SINGULARITIES OF DIRICHLET SERIES WITH COMPLEX EXPONENTS

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1. INTRODUCTION

In this paper we are concerned with the function f(s) defined by

$$(1) \qquad f(s) = \sum_{n=1}^{\infty} a_n e^{-(\mu_n + i\nu_n)s},$$

where μ_n and ν_n are real numbers and μ_n increases and tends to infinity.

In order to ensure that the series (1) possesses an abscissa of convergence, and that this coincides with the abscissa of absolute convergence, we shall assume that

$$(2) \qquad \nu_n = o(\mu_n)$$

and that

$$(3) \qquad \frac{\log n}{\mu_n} \to 0 \text{ as } n \to \infty .$$

It is easy to see that if condition (3) is not satisfied, then the series (1) may fail to possess an abscissa of convergence, even if $\{\nu_n\}$ is bounded; as an example, we consider the series

$$(4) \qquad \sum_{1}^{\infty} \exp\{-[\log n + (-1)^n 2\pi i] s\} .$$

For $s = 1/4$, this series becomes $i \sum (-1)^n n^{-1/4}$, which converges; but for $s = 1/2$, the series becomes $-\sum n^{-1/2}$, and this diverges.

Our purpose in this paper is to extend to the series (1), where μ_n and ν_n satisfy (2) and (3), the theorem of Vivanti [3], that if the arguments of the coefficients of a power series all lie within a fixed angle less than π, then the positive point on the circle of convergence is a singularity of the sum of the series.

THEOREM 1. *Suppose that the series in (1) satisfies condition (3), that the abscissa of convergence* σ_c *is finite, and that*

$$(5) \qquad |\arg a_n - \sigma_c \nu_n| \leq \chi < \pi/2 ,$$

where $-\pi < \arg a_n \leq \pi$. *If* $\sigma_c = 0$, *suppose also that*

$$(6) \qquad |\nu_n| \leq M,$$

where M *is some positive constant. Then* f(s) *has a singularity at* $s = \sigma_c$.

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Condition (5) is in fact a natural condition, since $\arg a_n - \sigma_c \nu_n$ is the argument of the nth term of the series (1) at the point $s = \sigma_c$. We note that if $\sigma_c \neq 0$, (6) follows from (5).

Lunc [2] has pointed out that the series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} = e^{-1} e^{-(1+\pi i/2)z} + e^{-1} e^{-(1-\pi i/2)z} + \cdots + e^{-k} e^{-(k+\pi i/2)z} + e^{-k} e^{-(k-\pi i/2)z} + \cdots,$$

for which $|\arg a_n - \sigma_c \nu_n| \leq \pi/2$, does not possess a singularity at the real point on the abscissa of convergence. We give as our second theorem a slight modification of Lunc's result.

THEOREM 2 (compare Lunc [2]). *For each* $\sigma_c \neq 0$, *there exists a Dirichlet series* (1) *with abscissa of convergence* σ_c *such that*

(i) $\mu_{n+1} - \mu_n \geq q > 0$,

(ii) $\{\nu_n\}$ *is bounded,*

(iii) $|\arg a_n - \sigma_c \nu_n| \leq \pi/2$,

(iv) *the point* $s = \sigma_c$ *is not a singularity of* $f(s)$.

Theorem 2 shows that, in the case $\sigma_c \neq 0$, Theorem 1 is the best possible result, in the sense that Theorem 1 becomes false if we relax condition (5) to allow $|\arg a_n - \sigma_c \nu_n| \leq \pi/2$. In imposing condition (i), we have chosen the case most likely to yield Theorem 1, in that $|\nu_n/\mu_n| = O(1/n)$ and $\{\nu_n/(\mu_n - \mu_{n-1})\}$ is bounded. It is conceivable that when $\sigma_c = 0$, Theorem 1 is true for some class of unbounded sequences $\{\nu_n\}$. Whether this is the case is not known.

2. PRELIMINARIES TO PROOF OF THEOREM 1

In our proof we shall use the representation of the series (1) as a Stieltjes integral. Thus we write $a_n = a'_n e^{i\theta_n}$, where $a'_n \geq 0$, and we define the quantities $\phi(t)$, $\theta(t)$, and $\alpha(t)$ as follows: Let ϕ and θ be continuous functions of t $(0 < t < \infty)$ such that

(i) ϕ is bounded and $\phi(t) = 0$ for all sufficiently small t;

(ii) $\phi(\mu_n) = \nu_n$, $\theta(\mu_n) = \theta_n$;

(iii) there exists a number χ' with $\chi \leq \chi' < \pi/2$ such that $|\theta(t) - \sigma_c \phi(t)| \leq \chi'$ for all t $(0 < t < \infty)$.

Finally, let $\alpha(t) = \sum_{\mu_n \leq t} a'_n$. Then

$$(7) \qquad f(s) = \sum_1^{\infty} a_n e^{-(\mu_n + i\nu_n)s} \doteq \int_0^{\infty} \exp\{-s[t + i\phi(t)] + i\theta(t)\}\, d\alpha(t).$$

It is easy to see that $\phi(t)$, $\theta(t)$, and $\alpha(t)$ can be chosen to satisfy our requirements. Also, if $\phi(t)$ is chosen as above, then $\phi(t)/t$ is bounded and tends to zero as t tends to infinity. Let

$$M_1 = \sup |\phi(t)/t|. \tag{8}$$

Our proof is derived from a method used by Delange [1] to obtain a theorem concerning the singularities of a class of Laplace-Stieltjes integrals.

LEMMA 1 (Delange [1]). *Let* x, ℓ, λ, *and* μ *be positive numbers* $(0 < \lambda < 1 < \mu)$, *and suppose that* $|y| \leq \ell$. *Let* $\alpha_1(t)$ *be an increasing function such that the abscissa of convergence of the integral* $\int_0^\infty \exp(-st)\,d\alpha_1(t)$ *is zero. Then to each positive* ε *there corresponds an integer* $N_0(\varepsilon)$ *such that for* $n \geq N_0(\varepsilon)$

$$\int_0^{\lambda n/x} \exp[-(x+iy)t]\,t^n\,d\alpha_1(t) \leq n!\left[\frac{\lambda e^{1-\lambda}}{x}(1+\varepsilon)\right]^n \tag{9}$$

and

$$\int_{\mu n/x}^\infty \exp[-(x+iy)t]\,t^n\,d\alpha_1(t) \leq n!\left[\frac{\mu e^{1-\mu}}{x}(1+\varepsilon)\right]^n. \tag{10}$$

Our second lemma is a generalization of this in the case $y = 0$.

LEMMA 2. *If* x, λ, *and* μ *are as in Lemma* 1 *and the conditions of Theorem* 1 *are satisfied, then to each positive* ε *there corresponds an integer* $N(\varepsilon)$ *such that for* $n \geq N(\varepsilon)$

$$\text{(a)}\quad \begin{aligned}|A_n(x)| &= \left|\int_0^{\lambda n/x} \exp\{-\zeta(t)(x+\sigma_c)+i\,\theta(t)\}\,[\zeta(t)]^n\,d\alpha(t)\right| \\ &\leq n!\left[\frac{\lambda e^{1-\lambda}}{x}(1+\varepsilon)\right]^n,\end{aligned}$$

$$\text{(b)}\quad \begin{aligned}|B_n(x)| &= \left|\int_{\mu n/x}^\infty \exp\{-\zeta(t)(x+\sigma_c)+i\,\theta(t)\}\,[\zeta(t)]^n\,d\alpha(t)\right| \\ &\leq n!\left[\frac{\mu e^{1-\mu}}{x}(1+\varepsilon)\right]^n,\end{aligned}$$

where $\zeta(t) = t + i\phi(t)$.

Since $\int_0^\infty \exp\{-s\,\zeta(t)+i\,\theta(t)\}\,d\alpha(t)$ is absolutely convergent for $\Re s > \sigma_c$, it follows that $\int_0^\infty \exp[-(x+\sigma_c)t]\,d\alpha(t)$ converges for $x > 0$. We write

$$\int_0^\infty \exp[-(x+\sigma_c)t]\,d\alpha(t) = \int_0^\infty \exp(-xt)\,d\beta(t),$$

where $\beta(t) = \int_0^t \exp(-\sigma_c u)\,d\alpha(u)$. Now let

$$z_n(t) = (1 + i\,\phi(t)/t)^n. \tag{11}$$

From (8), we obtain for all $t \geq 0$ the inequalities

$$|z_n(t)| \leq (1 + |\phi(t)/t|)^n \leq (1 + M_1)^n, \tag{12}$$

while for $t \geq \dfrac{\lambda n}{(1+M_1)ex}$ and $n \geq n_0(\varepsilon)$

$$|z_n(t)| \leq (1 + \varepsilon/3)^n, \tag{13}$$

since $|\phi(t)/t| \to 0$ as $t \to \infty$. Now

$$\left| \int_0^{\lambda n/x} [\zeta(t)]^n \exp\{-(x+\sigma_c)\zeta(t) + i\,\theta(t)\}\, d\alpha(t) \right|$$

$$= \left| \int_0^{\lambda n/x} t^n z_n(t) \exp\{-xt + i[\theta(t) - (x+\sigma_c)\phi(t)]\}\, d\beta(t) \right|$$

(14)

$$\leq \int_0^{\lambda n/[(1+M_1)ex]} t^n |z_n(t)| \exp(-xt)\, d\beta(t) + \int_{\lambda n/(1+M_1)ex}^{\lambda n/x} t^n |z_n(t)| \exp(-xt)\, d\beta(t)$$

$$\leq (1+M_1)^n \int_0^{\lambda n/[(1+M_1)ex]} t^n \exp(-xt)\, d\beta(t) + (1+\varepsilon/3)^n \int_0^{\lambda n/x} \exp(-xt)\, t^n d\beta(t).$$

Applying Lemma 1 to the second integral in the last member of (14), we obtain the inequality

$$\int_0^{\lambda n/x} \exp(-xt)\, t^n d\beta(t) \leq n! \left[\frac{\lambda e^{1-\lambda}}{x} (1+\varepsilon/3) \right]^n \tag{15}$$

for $n \geq N_0(\varepsilon/3)$.

Also, since $\int_0^\infty \exp(-\varepsilon_1 t)\, d\beta(t)$ converges for every $\varepsilon_1 > 0$,

$$\int_0^{\lambda n/[(1+M_1)ex]} d\beta(t) \leq K_1 \exp\left\{ \frac{\varepsilon_1 \lambda n}{(1+M_1)ex} \right\},$$

where K_1 is a constant. Since $\exp(-xt)t^n$ increases for $0 \leq t \leq n/x$, it follows that

$$\int_0^{\lambda n/(1+M_1)ex} \exp(-xt)\, t^n d\beta(t) \leq K_1 \exp\left[\frac{(\varepsilon_1/x - 1)\lambda n}{(1+M_1)e} \right] \left[\frac{\lambda n}{(1+M_1)ex} \right]^n$$

(16)

$$\leq n! \left[\frac{\lambda e^{1-\lambda}}{x} (1+\varepsilon/3) \right]^n \left\{ \frac{\exp\left[\lambda \left(1 - \frac{1}{(1+M_1)e} \right) \right]}{(1+M_1)e} \right\}^n$$

$$\leq n! \left[\frac{\lambda e^{1-\lambda}}{x} (1+\varepsilon/3) \right]^n \frac{1}{(1+M_1)^n},$$

provided we choose ε_1 so that $K_1^{1/n}\exp\left\{\frac{\varepsilon_1 \lambda}{(1+M_1)ex}\right\} \leq 1 + \varepsilon/3$.

Therefore, from (14), (15), and (16) we see that for n sufficiently large

$$\left|\int_0^{\lambda n/x} [\zeta(t)]^n \exp[-(x+\sigma_c)\zeta(t) + i\,\theta(t)]\,d\alpha(t)\right|$$

$$\leq [1 + (1+\varepsilon/3)^n]n!\left[\frac{\lambda e^{1-\lambda}}{x}(1+\varepsilon/3)\right]^n$$

$$\leq n!\left[\frac{\lambda e^{1-\lambda}}{x}(1+\varepsilon/3)\right]^n + n!\left[\frac{\lambda e^{1-\lambda}}{x}(1+3\varepsilon/4)\right]^n \qquad (\varepsilon < 3/4)$$

$$\leq 2n!\left[\frac{\lambda e^{1-\lambda}}{x}(1+3\varepsilon/4)\right]^n \leq n!\left[\frac{\lambda e^{1-\lambda}}{x}(1+\varepsilon)\right]^n,$$

which establishes equation (a). Turning our attention to relation (b), we see from (13) and Lemma 1 that if $\varepsilon < 1$ and $n \geq N(\varepsilon)$, then

$$\left|\int_{\mu n/x}^{\infty} [\zeta(t)]^n \exp[-(x+\sigma_c)\zeta(t) + i\,\theta(t)]\,d\alpha(t)\right|$$

$$\leq \int_{\mu n/x}^{\infty} t^n\,|z_n(t)|\,\exp[-(x+\sigma_c)t]\,d\alpha(t)$$

$$\leq n!\left[\frac{\mu e^{1-\mu}}{x}(1+\varepsilon/2)\right]^n (1+\varepsilon/3)^n \leq n!\left[\frac{\mu e^{1-\mu}}{x}(1+\varepsilon)\right]^n.$$

This establishes relation (b) and completes the proof of Lemma 2.

Completion of Proof of Theorem 1. Now suppose that f(s) is regular at the point $s = \sigma_c$. Then there exists a number $\rho > 1$ such that

$$f(\sigma_c + s) = \int_0^{\infty} \exp[-(\sigma_c + s)\zeta(t) + i\,\theta(t)]\,d\alpha(t)$$

is regular in the circle with centre at the point $s = 1$ and with radius ρ. Then

$$\int_{\lambda n}^{\mu n} \exp[-(1+\sigma_c)\zeta(t) + i\,\theta(t)][\zeta(t)]^n\,d\alpha(t) = (-1)^n f^{(n)}(1) - A_n(1) - B_n(1),$$

and therefore

$$\left|\int_{\lambda n}^{\mu n} \exp[-(1+\sigma_c)\zeta(t) + i\,\theta(t)][\zeta(t)]^n\,d\alpha(t)\right| \leq |f^{(n)}(1)| + |A_n| + |B_n|. \tag{17}$$

Let M_0 be the maximum of $|f(\sigma_c + s)|$ on the circle (centre 1 and radius $\rho/(1+\varepsilon/2)$), and let

$$\delta = \operatorname{Max}\left(\frac{1}{\rho}, \lambda e^{1-\lambda}, \mu e^{1-\mu}\right).$$

Clearly, $\delta < 1$, and by Cauchy's inequality,

$$|f^{(n)}(1)| \le n!\, M_0 \left(\frac{1+\varepsilon/2}{\rho}\right)^n \le n!\, M_0 [(1+\varepsilon/2)\,\delta]^n. \tag{18}$$

Therefore (12) and Lemma 2 imply that

$$\left|\int_{\lambda n}^{\mu n} \exp[-(1+\sigma_c)\zeta(t) + i\phi(t)][\zeta(t)]^n d\alpha(t)\right| \le (M_0+2) n!\, [(1+\varepsilon/2)\delta]^n \le n!\, [(1+\varepsilon)\delta]^n \tag{19}$$

when n is sufficiently large.

But equation (19) cannot hold. To see this, we proceed as follows.

$$\begin{aligned}\left|\int_{\lambda n}^{\mu n} \exp[-(1+\sigma_c)\zeta(t) + i\,\phi(t)][\zeta(t)]^n d\alpha(t)\right| &= \left|\int_{\lambda n}^{\mu n} \exp\{-\zeta(t) + i[\theta(t) - \sigma_c\phi(t)]\}\,[\zeta(t)]^n d\beta(t)\right| \\ &\ge \int_{\lambda n}^{\mu n} \Re\,\{e^{-\zeta(t)}\exp\{i[\theta(t) - \sigma_c\phi(t)]\}\}\;[\zeta(t)]^n d\beta(t).\end{aligned} \tag{20}$$

Now

$$\arg\{\exp[-\zeta(t) + i[\theta(t) - \sigma_c\phi(t)]][\zeta(t)]^n\} = -\phi(t) + \theta(t) - \sigma_c\phi(t) + n\arg\zeta(t). \tag{21}$$

But $\arg\zeta(t) = \tan^{-1}(\phi(t)/t)$, and for $t \ge \lambda n \ge \lambda n_1(\varepsilon)$,

$$\left|\frac{\phi(t)}{t} - \tan^{-1}\left(\frac{\phi(t)}{t}\right)\right| < \varepsilon_1\left|\frac{\phi(t)}{t}\right|.$$

Hence, for $\lambda n \le t \le \mu n$,

$$|n\arg\zeta(t) - \phi(t)| \le |\phi(t)| \max\left\{\frac{1+\varepsilon_1}{\lambda} - 1,\ 1 - \frac{(1-\varepsilon_1)}{\mu}\right\}. \tag{22}$$

Thus, for each $\varepsilon > 0$, we can choose λ and μ sufficiently close to 1 to ensure that

$$|n\arg\zeta(t) - \phi(t)| < \varepsilon\,|\phi(t)|. \tag{23}$$

Now we have chosen $\theta(t)$ and $\phi(t)$ so that $|\theta(t) - \sigma_c\phi(t)| \le \chi'$ and $\phi(t)$ is bounded. Let $M_2 = \sup|\phi(t)|$, and choose $\varepsilon < \frac{1}{2M_2}\left(\frac{1}{2}\pi - \chi'\right)$. Then, by (21) and (23),

$$\begin{aligned}(24)\qquad &|\arg\{e^{-\zeta(t)}\exp\{i[\theta(t)-\sigma_c\phi(t)]\}[\zeta(t)]^n\}|\\ &\le |n\arg\zeta(t)-\phi(t)| + |\theta(t)-\sigma_c\phi(t)|\\ &\le \chi' + \frac{1}{2}(\pi/2-\chi') = \frac{1}{2}(\pi/2+\chi') = \chi'' < \pi/2.\end{aligned}$$

Therefore

$$\begin{aligned}(25)\qquad &\Re\{\exp[-\zeta(t)+i\{\theta(t)-\sigma_c\phi(t)\}][\zeta(t)]^n\}\\ &\ge \cos\chi''\,|\exp[-\zeta(t)+i(\theta(t)-\sigma_c\phi(t))][\zeta(t)]^n|\\ &\ge K\exp(-t)\cdot(1-\varepsilon)^n t^n,\end{aligned}$$

where K is a positive constant. Therefore, by (20) and (25),

$$\begin{aligned}(26)\qquad &\left|\int_{\lambda n}^{\mu n}\exp\{-\zeta(t)+i[\theta(t)-\sigma_c\phi(t)]\}[\zeta(t)]^n d\beta(t)\right|\\ &\ge K(1-\varepsilon)^n\int_{\lambda n}^{\mu n}\exp(-t)t^n d\beta(t).\end{aligned}$$

We now choose $n \ge (\mu-1)^{-1}$ so that $\mu n \ge n+1$. Then

$$\begin{aligned}(27)\qquad &\left|\int_{\lambda n}^{\mu n}\exp\{-\zeta(t)+i[\theta(t)-\sigma_c\phi(t)]\}[\zeta(t)]^n d\beta(t)\right|\\ &\ge K(1-\varepsilon)^n\int_n^{n+1}\exp(-t)\,t^n d\beta(t) \ge K(1-\varepsilon)^n e^{-n-1}n^n\int_n^{n+1} d\beta(t).\end{aligned}$$

Now, if (19) holds, we obtain from (27) the relations

$$\int_n^{n+1} d\beta(t) \le K e^{n+1} n^{-n}(1+2\varepsilon)^n n!\,[(1+\varepsilon)\delta]^n \le K n^{1/2}[(1+4\varepsilon)\delta]^n,$$

for sufficiently large n, and hence

$$(28)\qquad \limsup_{n\to\infty}\left\{\int_n^{n+1}\exp(-\sigma_c t)\,d\alpha(t)\right\}^{1/n} < (1+4\varepsilon)\delta.$$

But (28) holds for every $\varepsilon > 0$, and therefore we have only to choose ε so that $(1+4\varepsilon)\delta < 1$ to ensure that

$$\limsup_{n\to\infty}\left\{\int_n^{n+1}\exp(-\sigma_c t)\,d\alpha(t)\right\}^{1/n} < 1.$$

This being so, there exists an $\eta > 0$ such that

$$\limsup_{n\to\infty} \left\{ \int_n^{n+1} \exp(-\sigma_c t)\, d\alpha(t) \right\}^{1/n} < e^{-\eta},$$

and therefore, since

$$\int_n^{n+1} \exp[(\eta - \sigma_c)t]\, d\alpha(t) < e^{\eta(n+1)} \int_n^{n+1} \exp(-\sigma_c t)\, d\alpha(t),$$

we have the inequalities

$$\begin{aligned} &\limsup_{n\to\infty} \left\{ \int_n^{n+1} \exp[(\eta - \sigma_c)t]\, d\alpha(t) \right\}^{1/n} \\ (29) \qquad &\leq e^{\eta} \limsup_{n\to\infty} \left\{ \int_n^{n+1} \exp(-\sigma_c t)\, d\alpha(t) \right\}^{1/n} < 1. \end{aligned}$$

Therefore the series $\sum_1^\infty \int_n^{n+1} \exp[(\eta - \sigma_c)t]\, d\alpha(t)$ converges, and hence the integral $\int_0^\infty \exp[(\eta - \sigma_c)t]\, d\alpha(t)$ also converges; in other words, the integral

$$\int_0^\infty \exp(-s\,\zeta(t) + i\,\theta(t))\, d\alpha(t)$$

is absolutely convergent at the point $s = \sigma_c - \eta$. But this contradicts the definition of σ_c as the abscissa of convergence of this integral. It follows that f(s) must have a singularity at $s = \sigma_c$, and Theorem 1 is thus established.

3. PROOF OF THEOREM 2

Consider the series $\sum_{n=1}^\infty a_n e^{-(\mu_n + i\nu_n)s} = f(s)$, where

$$a_n = \exp\frac{\sigma_c n}{2}, \qquad \mu_n = n/2, \qquad \nu_n = \frac{(-1)^n \pi}{2\sigma_c}.$$

Here

$$(30) \qquad f(s) = \sum_1^\infty \exp\left\{-\frac{n}{2}(s - \sigma_c) - \frac{(-1)^n i\pi s}{2\sigma_c}\right\}.$$

If n is even, we write $n = 2r$, combine the $(2r - 1)$st and $2r$-th terms, and obtain the relations

$$f(s) = \sum_{r=1}^{\infty} e^{-r(s-\sigma_c)} \{e^{(s-\sigma_c)/2} e^{i\pi s/2\sigma_c} + e^{-i\pi s/2\sigma_c}\}$$

$$(31) \qquad = \sum_{r=1}^{\infty} e^{-r(s-\sigma_c)} \left\{ \cos \frac{\pi s}{2\sigma_c} [e^{(s-\sigma_c)/2} + 1] + i \sin \frac{\pi s}{2\sigma_c} [e^{(s-\sigma_c)/2} - 1] \right\}$$

$$= \frac{\cos \frac{\pi s}{2\sigma_c} [e^{(s-\sigma_c)/2} + 1] + i \sin \frac{\pi s}{2\sigma_c} [e^{(s-\sigma_c)/2} - 1]}{e^{s-\sigma_c} - 1}.$$

By choice of a_n, the series clearly has abscissa of convergence σ_c, and we see from (31) that f(s) is regular at the point σ_c.

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