values. If this is not the case, ζ_0 is the endpoint of an arc γ of abnormal points of C, and if ζ_0 is a point of the set E of Theorem 7, so that $f(z) \neq w_0, w_0'$ in some neighbourhood U of ζ_0, then the same is true at each point of γ in U. Thus in this case E contains a whole arc of points, none of which is an endpoint of an asymptotic path. If such a situation is possible, the conclusion of Theorem 7 certainly fails.

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